Optimal execution in limit order books with stochastic liquidity

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Bachelier meeting, Toronto, June 2010
Problem: Minimize impact on execution prices (as in Predoiu, Shaikhet, Shreve)
Limit order book model with *stochastic liquidity*
Structure of optimal strategies
Examples and numerical implementation
Market buy order of $x_0$ shares at $t = 0$ has linear price impact.

- Ask price $A_t$ martingale and bid $B_t < A_t$
  - Effect of $A$ can be neglected for risk neutral investor
- Dynamic of price displacement $D$ with resilience speed $\rho > 0$
  \[ dD_t = \frac{1}{q_t} d\Theta_t - \rho D_t dt \]
- Impact cost at $t$: \[ \left( D_t + \frac{1}{2q_t} x_t \right) x_t \]
Model with stochastic liquidity

- Dynamic order book height: \( K_t := \frac{1}{q_t} \) e.g. positive diffusion
- Risk-neutral investor wants to purchase \( x \) shares on \([t, T]\)

Singular control problem in continuous time

\[
U(t, \delta, x, \kappa) := \inf_{\Theta \in \mathcal{A}(x)} J(t, \delta, \Theta, \kappa)
\]

Admissible strategies \( \mathcal{A}(x) \)

\( \Theta : \Omega \times [t, T] \rightarrow [0, x] \) adapted, increasing, càglàd, \( \Theta_t = 0, \Theta_{T^+} = x \) a.s.

Trading costs \( (\Delta \Theta_s := \Theta_{s^+} - \Theta_s) \)

\[
J(\Theta) := J(t, \delta, \Theta, \kappa) := \mathbb{E} \left[ \int_{[t, T]} \left( D_s + \frac{K_s}{2} \Delta \Theta_s \right) d\Theta_s \bigg| D_t = \delta, K_t = \kappa \right]
\]
Intuition: Wait and Buy region

- Scaling property of value function reduces dimension:

\[ U(t, a\delta, ax, \kappa) = a^2 U(t, \delta, x, \kappa) \text{ for } a \in \mathbb{R}_{\geq 0} \]

\[ \frac{a}{\delta} = 1 \Rightarrow U(t, \delta, x, \kappa) = \delta^2 U(t, 1, \frac{x}{\delta}, \kappa) \]
Intuition: Wait and Buy region

- Scaling property of value function reduces dimension:

$$U(t, a\delta, a\kappa) = a^2 U(t, \delta, x, \kappa) \text{ for } a \in \mathbb{R}_{\geq 0}$$

$$a = \frac{1}{\delta} \Rightarrow U(t, \delta, x, \kappa) = \delta^2 U(t, 1, \frac{x}{\delta}, \kappa)$$

- How could optimal strategy look like for fixed $t$ and $\kappa$?
- Wait if $\frac{x}{\delta}$ is small, say $\frac{x}{\delta} \leq c \in (0, \infty]$.
- Otherwise buy $\xi > 0$ shares s.t. $\frac{x-\xi}{\delta+\frac{\xi}{q}} = \frac{1}{c}$.
Binomial model and resilience = 2

Scenario A

Scenario B

\[ \kappa_t \]

\[ \kappa_0 = 2.1 \]

\[ p = 1/2 \]

\[ t_0 = 0 \quad t_1 = 0.0001 \quad T = 1 \]

Discrete Trading

Non-uniqueness

No Trading

\[ 0 \quad 2.1 \quad \kappa_0 \]

\[ X/D \]
Theorem (F./Schöneborn/Urusov)

\[ dK_s = \mu(s, K_s)ds + \sigma(s, K_s)dW_s \]

Let \( K \) be a positive, continuous diffusion satisfying

i) \( \eta_s := \frac{2}{K_s} + \frac{\mu(s, K_s)}{K_s^2} - \frac{\sigma^2(s, K_s)}{K_s^3} > 0 \quad \text{for all } s \in [t, T] \)

ii) \( \mathbb{E}\left[ \sup_{s \in [t, T]} \frac{K_s^2}{\inf_{s \in [t, T]} K_s} \right] < \infty \)

iii) \( \mathbb{E}\left[ \left( \int_t^T |\eta_s|ds \right) \left( \sup_{s \in [t, T]} K_s^2 \right) \right] < \infty \)

Then \( J(\Theta) \) is strictly convex and there exists a unique optimal strategy \( \Theta^* \).
Theorem (F./Schöneborn/Urusov)

\[ dK_s = \mu(s, K_s)ds + \sigma(s, K_s)dW_s \]

Let \( K \) be a positive, continuous diffusion satisfying

i) \( \eta_s := \frac{2\rho}{K_s} + \frac{\mu(s, K_s)}{K_s^2} - \frac{\sigma^2(s, K_s)}{K_s^3} > 0 \quad \text{for all } s \in [t, T] \)

ii) \( \mathbb{E} \left[ \sup_{s \in [t, T]} K_s^2 \frac{\inf_{s \in [t, T]} K_s}{K_s} \right] < \infty \)

iii) \( \mathbb{E} \left[ (\int_t^T |\eta_s| ds) \left( \sup_{s \in [t, T]} K_s^2 \right) \right] < \infty \)

Then \( J(\Theta) \) is strictly convex and there exists a unique optimal strategy \( \Theta^* \).

Idea:

- Strict convexity: rewrite \( J \) in terms of \( D \) via \( dD_s = K_s d\Theta_s - \rho D_s ds \)
  \( J(\Theta) \approx \mathbb{E} \left[ \int_{[t, T]} \eta_s D_s^2 ds \right] \sim \text{Assumption i)} \)
- Existence: Komlos argument
Uniqueness ensures WR-BR structure

Theorem (F./Schöneborn/Urusov)

Under the above assumptions there exists a unique barrier function $c : [0, T] \times (0, \infty) \rightarrow (0, \infty]$ with $c(T, \kappa) \equiv 0$ such that

$$\Delta \Theta^*_t(t, \delta, x, \kappa) = \max \left\{ 0, \frac{x - c(t, \kappa)\delta}{1 + \kappa c(t, \kappa)} \right\}. \quad (1)$$
Theorem (F./Schöneborn/Urusov)

Under the above assumptions there exists a unique barrier function $c : [0, T] \times (0, \infty) \to (0, \infty]$ with $c(T, \kappa) \equiv 0$ such that

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Idea:

- Trade splitting argument
- Exclude upper WR by uniqueness
Example 1/3: $K$ deterministic

- $K$ càglàd, bounded ensures WR-BR structure
- Obizhaeva/Wang ($dK_t = 0$) gives $c(t, \kappa) = \frac{\rho(T-t)+1}{\kappa}$
- Explicit barrier via Euler-Lagrange formalism, e.g., $K_t = K_0 e^{\nu \rho t}$ gives

$$c(t, \kappa) = \begin{cases} 
\infty & \text{if } \nu < -1 \\
\frac{1+\nu - e^{-\rho \nu (T-t)}}{\nu(1+\nu)\kappa} & \text{otherwise}
\end{cases}$$
Example 2/3: \( K \) GBM

\[
dK_t = K_t(\mu_t dt + \sigma_t dW_t)
\]

- WR-BR-WR examples exist for time-inhomogeneous GBM
- If WR-BR structure holds: \( c(t, \kappa_t) = \frac{c(t)}{\kappa_t} \) via scaling property, 'bad model' due to passive in the liquidity behavior
Example 3/3: K CIR- numerical scheme

\[ dK_s = \mu(K - K_s)ds + \sigma \sqrt{K_s} dW_s \]

1. **Possible idea:**
   Implement HJB equation (QVI) by finite difference scheme
   \[
   \min \left\{ \kappa U_D - U_X + D, \ U_t - \rho D U_D + \mu (\kappa - \kappa) U_\kappa + \frac{\sigma^2}{2} \kappa U_{\kappa\kappa} \right\} = 0
   \]

2. **Here:**
   Approximate state space diffusion by a Markov chain à la Kushner
   - Code is essentially the same as in 1.
   - Convergence proof by probabilistic methods, i.e. no use of HJB eq./verification argument or convexity/smoothness/growth conditions
Example 3/3: $K$ CIR- WR-BR-WR example (for large vola)

Non-uniqueness of optimal strategy
Example 3/3: $K$ CIR- aggressive in the liquidity behavior (for high mean-reversion)
Market microstructure model of order book to study optimal execution problem

Stochastic liquidity $\sim$ differential order placement

Wait/Buy Region structure does not always hold!

Numerical analysis via Markov chain implementation: Aggressive/passive in the liquidity behavior


Thank you for your attention!