# Option Pricing under a Mixed-Exponential Jump Diffusion Model

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# Outline

- A flexible mixed-exponential jump diffusion model (MEM) for option pricing
- Deriving distributions of first passage times by solving an OIDE explicitly
- Pricing lookback and barrier options analytically
- Numerical results, including an example to price lookback and barrier options approximately under Merton's jump diffusion model using the MEM

# Motivation I: Empirical Features

- The main empirical motivation comes from the asymmetric leptokurtic feature, i.e., the asset return distribution has a higher peak and two heavier tails than those of normal distributions.
- It implies that extreme asset returns occur more frequently in reality than predicted by the Black-Scholes model (BSM).
- Jump diffusion models are proposed to better capture the leptokurtic feature; e.g., Merton's model (1976), Kou's model (2002).

# Motivation I: Empirical Features

- However, it is not clear at all how heavy the tails of the asset return distributions are.
- Moreover, empirically it is almost impossible to distinguish some heavy tails, e.g. power tails from exponential tails. See Heyde and Kou (2004).
- Question: How to capture the uncertainty about the heaviness of asset return tails?
- Question: What distributions should we use for jump sizes in jump diffusion modeling?

# Motivation I: Empirical Features

- The jump size distribution is expected to be flexible enough to incorporate various heavy-tailed distributions.
- Consider using a mixed-exponential jump diffusion process (MEP) to model the asset return:

$$X_t = X_0 + \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad \text{for any } t \geq 0,$$

# Motivation I: Empirical Features

- Jump sizes $\{Y_i\}$ assume a mixed-exponential distribution (ME) with pdf

$$f_Y(x) = \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i x} I_{\{x \geq 0\}} + \sum_{j=1}^{n} q_j \theta_j e^{\theta_j x} I_{\{x<0\}}.$$

- The weights $p_i$ and $q_j$ can be negative.
- The class of ME distributions is dense w.r.t. that of all the distributions.
- The MEMs can approximate jump diffusion models with arbitrary jump size distributions, including various heavy-tailed distributions.

# Motivation I: Empirical Features

- Comparison with the hyper-exponential jump diffusion model (HEM).
  - See, e.g., Lipton (2002), Asmussen, Avram and Pistorius (2004), Boyarchenko (2006), Jeannin and Pistorius (2008), Boyarchenko and Levendorskiĭ (2008), Boyarchenko and Boyarchenko (2008), Crosby, Saux and Mijatovic (2009), Carr and Crosby (2010), $\cdots$

- In particular, the MEMs can approximate Merton's model.

- Compared with the phase-type distributions (Asmussen et al. (2004)), the representation of the ME distribution is unique.

- The class of phase-type distributions and that of ME distributions do not contain each other.

# Motivation II–Analytical Tractability

- Additionally, MEM can lead to analytical solutions to pricing problems for lookback and barrier options.
- This is primarily because we can obtain distributions of the first passage time $\tau_b$ of the MEP.
  - $\tau_b := \inf\{t \geq 0 : X_t \geq b\}$.

# An exponent equation

- The moment generating function of the MEP $\{X_t\}$: $E^{X_0}[e^{xX_t}] = e^{xX_0 + G(x)t}$, where the exponent $G(x)$ is given by

$$G(x) = \frac{\sigma^2}{2}x^2 + \mu x + \lambda\left(\sum_{i=1}^{m}\frac{p_i\eta_i}{\eta_i - x} + \sum_{j=1}^{n}\frac{q_j\theta_j}{\theta_j + x} - 1\right).$$

- Introduce an exponent equation

$$G(x) = \alpha, \qquad \text{for any sufficiently large } \alpha > 0.$$

- It has exactly $(m + n + 2)$ real roots such that

$$-\infty < \gamma_{n+1,\alpha} < \gamma_{n,\alpha} < \cdots < \gamma_{2,\alpha} < \gamma_{1,\alpha} < 0$$

$$0 < \beta_{1,\alpha} < \beta_{2,\alpha} < \cdots < \beta_{m,\alpha} < \beta_{m+1,\alpha} < +\infty.$$

# An exponent equation

- Plot of the exponent $G(x)$

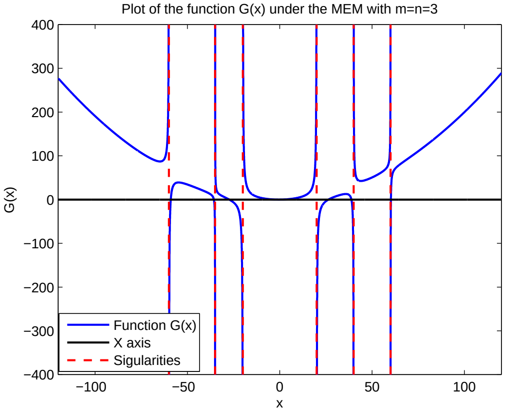


Figure: Plot of the function $G(x)$ with $m = n = 3$. Related parameters are $\mu = 0.05$, $\sigma = 0.2$, $\lambda = 5$, $(\eta_1\ \eta_2\ \eta_3)$=(20 40 60), $(\theta_1\ \theta_2\ \theta_3)$=(20 35 60), $p_u = q_d = 0.5$, $(p_1\ p_2\ p_3)$=(1.2 -0.3 0.1) and $(q_1\ q_2\ q_3)$=(1.3 0.1 -0.4)

# Lookback and barrier options

- Lookback options depend on running maximum (or minimum) of the asset prices in $[0, T]$:
  - E.g., a floating-strike lookback put's payoff: $(\max_{0 \le u \le T} S_u - S_T)^+$.
- Barrier options depend on first passage times of the asset prices in $[0, T]$:
  - E.g., an up-and-in call barrier's payoff: $(S_T - K)^+ I_{\{\tau_H < T\}}$.
- Therefore, distributions of running maxima or first passage times play a key role.

# Comparison with literature

- In the literature, Wiener-Hopf factorization is a usual tool to study first passage times or/and price barrier options.
  - See, e.g., Rogers (2000), Boyarchenko and Levendorskii (2002), Jeannin and Pistorius (2008), Boyarchenko and Boyarchenko (2008), Crosby, Saux and Mijatovic (2009) $\cdots$
- However, we derive the distribution of the first passage time by solving an OIDE explicitly and by applying a martingale method similar as in Kou and Wang (2003).

# First passage times

- Objective: For any sufficiently large $\alpha > 0$, $\theta < \eta_1$ and $X_0 = x \in \mathbb{R}$, derive

$$v(x) := E^x[e^{-\alpha\tau_b + \theta X_{\tau_b}}].$$

- Roughly speaking, we should have

$$v(x) \begin{cases} = e^{\theta x} & \text{if } x \geq b \\ \text{solves the high-order OIDE } (Lv)(x) = \alpha v(x) & \text{if } x < b, \end{cases}$$

- Here $L$ is the infinitesimal generator of the MEP $\{X_t\}$:

$$(Lu)(x) = \frac{\sigma^2}{2} u''(x) + \mu u'(x) + \lambda \sum_{i=1}^{m} \int_0^{+\infty} [u(x+y) - u(x)] p_i \eta_i e^{-\eta_i \cdots}$$

$$+ \lambda \sum_{j=1}^{n} \int_{-\infty}^{0} [u(x+y) - u(x)] q_j \theta_j e^{\theta_j y} dy.$$

# Solve an OIDE explicitly

**Theorem (Cai and Kou (2008))**

*Any solution $u(x)$ of the OIDE $(Lu)(x) = \alpha u(x)$ is also the solution of a $(m+n+2)$ order homogeneous linear ODE with constant coefficients, whose characteristic equation is given by $(G(x)-\alpha)\prod_{i=1}^{m}(\eta_i - x)\prod_{j=1}^{n}(\theta_j + x) = 0$.*

- Consequently, the general solution of the OIDE $(Lu)(x) = \alpha u(x)$ is given by:

$$u(x) = \sum_{i=1}^{m+1} c_i e^{\beta_i x} + \sum_{j=1}^{n+1} d_j e^{\gamma_j x},$$

where $\beta_1, \beta_2, \cdots, \beta_{m+1}, \gamma_1, \gamma_2, \cdots, \gamma_{n+1}$ are $(m+n+2)$ roots of the exponent equation $G(x) = \alpha$.

# Derivation of $E^x[e^{-\alpha\tau_b+\theta X_{\tau_b}}]$

## Theorem (Cai and Kou (2008))

*For any sufficiently large $\alpha > 0$, $\theta < \eta_1$ and $x, b \in \mathbb{R}$, we have:*

$$E^x[e^{-\alpha\tau_b+\theta X_{\tau_b}}] = \begin{cases} e^{\theta x} & \text{if } x \geq b \\ \sum_{l=1}^{m+1} w_l e^{\beta_l x} & \text{if } x < b, \end{cases}$$

*where $\beta_1, \cdots, \beta_{m+1}$ are $m+1$ positive roots of the exponent equation $G(x) = \alpha$. Here $w := (w_1, w_2, \cdots, w_{m+1})'$ is uniquely determined by the following linear system*

$$ABw = J.$$

- $J = e^{\theta b}(1, \frac{\eta_1}{\eta_1-\theta}, \frac{\eta_2}{\eta_2-\theta}, \cdots, \frac{\eta_m}{\eta_m-\theta})'$,
  $B = Diag\{e^{\beta_1 b}, e^{\beta_2 b}, \cdots, e^{\beta_{m+1} b}\}$.

# Derivation of $E^x[e^{-\alpha\tau_b+\theta X_{\tau_b}}]$

- Here $A$ is a $(m+1)\times(m+1)$ nonsingular matrix

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{\eta_1}{\eta_1-\beta_1} & \frac{\eta_1}{\eta_1-\beta_2} & \cdots & \frac{\eta_1}{\eta_1-\beta_{m+1}} \\ \frac{\eta_2}{\eta_2-\beta_1} & \frac{\eta_2}{\eta_2-\beta_2} & \cdots & \frac{\eta_2}{\eta_2-\beta_{m+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\eta_m}{\eta_m-\beta_1} & \frac{\eta_m}{\eta_m-\beta_2} & \cdots & \frac{\eta_m}{\eta_m-\beta_{m+1}} \end{pmatrix}$$

- Analytical solutions of lookback and barrier option prices in terms of single or double Laplace transforms can be derived.

# Numerical examples of lookback option prices

- Lookback put option pricing under MEM

| Pricing lookback options under the MEM | | | | | |
|---|---|---|---|---|---|
| $M$ | $\sigma$ | $\lambda$ | EI value | MC value | Std Err |
| 105 | 0.1 | 1 | 7.13774 | 7.14438 | 0.00717 |
| | | 3 | 8.38446 | 8.39709 | 0.00737 |
| | | 5 | 9.54401 | 9.55256 | 0.00747 |
| 105 | 0.2 | 1 | 15.11777 | 15.12439 | 0.00676 |
| | | 3 | 15.96876 | 15.97562 | 0.00679 |
| | | 5 | 16.79455 | 16.80643 | 0.00683 |
| 107 | 0.1 | 1 | 7.69091 | 7.69950 | 0.01046 |
| | | 3 | 8.91452 | 8.93285 | 0.01084 |
| | | 5 | 10.05471 | 10.06516 | 0.01108 |
| 107 | 0.2 | 1 | 15.48878 | 15.49762 | 0.01038 |
| | | 3 | 16.33422 | 16.34394 | 0.01043 |
| | | 5 | 17.15486 | 17.16885 | 0.01052 |

# Numerical examples of barrier option prices

- Barrier option pricing

Pricing barrier options under the MEM

| $\lambda$ | $\sigma$ | $K$ | EI value | MC value | Std Err |
|---|---|---|---|---|---|
| | | 101 | 10.02579 | 10.03530 | 0.02077 |
| 2 | 0.2 | 105 | 8.35828 | 8.36193 | 0.02072 |
| | | 109 | 6.84158 | 6.84678 | 0.02073 |
| | | 101 | 13.94197 | 13.91411 | 0.02840 |
| 2 | 0.3 | 105 | 12.25099 | 12.24949 | 0.02935 |
| | | 109 | 10.70121 | 10.67303 | 0.02984 |
| $\lambda$ | $\sigma$ | $H$ | EI value | MC value | Std Err |
| | | 105 | 10.09973 | 10.10853 | 0.02004 |
| 3 | 0.2 | 110 | 10.04892 | 10.08167 | 0.02035 |
| | | 115 | 9.83030 | 9.83137 | 0.02117 |
| | | 105 | 13.75881 | 13.75839 | 0.02859 |
| 3 | 0.3 | 110 | 13.74182 | 13.78379 | 0.02877 |
| | | 115 | 13.66264 | 13.68834 | 0.02910 |

# Approximating Merton's model with the MEMs

- The MEMs can approximate Merton's model in the sense of weak convergence.
- Consider using the MEM with the following jump size pdf to approximate Merton's model with jump size distribution with mean zero and standard deviation 0.01.

$$
\begin{aligned}
f_Y(x) = 0.5 \times \Big( & 8.7303 \times 213.0215 \times e^{-213.0215|x|} \\
& + 2.1666 \times 236.0406 \times e^{-236.0406|x|} \\
& - 10 \times 237.1139 \times e^{-237.1139|x|} \\
& + 0.0622 \times 939.7441 \times e^{-939.7441|x|} \\
& + 0.0409 \times 939.8021 \times e^{-939.8021|x|} \Big). \qquad (1)
\end{aligned}
$$

# Approximating Merton's model with the MEMs

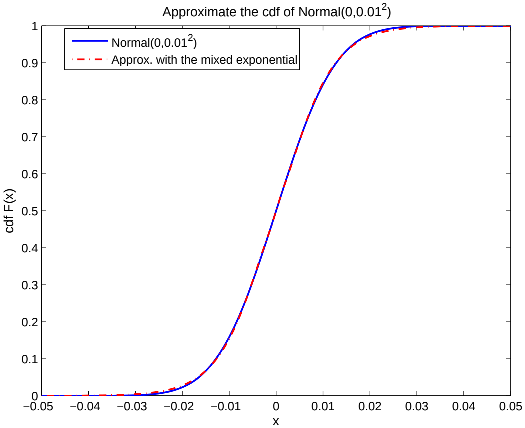


Figure: Approximate the normal distribution $N(0, 0.01^2)$ using the mixed-exponential distribution with pdf (1).

| Pricing lookback options under Merton's model | | | | | |
|---|---|---|---|---|---|
| $\sigma$ | $\lambda$ | $M$ | EI value | MC value | Std Err |
| 0.1 | 1 | 105 | 6.51320 | 6.50105 | 0.01049 |
| | | 107 | 7.07855 | 7.06909 | 0.01041 |
| | | 109 | 7.84251 | 7.83448 | 0.01026 |
| 0.1 | 5 | 105 | 6.66681 | 6.65129 | 0.01077 |
| | | 107 | 7.22694 | 7.21430 | 0.01068 |
| | | 109 | 7.98273 | 7.97271 | 0.01051 |
| 0.2 | 1 | 105 | 14.70418 | 14.67465 | 0.02287 |
| | | 107 | 15.07787 | 15.05112 | 0.02270 |
| | | 109 | 15.57228 | 15.54649 | 0.02253 |
| 0.2 | 5 | 105 | 14.79288 | 14.77413 | 0.02311 |
| | | 107 | 15.16551 | 15.15086 | 0.02293 |
| | | 109 | 15.65840 | 15.64581 | 0.02275 |

# Approximating Merton's model with the MEMs

| Pricing barrier options under Merton's model | | | | | |
|---|---|---|---|---|---|
| $\lambda$ | $\sigma$ | $K$ | EI value | MC value | Std Err |
| | | 101 | 9.54594 | 9.54397 | 0.01988 |
| 1 | 0.2 | 105 | 7.88753 | 7.87605 | 0.01966 |
| | | 109 | 6.38154 | 6.39107 | 0.01952 |
| | | 101 | 13.63140 | 13.64003 | 0.02774 |
| 1 | 0.3 | 105 | 11.93684 | 11.93757 | 0.02855 |
| | | 109 | 10.38581 | 10.41109 | 0.02922 |
| $\lambda$ | $\sigma$ | $H$ | EI value | MC value | Std Err |
| | | 105 | 9.45243 | 9.44670 | 0.01839 |
| 3 | 0.2 | 110 | 9.39394 | 9.38877 | 0.01863 |
| | | 115 | 9.14142 | 9.12743 | 0.01979 |
| | | 105 | 13.31261 | 13.29997 | 0.02754 |
| 3 | 0.3 | 110 | 13.29437 | 13.30942 | 0.02754 |
| | | 115 | 13.20890 | 13.25942 | 0.02804 |

# Summary

- Proposing a mixed-exponential jump diffusion model
  - The flexibility of the MEM
    $\Longrightarrow$
    (1) Better capture the leptokurtic feature.
    (2) Approximate Merton's model.
  - Distributions of first passage times
    $\Longrightarrow$
    Analytical solutions to pricing of lookback and barrier options.
- Numerical results indicate (1) our pricing algorithms are accurate and efficient; and (2) approximating Merton's model with the MEMs can lead to accurate approximations to lookback and barrier option prices under Merton's model.

- Thank you!