Option Pricing under a Mixed-Exponential Jump Diffusion Model

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June, 2010
A flexible mixed-exponential jump diffusion model \textit{(MEM)} for option pricing

Deriving distributions of \textit{first passage times} by solving an OIDE \textit{explicitly}

Pricing lookback and barrier options analytically

Numerical results, including an example to price lookback and barrier options approximately under \textit{Merton’s jump diffusion model} using the MEM
The main empirical motivation comes from the asymmetric leptokurtic feature, i.e., the asset return distribution has a higher peak and two heavier tails than those of normal distributions.

It implies that extreme asset returns occur more frequently in reality than predicted by the Black-Scholes model (BSM).

Jump diffusion models are proposed to better capture the leptokurtic feature; e.g., Merton’s model (1976), Kou’s model (2002).
However, it is not clear at all how heavy the tails of the asset return distributions are.

Moreover, empirically it is almost impossible to distinguish some heavy tails, e.g. power tails from exponential tails. See Heyde and Kou (2004).

**Question**: How to capture the uncertainty about the heaviness of asset return tails?

**Question**: What distributions should we use for jump sizes in jump diffusion modeling?
The jump size distribution is expected to be flexible enough to incorporate various heavy-tailed distributions.

Consider using a mixed-exponential jump diffusion process (MEP) to model the asset return:

\[ X_t = X_0 + \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad \text{for any } t \geq 0, \]
Jump sizes \( \{Y_i\} \) assume a mixed-exponential distribution (ME) with pdf

\[
f_Y(x) = \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i x} I_{\{x \geq 0\}} + \sum_{j=1}^{n} q_j \theta_j e^{\theta_j x} I_{\{x < 0\}}.
\]

- The weights \( p_i \) and \( q_j \) can be negative.
- The class of ME distributions is dense w.r.t. that of all the distributions.
- The MEMs can approximate jump diffusion models with arbitrary jump size distributions, including various heavy-tailed distributions.
Motivation I: Empirical Features

- Comparison with the hyper-exponential jump diffusion model (HEM).

- In particular, the MEMs can approximate Merton’s model.

- Compared with the phase-type distributions (Asmussen et al. (2004)), the representation of the ME distribution is unique.

- The class of phase-type distributions and that of ME distributions do not contain each other.
Additionally, MEM can lead to analytical solutions to pricing problems for lookback and barrier options.

This is primarily because we can obtain distributions of the first passage time $\tau_b$ of the MEP.

$\tau_b := \inf\{t \geq 0 : X_t \geq b\}$. 
An exponent equation

- The moment generating function of the MEP \( \{X_t\} \):
  \[
  E^{X_0}[e^{X_t}] = e^{X_0 + G(x)t}, \text{ where the exponent } G(x) \text{ is given by}
  \]
  \[
  G(x) = \frac{\sigma^2}{2} x^2 + \mu x + \lambda \left( \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - x} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + x} - 1 \right).
  \]

- Introduce an exponent equation
  \[
  G(x) = \alpha, \quad \text{for any sufficiently large } \alpha > 0.
  \]

- It has exactly \((m + n + 2)\) real roots such that
  \[
  -\infty < \gamma_{n+1,\alpha} < \gamma_{n,\alpha} < \cdots < \gamma_{2,\alpha} < \gamma_{1,\alpha} < 0
  \]
  \[
  0 < \beta_{1,\alpha} < \beta_{2,\alpha} < \cdots < \beta_{m,\alpha} < \beta_{m+1,\alpha} < +\infty.
  \]
An exponent equation

Plot of the exponent $G(x)$

Plot of the function $G(x)$ under the MEM with $m=n=3$

**Figure:** Plot of the function $G(x)$ with $m = n = 3$. Related parameters are $\mu = 0.05$, $\sigma = 0.2$, $\lambda = 5$, $(\eta_1 \eta_2 \eta_3) = (20 \ 40 \ 60)$, $(\theta_1 \ \theta_2 \ \theta_3) = (20 \ 35 \ 60)$, $p_u = q_d = 0.5$, $(p_1 \ p_2 \ p_3) = (1.2 \ -0.3 \ 0.1)$ and $(q_1 \ q_2 \ q_3) = (1.3 \ 0.1 \ -0.4)$.
Lookback and barrier options

- Lookback options depend on running maximum (or minimum) of the asset prices in $[0, T]$:
  - E.g., a floating-strike lookback put’s payoff: 
    $$ (\max_{0 \leq u \leq T} S_u - S_T)^+ . $$

- Barrier options depend on first passage times of the asset prices in $[0, T]$:
  - E.g., an up-and-in call barrier’s payoff: 
    $$ (S_T - K)^+ I_{\{\tau_H < T\}}. $$

Therefore, distributions of running maxima or first passage times play a key role.
Comparison with literature

- In the literature, Wiener-Hopf factorization is a usual tool to study first passage times or/and price barrier options.

- However, we derive the distribution of the first passage time by solving an OIDE explicitly and by applying a martingale method similar as in Kou and Wang (2003).
First passage times

**Objective:** For any sufficiently large $\alpha > 0$, $\theta < \eta_1$ and $X_0 = x \in \mathbb{R}$, derive

$$v(x) := E_x^x [e^{-\alpha \tau_b + \theta X_{\tau_b}}].$$

Roughly speaking, we should have

$$v(x) \begin{cases} 
= e^{\theta x} & \text{if } x \geq b \\
\text{solves the high-order OIDE } (Lv)(x) = \alpha v(x) & \text{if } x < b,
\end{cases}$$

Here $L$ is the infinitesimal generator of the MEP $\{X_t\}$:

$$(Lu)(x) = \frac{\sigma^2}{2} u''(x) + \mu u'(x) + \lambda \sum_{i=1}^{m} \int_0^{+\infty} [u(x + y) - u(x)] p_i \eta_i e^{-\eta_i y} dy + \lambda \sum_{j=1}^{n} \int_{-\infty}^{0} [u(x + y) - u(x)] q_j \theta_j e^{\theta_j y} dy.$$
Theorem (Cai and Kou (2008))

Any solution $u(x)$ of the OIDE $(Lu)(x) = \alpha u(x)$ is also the solution of a $(m + n + 2)$ order homogeneous linear ODE with constant coefficients, whose characteristic equation is given by

$$(G(x) - \alpha) \prod_{i=1}^{m} (\eta_i - x) \prod_{j=1}^{n} (\theta_j + x) = 0.$$ 

Consequently, the general solution of the OIDE $(Lu)(x) = \alpha u(x)$ is given by:

$$u(x) = \sum_{i=1}^{m+1} c_i e^{\beta_i x} + \sum_{j=1}^{n+1} d_j e^{\gamma_j x},$$

where $\beta_1, \beta_2, \cdots, \beta_{m+1}, \gamma_1, \gamma_2, \cdots, \gamma_{n+1}$ are $(m + n + 2)$ roots of the exponent equation $G(x) = \alpha$. 
Derivation of $E^x[e^{-\alpha \tau b + \theta X_{\tau b}}]$

### Theorem (Cai and Kou (2008))

For any sufficiently large $\alpha > 0$, $\theta < \eta_1$ and $x, b \in \mathbb{R}$, we have:

$$E^x[e^{-\alpha \tau b + \theta X_{\tau b}}] = \begin{cases} e^{\theta x} & \text{if } x \geq b \\ \sum_{l=1}^{m+1} w_l e^{\beta_l x} & \text{if } x < b, \end{cases}$$

where $\beta_1, \cdots, \beta_{m+1}$ are $m + 1$ positive roots of the exponent equation $G(x) = \alpha$. Here $w := (w_1, w_2, \cdots, w_{m+1})'$ is uniquely determined by the following linear system

$$ABw = J.$$

- $J = e^{\theta b} (1, \frac{\eta_1}{\eta_1 - \theta}, \frac{\eta_2}{\eta_2 - \theta}, \cdots, \frac{\eta_m}{\eta_m - \theta})'$,

- $B = \text{Diag}\{e^{\beta_1 b}, e^{\beta_2 b}, \cdots, e^{\beta_{m+1} b}\}$. 
Derivation of $E^x[e^{-\alpha \tau_b + \theta X_{\tau_b}}]$

1. Here $A$ is a $(m + 1) \times (m + 1)$ nonsingular matrix

$$A = \begin{pmatrix}
\frac{1}{\eta_1} & \frac{1}{\eta_1-\beta_1} & \cdots & \frac{1}{\eta_1-\beta_{m+1}} \\
\eta_1 & \eta_1-\beta_1 & \cdots & \eta_1-\beta_{m+1} \\
\eta_2 & \eta_2-\beta_1 & \cdots & \eta_2-\beta_{m+1} \\
\eta_2 & \eta_2-\beta_2 & \cdots & \eta_2-\beta_{m+1} \\
\vdots & \vdots & \ddots & \vdots \\
\eta_m & \eta_m & \cdots & \eta_m-\beta_{m+1} \\
\eta_m & \eta_m-\beta_1 & \cdots & \eta_m-\beta_{m+1}
\end{pmatrix}$$

2. Analytical solutions of lookback and barrier option prices in terms of single or double Laplace transforms can be derived.
### Lookback put option pricing under MEM

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### Numerical examples of barrier option prices

- **Barrier option pricing**

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Approximating Merton’s model with the MEMs

- The MEMs can approximate Merton’s model in the sense of weak convergence.

- Consider using the MEM with the following jump size pdf to approximate Merton’s model with jump size distribution with mean zero and standard deviation 0.01.

\[
f_Y(x) = 0.5 \times \left( 8.7303 \times 213.0215 \times e^{-213.0215|x|} \\
+ 2.1666 \times 236.0406 \times e^{-236.0406|x|} \\
- 10 \times 237.1139 \times e^{-237.1139|x|} \\
+ 0.0622 \times 939.7441 \times e^{-939.7441|x|} \\
+ 0.0409 \times 939.8021 \times e^{-939.8021|x|} \right). \quad (1)
\]
Approximating Merton’s model with the MEMs

Figure: Approximate the normal distribution $N(0, 0.01^2)$ using the mixed-exponential distribution with pdf (1).
### Pricing lookback options under Merton’s model

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### Pricing barrier options under Merton’s model

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Proposing a mixed-exponential jump diffusion model

The flexibility of the MEM

(1) Better capture the leptokurtic feature.
(2) Approximate Merton’s model.

Distributions of first passage times

Analytical solutions to pricing of lookback and barrier options.

Numerical results indicate (1) our pricing algorithms are accurate and efficient; and (2) approximating Merton’s model with the MEMs can lead to accurate approximations to lookback and barrier option prices under Merton’s model.
Thank you!