# Forest of Stochastic Trees: A New Method for Valuing High Dimensional Swing Options

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Bachelier 2010 - Toronto

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June 24, 2010

# Executive Summary

- Algorithm for pricing swing opitions with a high-dimensional underlying and modest number of exercise opportunities and rights.
- Easily accommodates general price processes and payoffs.
- Generates high- and low-biased estimators.
- Estimators converge in the $p$-norm and are consistent.
- Confidence intervals for the true option value.
- Computationally intensive.

# American-style Option

- An *American-style* option allows exercise any time prior to and at maturity.
- Given that the option has not yet been exercised at time $t$, its time-$t$ value is

$$B_t = \sup_{t \leq \tau \leq T} \mathbb{E}[P_\tau | \mathcal{F}_t]$$

where $P_t$ is the discounted exercise value at $t$.

# Valuation of American-style Options

- Valuation is done via dynamic programming through the recursive equations

$$H_k = \mathbb{E}\left[ B_{k+1} \middle| \mathcal{F}_k \right] \qquad \text{and}$$
$$B_k = \max(H_k, P_k),$$

where

- $H_k$ is the hold value of the option;
- $P_k$ is the value if exercised;
- $B_k$ is the current value of the option;
- the terminal condition is $H_M = 0$;
- $M$ is option expiry; and
- $k = k\Delta T$ denotes time.

# Motivation for Monte Carlo Methods

Monte Carlo methods:

- Convergence rate is independent of the dimension.
- Flexible in terms of underlying processes used.
- Easy to use multi-factor models.

# The Stochastic Tree

- In order to value the option we must simulate paths of the underlying asset.
- The tree method does this by beginning with an initial value and then generating successive *iid* branches from this node. From each of these nodes more *iid* branches are generated and so on (Broadie and Glasserman 1997).

# The Stochastic Tree


Figure: Stochastic Tree at timestep 0, $b = 3$

# The Stochastic Tree

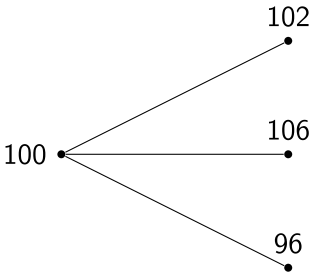


Figure: Stochastic Tree at timestep 1, $b = 3$

# The Stochastic Tree

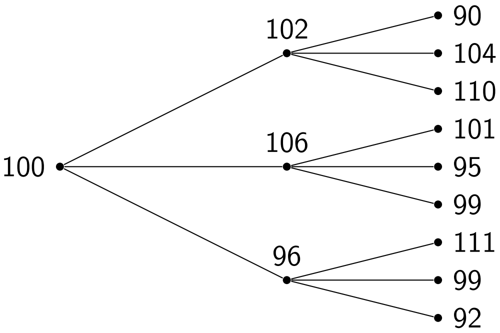


Figure: Stochastic Tree at timestep 2, $b = 3$

# Evaluation Via Dynamic Programming

The valuation process can be summarized by the following recursive relation:

$$\hat{B}_M^j = P_M^j$$
$$\hat{H}_k^j = \frac{1}{b}\sum_{i=1}^{b} \hat{B}_{k+1}^i$$
$$\hat{B}_k^j = \max\left\{P_k^j, \hat{H}_k^j\right\},$$
$$k = 0, \ldots, M-1.$$

- $P_k^j$ the exercise value at time $k$ in state $j$.
- $\hat{B}_0$ is a biased estimate to the true value and the bias is positive.

Discounting factor omitted.

# Estimators

- In addition a low- (negatively) biased estimate may also be constructed.
- Both estimators converge in the $p$-norm and are consistent.
- Averaging over independent repeated valuations gives:
  - High- and low-biased estimates to the true value.
  - These may be used to construct confidence intervals for the option price.

# What is a Swing Option?

- Swing options or take-and-pay options may be considered as a generalization of American-style options as they provide the holder multiple exercise rights (call and/or put-style) at predetermined prices ($K_u$ and $K_d$).
- They allow the holder control of both the timing and amount of delivery of the underlying asset at predetermined prices.

# What is a Swing Option?

- Swing options have typically been used in energy markets to help producers manage the raw materials used in energy production in the face of uncertain demand.
- They are typically part of a larger contract structure which would also include a futures portion to deliver a base amount of the underlying at specific intervals.
- They are OTC.

# Why are they difficult to price?

- The valuation of swing options is complicated by the fact that the holder has multiple exercise rights and with each exercise right, there is a choice in the amount exercised.
- As with the pricing of American-style options, the valuation of swing options is a problem in stochastic optimal control with three relevant state variables:
  - usage level
  - number of rights remaining
  - spot price
- In addition these contracts may also include penalties.

# Recursive Equations for Swing Option Pricing

- When exercised, assume the choice of two volumes, $v_1, v_2$.
- $B_k(S_k, \mathcal{N}_k, V_k)$ — the time-$k$ option value.
- $P_k(S_k, \mathcal{N}_k, V_k, v)$ — the payoff from exercising $v$ units at $k$.
- continuation value

$$H_k(S_k, \mathcal{N}_{k+1}, V_{k+1}) = \mathbb{E}[B_{k+1}(S_{k+1}, \mathcal{N}_{k+1}, V_{k+1}) | S_k, \mathcal{N}_{k+1}, V_{k+1}]$$

- Option value is given by

$$\begin{aligned} B_k = \max( & P_k(S_k, \mathcal{N}_k, V_k, v_1) + H_k(S_k, \mathcal{N}_k - 1, V_k + v_1), \\ & P_k(S_k, \mathcal{N}_k, V_k, v_2) + H_k(S_k, \mathcal{N}_k - 1, V_k + v_2), \\ & H_k(S_k, \mathcal{N}_k, V_k)), \end{aligned}$$

with the terminal conditions

$$\begin{aligned} B_N = \max( & P_N(S_N, \mathcal{N}_N - 1, V_N, v_1), P_N(S_N, \mathcal{N}_N - 1, V_N, v_2), \\ & P_N(S_N, \mathcal{N}_N, V_N, 0)) \end{aligned}$$

# Pricing Methods

- Solution to a system of HJB quasi-variational inequalities (Dahlgren, 2005)
  - Much more complex than Forest of Trees.
  - Price and derivatives estimates more accurate and stable.
- Monte Carlo Methods
  - Computation of the optimal exercise frontiers (Ibanez, 2004 and Barrera-Esteve et al., 2006)
  - Use of duality to generate high- and low-biased estimates (Meinshausen and Hambly, 2004).
  - Modified Least-squares to estimate continuation values (Barrera-Esteve et al., 2006)
- Forest of Trees (Lari-Lavassani et al., 2001 and Jaillet et al., 2004)
  - Discretize usage level and spot price.
  - Pricing is done using backward induction.

# Evaluating the forest of stochastic trees

- For swing options we extend the forest of trees method and create a stochastic tree for each *state* (swing rights remaining and usage level).
- Each stochastic tree therefore depends on the original asset price stochastic tree and on all other reachable states.
- The process for generating the stochastic tree containing asset values is identical to that done for American options.

# The Forest

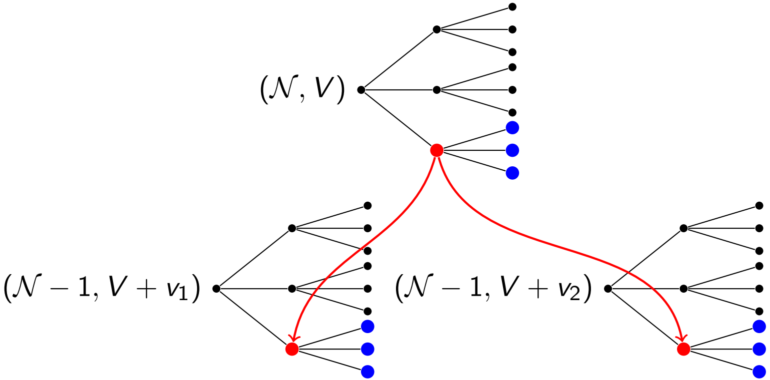


Figure: Section of a Forest, $\mathcal{N} = \#$ of Swing rights remaining, $V =$ usage level.

# Estimators for the forest

- High- (positively) and low- (negatively) biased estimate may be constructed.
- Both estimators converge in the $p$-norm and are consistent.
- Averaging over independent repeated valuations gives:
  - High- and low-biased estimates to the true value.
  - These may be used to construct confidence intervals for the option price.

# Example - Underlying Process

In our simulations the underlying assets are uncorrelated and their prices are taken to follow a risk neutralized GBM described by the Stochastic Differential Equation,

$$dS_t^k = S_t^k \left[ \left( r - \delta^k \right) dt + \sigma^k dZ_t^k \right]$$

# Results - 1D Swing Option parameters

$$T = 3.0 \text{ years} \qquad r = 0.05,\ \delta = 0.1$$

$$\sigma = 0.2 \qquad \text{base volume} = 1.0 \text{ units}$$

$$K_u = K_d = 40.0 \qquad \text{no penalties or volume choices}$$

$$N_u = N_d = 1 \qquad \text{number of ex. ops} = 4$$

Upon exercise the holder gets

$$\max\left(S_t - K_u, K_d - S_t, 0\right)$$

plus the continuation value with the corresponding # of exercise rights.

# Convergence of the estimators - 1D

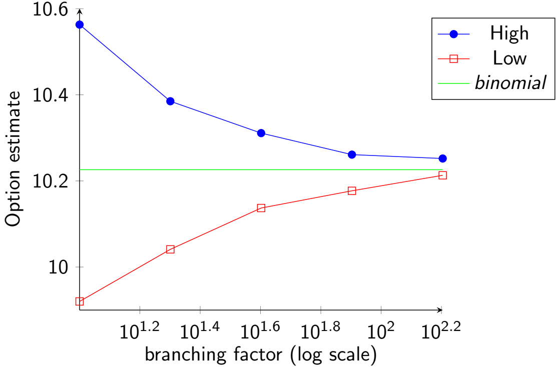


Figure: Mesh and path option-value estimators versus (log) mesh size. Std. Err. $\approx 0.09\%$. Repeated valuations $= 16384 \times \frac{10}{\text{branching factor}}$

# Results - 5D Swing Option parameters

Upon exercise the holder gets

$$\max\left(\max\left(S_t^1,\ldots,S_t^5\right)-K_u, K_d-\max\left(S_t^1,\ldots,S_t^5\right),0\right)$$

plus the continuation value with the corresponding # of exercise rights.

# Convergence of the estimators - 5D

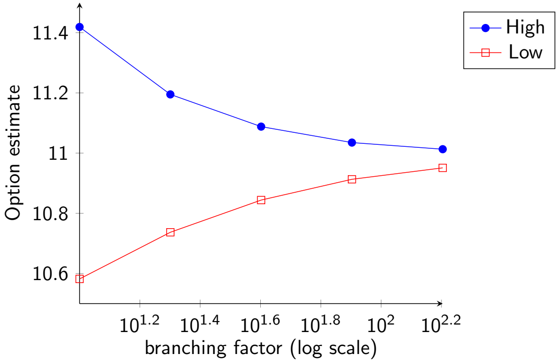


Figure: Mesh and path option-value estimators versus (log) mesh size. Std. Err. $\approx 0.09\%$. Repeated valuations $= 16348 \times \frac{10}{\text{branching factor}}$

# Summary and Future Work

- Method for valuing high dimensional swing options
- Easily accommodates general price processes and payoffs
- Consistent but biased estimators
- Forest of Stochastic Meshes
- Forest of Bias-reduced Stochastic Trees
- Forest of Bias-reduced Stochastic Meshes
- Hedge Parameters

# References


- Broadie, M., Glasserman, P., 1997. Pricing American-style securities using simulation. Journal of Economic Dynamics and Control 21, 1323-1352.
- Lari-Lavassani, A., Simchi, M., Ware, A., 2001. A Discrete Valuation of Swing Options. Canadian Applied Mathematics Quarterly 9, 35-73.
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# Acknowledgments

For resource support the speaker wishes to thank,

SHARCNET™

and for financial support,

NSERC CRSNG

Western

# Results - 1D Swing Option parameters

$$\begin{aligned} T &= 1.0 \text{ years} \\ \sigma &= 0.2 \\ K_u &= K_d = 40.0 \\ N_u &= N_d = 2 \end{aligned}$$

$r = 0.05, \ \delta = 0.1$
base volume = 1.0 units
no penalties or volume choices
number of ex. ops = 6

Upon exercise the holder gets

$$\max\left(S_t - K_u, K_d - S_t, 0\right)$$

plus the continuation value with the corresponding # of exercise rights.

# Convergence of the estimators - 1D

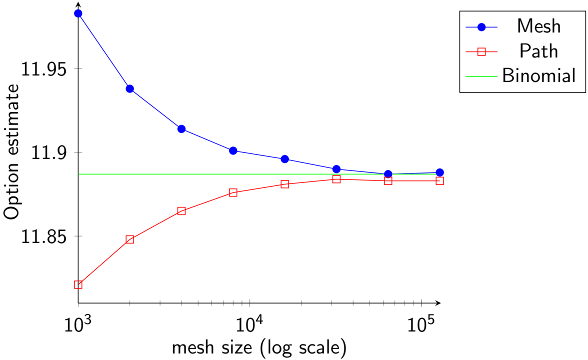


Figure: Mesh and path option-value estimators versus (log) mesh size. Std. Err. $\approx 0.01\%$. Repeated valuations $= 16384 \times \frac{1000}{\text{mesh size}}$

# Results - 5D Swing Option parameters

Upon exercise the holder gets

$$\max\left(\max\left(S_t^1,\ldots,S_t^5\right)-K_u, K_d - max\left(S_t^1,\ldots,S_t^5\right),0\right)$$

plus the continuation value with the corresponding # of exercise rights.

# Convergence of the estimators - 5D

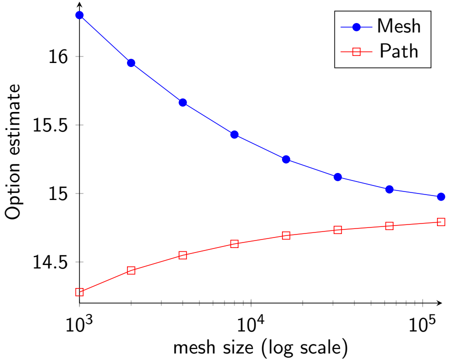


Figure: Mesh and path option-value estimators versus (log) mesh size. Std. Err. $\approx 0.01\%$. Repeated valuations $= 16384 \times \frac{1000}{\text{mesh size}}$

# Mathematical Discription of Swings

- Take 0 to be the time the contract is signed then the option is in effect for time $t \in [T_1, T_2]$, where $0 \leq T_1 < T_2$.
- During the contract the holder may exercise a given number of up and down swing rights ($N_u$ and $N_d$).
- Typically these rights can only be exercised at discrete set of times $\{\tau_1, \ldots, \tau_m\}$ with $T_1 \leq \tau_1 < \ldots < \tau_m \leq T_2$.

# Mathematical Discription of Swings

- When the holder chooses to swing in addition to the choice of up or down they may also have a choice of volumes of which to swing.
- These amounts maybe continuous or discrete but in either case the volumes at a given opportunity at $\tau_i$ will take the form $[u_i^1, u_i^2] \cup [u_i^3, u_i^4]$ for $1 \leq i \leq m$ and $u_i^1 \leq u_i^2 \leq 0 \leq u_i^3 \leq u_i^4$.

# Mathematical Discription of Swings

- Another feature that is included in these contracts are penalties which restrict the total volume which may be swung during the contract.
- Usage level, $U$, is restricted to a range $[U_{min}, U_{max}]$ at the completion of the contract.
- Usage outside of this range leads to penalties being applied to the holder at expiry.

# Mathematical Discription of Swings

Define, exercise and usage decision variables as,

$$\sigma_i^{\pm} = \begin{cases} 1 & \text{if swing up/down} \\ 0 & \text{otherwise} \end{cases}$$

$$v_i^{\pm} = \begin{cases} \text{volume bought/sold} & \text{if swing up/down} \\ 0 & \text{otherwise} \end{cases}$$

# Precise Discription of Swings

For all $1 \leq j < i \leq m$,

$$0 \leq \sigma_i^+ + \sigma_i^- \leq 1$$

$$\left(\sigma_j^+ + \sigma_j^-\right) + \left(\sigma_i^+ + \sigma_i^-\right) \leq 1 + \tfrac{\tau_i}{\tau_j + \Delta\tau}$$

$$0 \leq \sum_{i=1}^{m} \sigma_i^+ \leq N_u$$

$$0 \leq \sum_{i=1}^{m} \sigma_i^- \leq N_d$$

$$u_i^3 \sigma_i^+ \leq \upsilon_i^+ \leq u_i^4 \sigma_i^+$$

$$u_i^1 \sigma_i^- \leq \upsilon_i^- \leq u_i^2 \sigma_i^-$$

# Penalties

There are 2 general types of penalty structures for swing options:

- Local penalties: may be applied at the time of exercise.
- Global penalties: may be applied at expiry based on total volume swung.

# Penalties

For global penalties the general penalty structure is:

$$\phi(U) = \begin{cases} \mathcal{P}_1 & , \text{if } U(T_2) < U_{min} \\ 0 & , \text{if } U_{min} \leq U(T_2) \leq U_{max} \\ \mathcal{P}_2 & , \text{if } U(T_2) > U_{max} \end{cases}$$