Risk Preferences and their Robust Representation

Michael Kupper
(joint work with Samuel Drapeau)

Humboldt University Berlin

6th World Congress of the Bachelier Finance Society
Toronto – June 24th 2010
The goal is to understand “risk” in a context (setting) independent manner, focusing on some crucial invariant features:

- “diversification should not increase the risk”
- “the better for sure, the less risky”

We consider a structural approach to risk which is motivated by the former theory on preferences and risk

- von Neumann and Morgenstern and their theory on preference comparison and utility for lotteries.
- Artzner, Delbaen, Eber and Heath; Föllmer and Schied and their theory of monetary risk measures for random variables.

and three recent preprints by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio on

- Quasiconvex risk measures
- Complete quasiconvex duality theory
- Uncertainty averse preferences

Based on the concept of acceptance families we will give a robust representation for a huge class of risk orders.
Outline

1. Risk Orders, Risk Measures and Risk Acceptance Families
2. Robust Representation
3. Illustrative Setting
Outline

1. Risk Orders, Risk Measures and Risk Acceptance Families
2. Robust Representation
3. Illustrative Setting
Risky positions in $\mathcal{X}$ are ordered $\cdots \succcurlyeq x \succcurlyeq y \succcurlyeq z \cdots$ according to a total preorder $\succcurlyeq$.

The relation $x \succcurlyeq y$ means “$x$ is riskier than $y$”. 
Risky positions in $\mathcal{X}$ are ordered $\ldots \succcurlyeq x \succcurlyeq y \succcurlyeq z \ldots$ according to a total preorder $\succcurlyeq$.

The relation $x \succcurlyeq y$ means “$x$ is riskier than $y$”.

In this framework, the notions of “diversification should not increase the risk” and “the better for sure the less risky” translate into

**Definition (Risk Order)**

A total preorder $\succcurlyeq$ is a risk order if it is

- **Quasiconvex**: $x \succcurlyeq \lambda x + (1 - \lambda) y$ whenever $x \succcurlyeq y$,
- **Monotone**: $x \succcurlyeq y$ whenever $y \succcurlyeq x$.

- Diversification imposes $\mathcal{X}$ convex. (in fact a mixture space)
- $\succcurlyeq$ is a preorder expressing a kind of “better than ... for sure”.
Risk Orders, Risk Measures and Risk Acceptance Families
A Setting Dependant Interpretation of Risk

Possible settings by the specification of the convex set $\mathcal{X}$ and the monotonicity preorder $\succ$. 

- **Random variables** on $(\Omega, \mathcal{F}, P)$ with as preorder $\succ$ the "$\geq P$-almost surely".
- **Stochastic processes** modeling cumulative wealth processes $X = X_0, X_1, \ldots, X_T$ with as preorder $\succ$ the cash flow monotonicity "$X_t - X_{t-1} := \Delta X_t \geq \Delta Y_t$".
- **Probability distributions** (lotteries) $\mathcal{M}_1$ is a convex set with standard monotonicity preorders $\succ$ either the first or second stochastic order.
- **Cumulative consumption streams** are right continuous non decreasing functions $c : [0, 1] \rightarrow \mathbb{R}^+$ building a convex cone. Here $c^{(1)}$ is "better for sure" than $c^{(2)}$ if $c^{(1)} - c^{(2)}$ is still a cumulative consumption stream.
- **Stochastic kernels** are probability distributions subject to uncertainty, that is, mappings $\tilde{X} : \Omega \rightarrow \mathcal{M}_1$. Possible preorders $\succ$ are either the $P$-almost sure first or second stochastic order.

...
Risk Orders, Risk Measures and Risk Acceptance Families

Definitions

Definition (Risk Order)

A total preorder \( \succeq \) on \( \mathcal{X} \) is a risk order if it is

- **Quasiconvex**: \( x \succeq \lambda x + (1 - \lambda) y \) whenever \( x \succeq y \),
- **Monotone**: \( x \succeq y \) whenever \( y \succeq x \).

Total preorders can (separability) be represented by functions \( F : \mathcal{X} \to [-\infty, +\infty] \)

\[ x \succeq y \quad \iff \quad F(x) \geq F(y) \]

Numerical representations of risk orders inherit their properties and belong to the following class:

Definition (Risk Measure)

A function \( \rho : \mathcal{X} \to [-\infty, +\infty] \) is a risk measure if it is

- **Quasiconvex**: \( \rho(\lambda x + (1 - \lambda) y) \leq \max\{\rho(x), \rho(y)\} \),
- **Monotone**: \( \rho(x) \leq \rho(y) \) whenever \( x \succeq y \).
Risk Orders, Risk Measures and Risk Acceptance Families

Definitions

Definition (Risk Order)

A total preorder \( \succeq \) on \( X \) is a risk order if it is

- **Quasiconvex:** \( x \succeq \lambda x + (1 - \lambda) y \) whenever \( x \succeq y \),
- **Monotone:** \( x \succeq y \) whenever \( y \succ x \).

Definition (Risk Measure)

A function \( \rho : X \rightarrow [-\infty, +\infty] \) is a risk measure if it is

- **Quasiconvex:** \( \rho(\lambda x + (1 - \lambda) y) \leq \max\{\rho(x), \rho(y)\} \).
- **Monotone:** \( \rho(x) \leq \rho(y) \) whenever \( x \succ y \).

Any risk measure defines at any level of risk \( m \in \mathbb{R} \) a risk acceptance set

\[
A^m = \{x \mid \rho(x) \leq m\}
\]

of those positions with a risk below \( m \). Here again, the family, called *risk acceptance family*, gets properties from the risk measure and belongs to the following class.
## Definitions

### Risk Order

A total preorder $\succeq$ on $\mathcal{X}$ is a risk order if it is

- **Quasiconvex**: $x \succeq \lambda x + (1 - \lambda) y$ whenever $x \succeq y$,
- **Monotone**: $x \succeq y$ whenever $y \succ x$.

### Risk Measure

A function $\rho : \mathcal{X} \to [-\infty, +\infty]$ is a risk measure if it is

- **Quasiconvex**: $\rho(\lambda x + (1 - \lambda) y) \leq \max\{\rho(x), \rho(y)\}$.
- **Monotone**: $\rho(x) \leq \rho(y)$ whenever $x \succeq y$.

### Risk Acceptance Family

A family $\mathcal{A} = (\mathcal{A}^m)_{m \in \mathbb{R}}$ of subset of $\mathcal{X}$ is a risk acceptance family if it is

- **Convex**: $\mathcal{A}^m$ is convex,
- **Monotone**: $\mathcal{A}^m \subset \mathcal{A}^n$ and $x \succeq y$ for some $y \in \mathcal{A}^m$ implies $x \in \mathcal{A}^m$,
- **Right-Continuous**: $\mathcal{A}^m = \bigcap_{n>m} \mathcal{A}^n$. 
Theorem (Risk Orders ↔ Risk Measures ↔ Risk Acceptance Families)

- Any numerical representation \( \rho \) of a risk order \( \succeq \) is a risk measure. Any risk measure \( \rho \) defines a risk order \( \succeq \) through
  \[ x \succeq y \iff \rho(x) \geq \rho(y) \]
- Risk measures and risk acceptance families are related one to one through
  \[ A^m := \{ x \in \mathcal{X} \mid \rho(x) \leq m \} \quad \text{and} \quad \rho(x) = \inf \{ m \mid x \in A^m \} \]

Axioms of monotonicity or quasiconvexity for the risk orders are global!

Economic Index of Riskiness (Aumann and Serrano; Forster and Hart)

For a loss function \( l \), consider

\[ \lambda(X) = \sup \{ \lambda > 0 \mid E[l(-\lambda X)] \leq c \} \]

which represents the maximal exposure to a position \( X \) provided that the expected loss remains below and a threshold. The economic index of riskiness is then defined as

\[ \rho(X) := 1/\lambda(X) \implies A^m = \{ X \mid c \geq E[l(-X/m)] \} \]
Risk Orders, Risk Measures and Risk Acceptance Families
One-to-One Relation between Risk Orders, Risk Measures and Risk Acceptance Families

Theorem (Risk Orders ↔ Risk Measures ↔ Risk Acceptance Families)

- Any numerical representation \( \rho \) of a risk order \( \succeq \) is a risk measure. Any risk measure \( \rho \) defines a risk order \( \succeq \) through

\[
\begin{align*}
x \succeq y & \iff \rho(x) \geq \rho(y) \\
\end{align*}
\]

- Risk measures and risk acceptance families are related one to one through

\[
A^m := \{x \in X \mid \rho(x) \leq m\} \quad \text{and} \quad \rho(x) = \inf \left\{ m \mid x \in A^m \right\}
\]

Axioms of monotonicity or quasiconvexity for the risk orders are global!

Certainty Equivalent

- **Probability Distributions**: \( l : \mathbb{R} \to [-\infty, +\infty[ \) nondecreasing loss function

\[
\rho(\mu) = l^{-1} \left( \int l(-x) \mu(dx) \right) \quad \implies \quad A^m = \left\{ \mu \mid \int l(-x) \mu(dx) \leq l(m) \right\}
\]

- **Random Variables**: \( l : \mathbb{R} \to [-\infty, +\infty[ \) is a nondecreasing convex loss function.

\[
\rho(X) = l^{-1} (E[l(-X)]) \quad \implies \quad A^m = \left\{ X \mid E[l(-X)] \leq l(m) \right\}
\]
Risk Orders, Risk Measures and Risk Acceptance Families
Further Properties: Convexity, Positive Homogeneity and Scaling Invariance

**Proposition (Convexity, Positive Homogeneity and Scaling Invariance)**

(i) \( \rho \) is convex iff \( \lambda A^m + (1 - \lambda) A^{m'} \subset A^\lambda m + (1 - \lambda) m' \).

(ii) \( \rho \) is positive homogeneous iff \( \lambda A^m = A^\lambda m \) for \( \lambda > 0 \).

(iii) \( \rho \) is scaling invariant iff \( \lambda A^m = A^m \) for \( \lambda > 0 \).

(iv) \( \rho \) is affine iff \( \succsim \) is independent and archimedean.

These properties are no longer global!

**Examples (Savage; Markowitz; Sharpe; von Neumann and Morgenstern)**

- **Savage representation:** \( \rho (X) := E_Q [l(-X)] \) convex RM if \( l \) is a convex loss function.

- **Mean Variance:** \( \rho (X) = E [-X] + \frac{\gamma}{2} \text{Var} (X) \) convex RM monotone w.r.t. trivial order.

- **Sharpe Ratio:** \( \rho (X) = E [-X] / \sqrt{E [X^2 - E [X]^2]} \) scaling invariant RM monotone w.r.t. trivial order

- **von Neumann and Morgenstern:** \( \rho (\mu) = \int l(-x) \mu (dx) \) affine RM monotone w.r.t. the first stochastic order if \( l \) is a loss function.
Existence of a numéraire $\pi$. $\mathcal{X}$ is a vector space and $\succ$ a vector order.

**Definition (Cash Additive and Subadditive Risk Measures)**

A risk measure $\rho$ is

- **Cash Additive** if $\rho(x + m\pi) = \rho(x) - m$ for any $m \in \mathbb{R}$.
- **Cash Subadditive** if $\rho(x + m\pi) \geq \rho(x) - m$ for any $m > 0$.

**Proposition**

- $\rho$ is cash additive iff $\mathcal{A}^0 = \mathcal{A}^m + m\pi$.
- $\succ$ is cash additive iff
  1. $y \succ x \succ z$ implies the existence of a unique $m \in \mathbb{R}$ such that $x \sim m\pi$;
  2. $x \succ y$ implies $x + m\pi \succeq y + m\pi$.

- A cash additive risk measure $\rho$ is automatically convex.

**Classical cash additive monetary risk measures**

- **Average Value at Risk**: $\text{AV}_R q(X) = \sup_Q \{E_Q[-X] \mid dQ/dP < 1/q\}$.
- **Entropie**: $\rho(X) = \ln(E[e^{-X}])$.
- **Optimized Certainty Equivalent**: $\rho(X) = -\sup_m \{m + E[f(X - m)]\}$. 
Outline

1 Risk Orders, Risk Measures and Risk Acceptance Families

2 Robust Representation

3 Illustrative Setting
Robust Representation of Risk Orders

Setup, Lower Semicontinuous Risk Orders

- $\mathcal{X}$ is a locally convex topological vector space with dual $\mathcal{X}^*$.
- $\succsim$ is a vector order: $x \succsim y$ iff $x - y \in \mathcal{K}$ closed convex cone with polar cone $\mathcal{K}^\circ$.

Examples

- $L^\infty$, $\mathcal{K} = L^\infty_+$, weak topology $\sigma(L^\infty, L^1)$, dual $L^1$, polar cone $\mathcal{K}^\circ = L^1_+$.
- $\mathcal{M}_{1,c} \subset ca_c = \mathcal{X}$, weak topology $\sigma(ca_c, C) \implies \mathcal{X}^* = C$.
  - First stochastic order: $\mathcal{K}^1 = \{\mu \mid \int f \, d\mu \geq 0, \text{ for all nondecreasing } f\}$.
  - Second stochastic order: $\mathcal{K}^2 = \{\mu \mid \int f \, d\mu \geq 0, \text{ for all nondecreasing concave } f\}$.

Definition (Lower Semicontinuous Risk Orders)

A risk order $\succsim$ is lower semicontinuous if $L(x) = \{y \in \mathcal{X} \mid x \succsim y\}$ is $\sigma(\mathcal{X}, \mathcal{X}^*)$-closed for any $x \in \mathcal{X}$.

Proposition (Metha)

A risk order $\succsim$ is separable and lower semicontinuous if and only if there exists a corresponding lower semicontinuous risk measure $\rho$.
Moreover, the class of corresponding lower semicontinuous risk measures is stable under lower semicontinuous increasing transformations.
Robust Representation of Risk Orders
Setup, Lower Semicontinuous Risk Orders

- $X$ is a locally convex topological vector space with dual $X^*$.
- $\succsim$ is a vector order: $x \succsim y$ iff $x - y \in \mathcal{K}$ closed convex cone with polar cone $\mathcal{K}^\circ$.

Examples

- $L^\infty$, $\mathcal{K} = L^\infty_+$, weak topology $\sigma(L^\infty, L^1)$, dual $L^1$, polar cone $\mathcal{K}^\circ = L^1_+$.
- $M_{1,c} \subset ca_c = X$, weak topology $\sigma(ca_c, C) \implies X^* = C$.
  - First stochastic order: $\mathcal{K}^1 = \{\mu \mid \int f \, d\mu \geq 0, \text{ for all nondecreasing } f\}$.
  - Second stochastic order: $\mathcal{K}^2 = \{\mu \mid \int f \, d\mu \geq 0, \text{ for all nondecreasing concave } f\}$.

Definition (Lower Semicontinuous Risk Orders)

A risk order $\succsim$ is lower semicontinuous if $L(x) = \{y \in X \mid x \succsim y\}$ is $\sigma(X, X^*)$-closed for any $x \in X$.

Proposition (Metha)

A risk order $\succsim$ is separable and lower semicontinuous if and only if there exists a corresponding lower semicontinuous risk measure $\rho$.
Moreover, the class of corresponding lower semicontinuous risk measures is stable under lower semicontinuous increasing transformations.
Robust Representation of Risk Orders

Main robust representation result:

**Theorem**

Any lower semicontinuous risk measure \( \rho : \mathcal{X} \rightarrow [-\infty, +\infty] \) has a robust representation:

\[
\rho(x) = \sup_{x^* \in \mathcal{K}^\circ} R(x^*, \langle x^*, -x \rangle)
\]

for a unique maximal risk function \( R \in \mathcal{R}^{\text{max}} \).

**Definition**

\( \mathcal{R}^{\text{max}} \) denotes the set of maximal risk functions

\[
R : \mathcal{K}^\circ \times \mathbb{R} \rightarrow [-\infty, +\infty]
\]

- nondecreasing and left-continuous in the second argument
- \( R \) is jointly quasiconcave,
- \( R(\lambda x^*, s) = R(x^*, s/\lambda) \) for any \( \lambda > 0 \),
- \( R \) has a uniform asymptotic minimum, \( \lim_{s \rightarrow -\infty} R(x^*, s) = \lim_{s \rightarrow -\infty} R(y^*, s) \),
- its right-continuous version, \( R^+(x^*, s) := \inf_{s' > s} R(x^*, s) \) is upper semicontinuous in the first argument.
Main robust representation result:

**Theorem**

Any lower semicontinuous risk measure \( \rho : \mathcal{X} \to [-\infty, +\infty] \) has a robust representation:

\[
\rho(x) = \sup_{x^* \in \mathcal{K}^o} R(x^*, \langle x^*, -x \rangle)
\]

for a unique maximal risk function \( R \in \mathcal{R}_{max} \).

**Example: The cash additive case on \( L^\infty \)**

\[
\rho(X) = \sup_Q \{ E_Q [-X] - \alpha_{\min}(Q) \}
\]

In this case:

\[
R(Q, s) = s - \alpha_{\min}(Q).
\]

Moreover, \( \rho \) and \( \alpha_{\min} \) are one-to-one.
Robust Representation of Risk Orders

Representation Theorem

Main robust representation result:

Theorem

Any lower semicontinuous risk measure \( \rho : \mathcal{X} \rightarrow [-\infty, +\infty] \) has a robust representation:

\[
\rho(x) = \sup_{x^* \in \mathcal{K}^o} R(x^*, \langle x^*, -x \rangle)
\]

for a unique maximal risk function \( R \in \mathcal{R}^\text{max} \).

Conversely, for any risk function \( R : \mathcal{K}^o \times \mathbb{R} \rightarrow [-\infty, +\infty] \) which is nondecreasing and left-continuous in the second argument

\[
\rho(x) = \sup_{x^* \in \mathcal{K}^o} R(x^*, \langle x^*, -x \rangle)
\]

is a lower semicontinuous risk measures

- The one-to-one relation between \( \rho \) and the risk function \( R \in \mathcal{R}^\text{max} \) is crucial!
  - makes comparative statics meaningful
  - Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio introduced the notion of complete quasiconvex duality.

- Cerreia-Vioglio et al. provide complete quasiconvex duality results on \( M \)-spaces with unit under further assumptions on the monotonicity
  \( \rightarrow (L^\infty, \| \cdot \|_\infty), \mathcal{K} = L^\infty_+ \)
Robust Representation of Risk Orders

Representation Theorem: Modifications

- In case the order is regular, i.e., there is $\pi$ with $\langle x^*, \pi \rangle > 0$ for all $x^* \in \mathcal{K}^\circ \setminus \{0\}$, one gets a robust representation:

$$\rho(x) = \sup_{x^* \in \mathcal{K}^\circ \pi} R(x^*, \langle x^*, -x \rangle)$$

where $\mathcal{K}^\circ \pi = \{x^* \in \mathcal{K}^\circ \mid \langle x^*, \pi \rangle = 1\}$.

- In case of random variables with $\pi = 1$, $\mathcal{K}^\circ_1$ is a set of probability measures.
- The first and second stochastic order are not regular

Similar robust representation results hold on open/closed convex sets rather than vector spaces

The setup is general and includes the following risk orders/preferences:

- Expected utilities (Von Neumann and Morgenstern)
- Mean variance preferences (Markowitz)
- Coherent and convex risk measures (Artzner et al. and Föllmer/Schied and Frittelli/Gianin)
- Performance measures such as the Sharpe ratio and their monotone versions (Cherny and Madan)
- Economic index of riskiness (Aumann and Serrano)
- Value at risk
- Intertemporal preference functionals (Hindy, Huan and Kreps)
- Multiprior maxmin expected utilities (Gilboa and Schmeidler)
- Variational preferences (Maccheroni et al.)
- Uncertainty averse preferences (Cerreia-Vioglio et al.)
- ...
Robust Representation of Risk Orders

Sketch of the proof and computation of the risk function

- Start with the risk acceptance family $\mathcal{A} = (\mathcal{A}^m)_{m \in \mathbb{R}}$ corresponding to $\rho$.

- For any risk level $m$ holds by means of the theory of \textbf{cash-additive risk measures}

  $$X \in \mathcal{A}^m \iff E_Q[-X] \leq \alpha_{\min}(Q, m) \quad \text{for all } Q$$

  where $\alpha_{\min}(Q, m) = \sup_{X \in \mathcal{A}^m} E_Q[-X]$ is a penalty function.

- Then

  $$\rho(X) = \inf \{ m \in \mathbb{R} \mid X \in \mathcal{A}^m \}$$

  $$= \inf \{ m \in \mathbb{R} \mid E_Q[-X] - \alpha_{\min}(Q, m) \leq 0 \quad \text{for all } Q \}$$

  $$= \sup_{Q} \inf \{ m \mid E_Q[-X] \leq \alpha_{\min}(Q, m) \}$$

  $$= \sup_{Q} R(Q, E_Q[-X])$$

  where $R(Q, \cdot)$ is the generalized left-inverse of $m \mapsto \alpha_{\min}(Q, m)$

- The difficult part of the proof is to show that the duality is complete.
Outline

1. Risk Orders, Risk Measures and Risk Acceptance Families
2. Robust Representation
3. Illustrative Setting
Illustrative Settings

Random Variables

Proposition

Any lower semicontinuous risk measure \( \rho : \mathbb{L}^\infty \to [-\infty, +\infty] \) has a robust representation:

\[
\rho(X) = \sup_{Q \in \mathcal{M}_{1,f}(P)} R(Q, E_Q [-X]), \quad X \in \mathbb{L}^\infty,
\]

for a unique \( R \in \mathcal{R}^{max} \). In particular, if \( \rho \) is cash additive,

\[
\rho(X) = \sup_{Q \in \mathcal{M}_{1,f}(P)} \{ E_Q [-X] - \alpha_{min}(Q, 0) \}.
\]

- In case that \( \rho \) has the Fatou property it is well known (Delbaen) that \( \rho \) is \( \sigma(\mathbb{L}^\infty, \mathbb{L}^1) \)-lower semicontinuous and the previous representations hold in terms of \( \mathcal{M}_1(P) \).
- Under adequate lower semicontinuity conditions, robust representations of risk measures on \( \mathbb{L}^p \) and Orlicz spaces work analogously.
Illustrative Settings

Random Variables

Robust Representation of the Certainty Equivalent $\rho(X) = l^{-1}(E[l(-X)])$.

- **Quadratic Function:** $l(s) = s^2/2 - s$ for $s \geq -1$ and $l(s) = -1/2$ elsewhere.

  $$\rho(X) = \sup_{Q \in \mathcal{M}_1(P)} \left\{ \frac{E_Q[-X] + 1}{\|dQ/dP\|_{L^2}} - J(E_Q[-X]) \right\}, \quad X \in L^{\infty},$$

  whereby $J(s) = 1$ if $s > 1$ and $J(s) = +\infty$ elsewhere.

- **Logarithm Function:** If $l(s) = -\ln(-s)$ for $s < 0$, then

  $$\rho(X) := -\exp(E[\ln(X)]) = \sup_{Q \in \mathcal{M}_1(P)} \left\{ \frac{E_Q[-X]}{\exp(E[\ln(dQ/dP)])} \right\}, \quad X \in L^{\infty}.$$

**Economic Index of Riskiness (Aumann and Serrano)**

$$\rho(X) = \sup_{Q} \frac{E_Q[-X]}{E_Q[\ln(c_0dQ/dP)]}, \quad X \in L^{\infty}.$$
THANK YOU FOR YOUR ATTENTION!