Optimal investment with inside information and parameter uncertainty[1]

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Outline

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- But enlarging a Brownian filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ with the information carried by an $\mathcal{F}_T$-measurable random variable alters the drift of the stock.
- So, natural to investigate the effect of inside information with the drift not known. Brownian inside information will then not be equivalent to stock price information, as the latter can be used to make a better estimate of the drift.
- Accurate drift estimation is difficult, natural to consider uncertain drift when dealing with information problems.

The model

- A stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$, with $\mathcal{F}_0$ not necessarily trivial.
- We call $\mathbb{F}$ the background filtration.
- An $\mathbb{F}$-Brownian motion $B = (B_t)_{0 \le t \le T}$ and a stock price $S = (S_t)_{0 \le t \le T}$:

$$dS_t = \sigma S_t(\lambda dt + dB_t).$$

- $\sigma > 0$ is a known constant and the interest rate is zero.
- For an agent with access to $\mathbb{F}$, λ is a known constant.
- Here, agents do not have access to $\mathbb{F}$, that is, they do not observe B.
- Hence λ is taken to be an $\mathcal{F}_0$-measurable Gaussian random variable:

$$\lambda \sim \mathrm{N}(\lambda_0, \mathrm{v}_0), \quad \text{independent of } B,$$

  for constants λ_0 and $\mathrm{v}_0 \ge 0$.
- Agents' trading strategies will be required to be adapted to the stock price filtration, enlarged by inside information.

The problem with no inside information

- Define the observation process $\xi = (\xi_t)_{0 \le t \le T}$ by

$$\xi_t := \frac{1}{\sigma} \int_0^t \frac{dS_s}{S_s} = \lambda t + B_t = \frac{1}{\sigma} \log\left(\frac{S_t}{S_0}\right) + \frac{1}{2}\sigma t, \quad 0 \le t \le T.$$

- Denote by $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)_{0 \le t \le T}$ the filtration generated by ξ:

$$\widehat{\mathcal{F}}_t := \sigma(\xi_s; 0 \le s \le t) = \sigma(S_s; 0 \le s \le t), \quad 0 \le t \le T.$$

- So $\widehat{\mathcal{F}}_0$ is the trivial σ-algebra, and and $\widehat{\mathcal{F}}_t \subseteq \mathcal{F}_t$, for all $t \in [0, T]$.
- The non-insider's observation filtration is $\widehat{\mathbb{F}}$.
- We take HARA prefernces:

$$U_p(x) := \begin{cases} x^p/p, & p < 1, \quad p \neq 0, \\ \log x, & p = 0. \end{cases}$$

- The non-insider's problem is

$$\mathbb{E} U_p(X_T) \to \text{max!} \quad \text{over } \widehat{\mathbb{F}}\text{-adapted investment strategies.}$$

Inside information

- The insider has knowledge at time zero of the value of an $\mathcal{F}$-measurable random variable L, representing (noisy) knowledge of an $\mathcal{F}_T$-measurable random variable.
- We consider two examples:
  1. Brownian inside information, for which
  $$L = L_B := aB_T + (1-a)\epsilon, \quad 0 < a \leq 1,$$
  where $\epsilon \sim N(0,1)$, independent of B and λ.
  2. Stock price inside information, for which
  $$L = L_S := a\xi_T + (1-a)\epsilon, \quad 0 < a \leq 1,$$
  where $\epsilon \sim N(0,1)$, independent of B, λ and ξ.
- Denote the enlarged filtration by $\mathbb{F}^L = (\mathcal{F}_t^L)_{0 \leq t \leq T}$, with
$$\mathcal{F}_t^L := \mathcal{F}_t \vee \sigma(L), \quad 0 \leq t \leq T.$$

The insider's problem

- The insider's observation filtration is $\widehat{\mathbb{F}}^L := (\widehat{\mathcal{F}}^L_t)_{0 \le t \le T}$, with

$$\widehat{\mathcal{F}}^L_t := \widehat{\mathcal{F}}_t \vee \sigma(L), \quad 0 \le t \le T.$$

- The insider's optimisation problem is

$$\mathbb{E}[U_p(X_T) | \widehat{\mathcal{F}}^L_0] \to \text{max!} \quad \text{over } \widehat{\mathbb{F}}^L\text{-adapted investment strategies } \theta^L.$$

- Admissible strategies:

$$\mathcal{A}(\widehat{\mathbb{F}}^L) := \{\theta^L : \widehat{\mathbb{F}}^L\text{-adapted}, \int_0^T (\theta^L_t)^2 dt < \infty, X^L_t \ge 0, \forall t \in [0, T], \text{ a.s.}\}$$

- Denote the value function for initial capital $x > 0$ by

$$u_L(x) := \sup_{\theta^L \in \mathcal{A}(\widehat{\mathbb{F}}^L)} \mathbb{E}[U_p(X^L_T) | \widehat{\mathcal{F}}^L_0],$$

and the optimal strategy by $\theta^{L,*}$, with $L \equiv 0$ denoting no inside information.

Methodology I

- Begin with
$$dS_t = \sigma S_t(\lambda dt + dB_t),$$
written with respect to the background filtration $\mathbb{F}$.
- If inside information is available, form the enlarged filtration $\mathbb{F}^L = (\mathcal{F}^L_t)_{0 \leq t \leq T}$.
- The semi-martingale decomposition of B with respect to $\mathbb{F}^L$ is
$$B_t = B^L_t + \int_0^t \nu^L_s ds, \quad 0 \leq t \leq T.$$
- The $\mathbb{F}^L$-adapted process $\nu^L = (\nu^L_t)_{0 \leq t \leq T}$ is the information drift, given by
$$\nu^L_t := \frac{\partial}{\partial y} \log p(t, L, B_t), \quad 0 \leq t \leq T,$$
where $p(t, x, y)$ is the conditional density of L given $\mathbb{F}$:
$$E[f(L)|\mathcal{F}_t] = \int_{\mathbb{R}} f(x) p(t, x, B_t) dx, \quad 0 \leq t \leq T.$$

Methodology II

- The stock price dynamics with respect to $\mathbb{F}^L$ are then

$$dS_t = \sigma S_t \left(\lambda_t^L dt + dB_t^L\right), \quad \text{with} \quad \lambda_t^L := \lambda + \nu_t^L.$$

- The insider considers λ^L as a signal process and ξ as an observation process, and these have linear $\mathbb{F}^L$-dynamics:

$$\begin{aligned} d\lambda_t^L &= -\frac{1}{T_a - t} dB_t^L, \quad \lambda_0^L = \lambda + \nu_0^L, \quad T_a := T + \left(\frac{1-a}{a}\right)^2, \\ d\xi_t &= \lambda_t^L dt + dB_t^L, \quad \xi_0 = 0. \end{aligned}$$

- The insider filters λ^L given her observation filtration $\widehat{\mathbb{F}}^L$, inferring

$$\widehat{\lambda}_t^L := \mathbb{E}[\lambda_t^L | \widehat{\mathcal{F}}_t^L], \quad V_t^L := \mathrm{var}[\lambda_t^L | \widehat{\mathcal{F}}_t^L], \quad 0 \le t \le T.$$

- The prior distribution of λ_0^L given $\widehat{\mathcal{F}}_0^L$ is derived from the prior for λ, and turns out to be Gaussian:

$$\mathrm{Law}(\lambda_0^L | \widehat{\mathcal{F}}_0^L) = \mathrm{N}(m^L, \Sigma^L) =: \mathrm{N}(\widehat{\lambda}_0^L, V_0^L), \quad \text{independent of } B^L,$$

for some $\widehat{\mathcal{F}}_0^L$-measurable mean m^L and variance $\Sigma^L \ge 0$, given in terms of λ_0, v_0, the parameters of the prior for λ.

Example ($L = L_B = aB_T + (1-a)\epsilon$)

$$\mathrm{Law}[L_B|\mathcal{F}_t] = \mathrm{N}(aB_t, a^2(T_a - t)), \quad T_a := T + \left(\frac{1-a}{a}\right)^2.$$

So the information drift in this case is $\nu^{L_B} \equiv \nu^B$, given by

$$\nu_t^B = \frac{L_B - aB_t}{a(T_a - t)} = \frac{a(B_T - B_t) + (1-a)\epsilon}{a(T_a - t)}, \quad 0 \le t \le T.$$

The prior distribution of the signal process λ_0^B given $\widehat{\mathcal{F}}_0^B$ is then

$$\mathrm{Law}(\lambda_0^B|\widehat{\mathcal{F}}_0^B) = \mathrm{Law}\left(\lambda + \nu_0^B \middle| \widehat{\mathcal{F}}_0^B\right) = \mathrm{N}\left(\lambda_0 + \frac{L_B}{aT_a}, \mathrm{v}_0\right), \quad \text{independent of } B^B.$$

so the initial conditonal mean and variance of the signal are

$$\widehat{\lambda}_0^B = \lambda_0 + \frac{L_B}{aT_a}, \quad V_0^B = \mathrm{v}_0.$$

Example ($L = L_S = a\xi_T + (1-a)\epsilon$)

$$\mathrm{Law}[L_S|\mathcal{F}_t] = \mathrm{N}(a(B_t + \lambda T), a^2(T_a - t))$$

So the information drift in this case is $\nu^{L_S} \equiv \nu^S$, given by

$$\nu_t^S = \frac{L_S - a(B_t + \lambda T)}{a(T_a - t)} = \frac{a(B_T - B_t) + (1-a)\epsilon}{a(T_a - t)}, \quad 0 \le t \le T.$$

The prior distribution of the signal process λ_0^S given $\widehat{\mathcal{F}}_0^S$ is then

$$\mathrm{Law}(\lambda_0^S|\widehat{\mathcal{F}}_0^S) = \mathrm{N}(\widehat{\lambda}_0^S, V_0^S),$$

with

$$\widehat{\lambda}_0^S = \frac{\lambda_0(1 - T/T_a) + (1 + \mathrm{v}_0 T)(L_S/(aT_a))}{1 + \mathrm{v}_0 T(T/T_a)}, \quad V_0^S = \frac{(1 - (T/T_a))^2 \mathrm{v}_0}{1 + \mathrm{v}_0 T(T/T_a)}.$$

Remark

- Note that
$$\nu^B = \nu^S.$$
- Under $\mathbb{F}^L$, λ is a known parameter, so advance knowledge of $\xi_T = B_T + \lambda T$ is indistinguishable from advance knowledge of B_T.
- The two types of information will become distinct when we transfer to the insider's observation filtration $\widehat{\mathbb{F}}^L$, since the additional stock price information can contribute to the estimation of the unknown drift.
- As a consequence, our results will differ from Pikovsky and Karatzas [2].

Filtering I

- With a Gaussian prior and linear signal-observation dynamics with respect to $\mathbb{F}^L$ we compute $(\widehat{\lambda}^L, V^L)$ using a Kalman-Bucy filter which incorporates the inside information.
- A slight departure from the usual Kalman filter, induced by the inside information, is that the initial observation σ-algebra is not trivial (this changes the initial conditions of the filtering equations). We get:

$$\widehat{\lambda}^L_t = \widehat{\lambda}^L_0 + \int_0^t \mathrm{v}^L_s d\widehat{B}^L_s, \quad 0 \leq t \leq T,$$

where v^L is an "effective variance" given by

$$\mathrm{v}^L_t := V^L_t - \frac{1}{T_a - t} = \frac{\mathrm{v}^L_0}{1 + \mathrm{v}^L_0 t}, \quad 0 \leq t \leq T,$$

and where $\widehat{B}^L$ is a $\widehat{\mathbb{F}}^L$-Brownian motion, the innovations process, given by

$$\widehat{B}^L_t = \xi_t - \int_0^t \widehat{\lambda}^L_s ds, \quad 0 \leq t \leq T.$$

Filtering II

- We recover a full information model with respect to $(\Omega, \widehat{\mathcal{F}}^L_T, \widehat{\mathbb{F}}^L, \mathbb{P})$:

$$dS_t = \sigma S_t(\widehat{\lambda}^L_t dt + d\widehat{B}^L_t),$$

  with a Gaussian risk premium process, and the ensuing utility maximisation problem is solved via convex duality.

- Some care is needed in the case $a = 1$.
  1. The coefficient of dB^L_t in the SDE for λ^L becomes unbounded as $t \to T$, so one can only apply the filtering algorithm up to some terminal time strictly less than T.
  2. One computes the limiting maximum utility for a sequence of problems with terminal times $(T_n)_{n \in \mathbb{N}}$ increasing to T.
  3. For $a = 1$ and $L = L_S$ the filterng procedure is redundant, since knowing the initial and final stock prices at time zero immmediately gives the insider her best estimate of the stock price drift.

Value of information and effect on optimal strategy

- Quantify the additional utility of the insider relative to the non-insider via the additional wealth needed by the non-insider to achieve, on average, the same expected utility as the insider:

$$\mathbb{E}[u_L(x)|\widehat{\mathcal{F}}_0] = u_0(x + \pi_L(x)).$$

- To compute π_L we need the distribution of $\widehat{\lambda}_0^L$ given (the trivial σ-algebra) $\widehat{\mathcal{F}}_0$, and this turns out to be given by

$$\widehat{\lambda}_0^L \sim \mathrm{N}(\lambda_0, \mathrm{v}_0 - \mathrm{v}_0^L).$$

- Finally, we carry out a comparison of the insider's optimal strategy relative to the non-insider's by computing R^L, defined by

$$R_t^L := \frac{\mathbb{E}[\theta_t^{L,*}|\widehat{\mathcal{F}}_t]}{\theta_t^{0,*}}, \quad 0 \leq t \leq T.$$

Optimal investment with partial information I

- Here $\widehat{\mathbb{F}} = \widehat{\mathbb{F}} \equiv \widehat{\mathbb{F}}^0$, the classical partial information scenario (e.g. Rogers[3]).
- The signal process is λ and the observation process is ξ, with $\mathbb{F}$-dynamics

$$d\lambda = 0, \quad d\xi_t = \lambda dt + dB_t.$$

- The conditional mean and variance of λ given $\widehat{\mathbb{F}}$ are defined by

$$\widehat{\lambda}_t := \mathbb{E}[\lambda|\widehat{\mathcal{F}}_t], \quad \mathrm{v}_t := \mathbb{E}[(\lambda - \widehat{\lambda}_t)^2|\widehat{\mathcal{F}}_t] = \mathbb{E}[(\lambda - \widehat{\lambda}_t)^2], \quad 0 \le t \le T.$$

- For $p \neq 0$, define q by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Theorem (Optimal investment with partial information)

$$u_0(x) = \begin{cases} (x^p/p)C_0, & p < 1, \quad p \neq 0, \\ \log x + K_0/2, & p = 0, \end{cases}$$

where C_0, K_0 are constants given by

$$\begin{aligned} C_0 &= \left[(1 + \mathrm{v}_0 T)^p (1 + q\mathrm{v}_0 T)^{1-p}\right]^{-1/2} \exp\left[-\frac{q\lambda_0^2 T}{2(1 + q\mathrm{v}_0 T)}\right], \\ K_0 &= (\lambda_0^2 + \mathrm{v}_0)T - \log(1 + \mathrm{v}_0 T). \end{aligned}$$

The optimal $\widehat{\mathbb{F}}$-adapted trading strategy is

$$\theta_t^{0,*} = \left. \begin{cases} \widehat{\lambda}_t[\sigma(1-p)(1 + q\mathrm{v}_t(T-t))]^{-1}, & p < 1, \quad p \neq 0, \\ \widehat{\lambda}_t/\sigma, & p = 0, \end{cases} \right\} \quad 0 \leq t \leq T,$$

where $\widehat{\lambda}, \mathrm{v}$ are given by

$$\widehat{\lambda}_t = \frac{\lambda_0 + \mathrm{v}_0 \xi_t}{1 + \mathrm{v}_0 t}, \quad \mathrm{v}_t = \frac{\mathrm{v}_0}{1 + \mathrm{v}_0 t}, \quad 0 \leq t \leq T.$$

Theorem (Optimal investment with Brownian information)

The value function when $\widehat{\mathbb{F}}^L = \widehat{\mathbb{F}}^{L_B} \equiv \widehat{\mathbb{F}}^B$ *and* $a \neq 1$ *is*

$$u_B(x) = \begin{cases} (x^p/p)C_B, & p < 1, \quad p \neq 0, \\ \log x + K_B/2, & p = 0, \end{cases}$$

The optimal $\widehat{\mathbb{F}}^B$*-adapted trading strategy is* $\theta^{B,*}$*, given by*

$$\theta_t^{B,*} = \left\{ \begin{array}{ll} \widehat{\lambda}_t^B[\sigma(1-p)(1+q\mathrm{v}_t^B(T-t))]^{-1}, & p < 1, \quad p \neq 0, \\ \widehat{\lambda}_t^B/\sigma, & p = 0, \end{array} \right\} \quad 0 \leq t \leq T,$$

where $\widehat{\lambda}^B, \mathrm{v}^B$ *are given by*

$$\widehat{\lambda}_t^B = \frac{\widehat{\lambda}_0^B + \mathrm{v}_0^B \xi_t}{1 + \mathrm{v}_0^B t}, \quad \mathrm{v}_t^B = \frac{\mathrm{v}_0^B}{1 + \mathrm{v}_0^B t}, \quad 0 \leq t \leq T.$$

For $a = 1$ *and* $\mathrm{v}_0 > 0$ *the above results still hold, while for* $a = 1$ *and* $\mathrm{v}_0 = 0$*,* $u_L(x)$ *is unbounded for* $p \in [0,1)$ *and equal to zero for* $p < 0$.

Optimal investment with Brownian inside information

- Here, C_B, K_B are $\widehat{\mathcal{F}}_0^B$-measurable random variables:

$$
\begin{aligned}
C_B &= \left[(1+\mathrm{v}_0^B T)^p (1+q\mathrm{v}_0^B T)^{1-p}\right]^{-1/2} \exp\left(-\frac{q(\widehat{\lambda}_0^B)^2 T}{2(1+q\mathrm{v}_0^B T)}\right), \\
K_B &= \left((\widehat{\lambda}_0^B)^2 + \mathrm{v}_0^B\right) T - \log(1+\mathrm{v}_0^B T),
\end{aligned}
$$

where $\widehat{\lambda}_0^B, \mathrm{v}_0^B$ are given by

$$
\widehat{\lambda}_0^B = \lambda_0 + \frac{L_B}{aT_a}, \quad \mathrm{v}_0^B = \mathrm{v}_0 - \frac{1}{T_a}.
$$

Remarks

- For $a = 1$ and $\mathrm{v}_0 > 0$, u_B and $\theta^{B,*}$ are well defined, even though the insider has precise knowledge of B_T. This is a direct consequence of drift uncertainty.

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- For $a = 1$ and $\mathrm{v}_0 > 0$, u_B and $\theta^{B,*}$ are well defined, even though the insider has precise knowledge of B_T. This is a direct consequence of drift uncertainty.
- Contrast this with Pikovsky and Karatzas [2], in which there is no drift parameter uncertainty ($\mathrm{v}_0 \to 0$) and exact knowledge of any kind at time zero leads to unbounded (logarithmic) utility.
- For $\mathrm{v}_0 \to 0$ we recover results consistent with [2]: since $\xi_T = B_T + \lambda T$, when the value of λ is known with certainty, exact Brownian information is equivalent to exact stock price information.

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- Contrast this with Pikovsky and Karatzas [2], in which there is no drift parameter uncertainty ($\mathrm{v}_0 \to 0$) and exact knowledge of any kind at time zero leads to unbounded (logarithmic) utility.
- For $\mathrm{v}_0 \to 0$ we recover results consistent with [2]: since $\xi_T = B_T + \lambda T$, when the value of λ is known with certainty, exact Brownian information is equivalent to exact stock price information.
- We shall see that, with $\mathrm{v}_0 > 0$, only exact terminal stock price knowledge leads to unbounded logarithmic utility.
- The intuition is: with the initial and terminal stock prices known at time zero, one immediately obtains the best possible estimate of λ at time zero, filtering is redundant, and utility explodes.

Additional utility of the insider

- To compute the value of the additional information, $\pi_{L_B} \equiv \pi_B$, we use $\mathrm{Law}(\widehat{\lambda}_0^B|\widehat{\mathcal{F}}_0) = \mathrm{Law}(\widehat{\lambda}_0^B)$, given by

$$\widehat{\lambda}_0^B \sim \mathrm{N}(\lambda_0, 1/T_a) = \mathrm{N}(\lambda_0, \mathrm{v}_0 - \mathrm{v}_0^B).$$

- This gives

$$\pi_B(x)/x = [(1+\mathrm{v}_0 T)(1+q\mathrm{v}_0^B T)]^{1/2}[(1+\mathrm{v}_0^B T)(1+q\mathrm{v}_0 T)]^{-1/2} - 1, \quad p < 1.$$

- We have
  1. $\pi_B(x) > 0$,
  2. $\pi_B(x) \to \infty$ for $a = 1, \mathrm{v}_0 = 0$.

Theorem (Optimal investment with stock price information)

The value function when $\widehat{\mathbb{F}}^L = \widehat{\mathbb{F}}^{L_S} \equiv \widehat{\mathbb{F}}^S$ *and* $a \neq 1$ *is*

$$u_S(x) = \begin{cases} (x^p/p)C_S, & p < 1, \quad p \neq 0, \\ \log x + K_S/2, & p = 0, \end{cases}$$

The optimal $\widehat{\mathbb{F}}^S$*-adapted trading strategy is* $\theta^{S,*}$*, given by*

$$\theta^{S,*}_t = \left\{ \begin{array}{ll} \widehat{\lambda}^S_t[\sigma(1-p)(1+q\mathrm{v}^S_t(T-t))]^{-1}, & p < 1, \quad p \neq 0, \\ \widehat{\lambda}^S_t/\sigma, & p = 0, \end{array} \right\} \quad 0 \leq t \leq T,$$

where $\widehat{\lambda}^S, \mathrm{v}^S$ *are given by*

$$\widehat{\lambda}^S_t = \frac{\widehat{\lambda}^S_0 + \mathrm{v}^S_0 \xi_t}{1 + \mathrm{v}^S_0 t}, \quad \mathrm{v}^S_t = \frac{\mathrm{v}^S_0}{1 + \mathrm{v}^S_0 t}, \quad 0 \leq t \leq T.$$

For $a = 1$*,* $u_L(x)$ *is unbounded for* $p \in [0,1)$ *and equal to zero for* $p < 0$*.*

Optimal investment with stock price inside information

- Here, $C_S.K_S$ are $\widehat{\mathcal{F}}_0^S$-measurable random variables:

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\begin{aligned}
C_S &= \left[(1+\mathrm{v}_0^S T)^p (1+q\mathrm{v}_0^S T)^{1-p}\right]^{-1/2} \exp\left(-\frac{q(\widehat{\lambda}_0^S)^2 T}{2(1+q\mathrm{v}_0^S T)}\right), \\
K_S &= \left((\widehat{\lambda}_0^S)^2 + \mathrm{v}_0^S\right) T - \log(1+\mathrm{v}_0^S T),
\end{aligned}
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where $\widehat{\lambda}_0^S, \mathrm{v}_0^S$ are given by

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\widehat{\lambda}_0^S = \frac{\lambda_0(1-T/T_a) + (1+\mathrm{v}_0 T)(L_S/(aT_a))}{1+\mathrm{v}_0 T(T/T_a)}, \quad \mathrm{v}_0^S = \frac{(1-T/T_a)^2 \mathrm{v}_0}{1+\mathrm{v}_0 T(T/T_a)} - \frac{1}{T_a}.
$$

Additional utility of the insider

- To compute the value of the additional information, $\pi_{L_S} \equiv \pi_S$, we use

$$\widehat{\lambda}_0^S \sim \mathrm{N}\left(\lambda_0, \frac{(1+\mathrm{v}_0 T)^2}{T_a(1+\mathrm{v}_0 T(T/T_a))}\right) = \mathrm{N}(\lambda_0, \mathrm{v}_0 - \mathrm{v}_0^S).$$

- This gives

$$\pi_S(x)/x = [(1+\mathrm{v}_0 T)(1+q\mathrm{v}_0^S T)]^{1/2}[(1+\mathrm{v}_0^S T)(1+q\mathrm{v}_0 T)]^{-1/2} - 1, \quad p < 1.$$

- $\pi_S(x) > 0$.

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- $\pi_S(x) \to \infty$ for $a \to 1$, for any $\mathrm{v}_0 \geq 0$.

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- This gives

$$\pi_S(x)/x = [(1+\mathrm{v}_0 T)(1+q\mathrm{v}_0^S T)]^{1/2}[(1+\mathrm{v}_0^S T)(1+q\mathrm{v}_0 T)]^{-1/2} - 1, \quad p < 1.$$

- $\pi_S(x) > 0$.
- $\pi_S(x) \to \infty$ for $a \to 1$, for any $\mathrm{v}_0 \geq 0$.
- For $a \neq 1$,

$$\pi_S(x) > \pi_B(x), \quad \text{for all } x,$$

  since $\mathrm{v}_0^S < \mathrm{v}_0^B$.
- *Stock price information is more valuable than Brownian information when the drift of the stock is unknown.*

Remarks I

- A direct consequence of parameter uncertainty. Under $\mathbb{F}^L$, when λ is known parameter, knowledge of $\xi_T = B_T + \lambda T$ is indistinct from knowledge of B_T.
- But under $\widehat{\mathbb{F}}^L$, the inside stock price information contributes to a better estimate of the unknown drift, so $\pi_S > \pi_B$.
- The distinction becomes extreme when $a = 1$ and $L_S = \xi_T$. The dynamics of S with respect to $\mathbb{F}^{S_T}$ are

$$\frac{dS_t}{\sigma S_t} = \left(dB_t^{S_T} + \frac{\xi_T - \xi_t}{T - t} dt \right).$$

  All terms except for B^{S_T} are manifestly adapted to $\widehat{\mathbb{F}}^{S_T}$, but not to $\widehat{\mathbb{F}}^{B_T}$.
- This suggests (and this can be made rigorous) that the $\mathbb{F}^{S_T}$-Brownian motion B^{S_T} is also a Brownian motion under the observation filtration $\widehat{\mathbb{F}}^{S_T}$:

$$B^{S_T} = \widehat{B}^{S_T},$$

  and the filtering procedure is redundant.

Remarks II

- Knowing the value of S_T at time zero immediately gives the insider the best estimate of λ.
- Recall: the best estimate of λ from continuous observations of $\xi_t = \lambda t + B_t$ over $[0, T]$ is $\bar{\lambda}(T) := \xi_T / T$. That is, only the initial and final observations matter. (The reason for the well-known difficulty of estimating the drift of a log=Brownian stock.)
- This leads to an explosion in expected utility for the case with perfect stock price (but not Brownian) information, in contrast to Pikovsky and Karatzas [2].

Comparison of trading strategies of regular trader and insider

- Using

$$\mathbb{E}[\widehat{\lambda}_t^L | \widehat{\mathcal{F}}_t] = \widehat{\lambda}_t, \quad 0 \le t \le T,$$

we obtain:

Proposition

For $p = 0$, $R_B = R_S = 1$, $\widehat{\mathbb{F}}$-a.s., while for $p \neq 0$, we have

$$R_t^B = \frac{1 + q\mathrm{v}_t(T - t)}{1 + q\mathrm{v}_t^B(T - t)}, \quad R_t^S = \frac{1 + q\mathrm{v}_t(T - t)}{1 + q\mathrm{v}_t^S(T - t)}, \quad 0 \le t \le T.$$

Remarks

- Since $\mathrm{v}_t^S < \mathrm{v}_t^B < \mathrm{v}_t$ for $t \in [0, T]$, for $p \neq 0$ we have (note $q < 0$ when $p \in (0,1)$)
$$qR_t^S > qR_t^B > q, \quad 0 \leq t \leq T.$$
- The insider takes a more aggressive holding in the stock than the non-insider when $p < 0$ (relative risk aversion larger than 1).
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- Due to the behaviour of $U_p(x)$ for $p \in (0,1)$ as $x \to 0$ and $x \to \infty$. For $p \in (0,1)$, $U_p(x)$ is unbounded as $x \to \infty$ (but finite if $p < 0$).

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- This does not result in greater expected utility. The uninformed agent's gamble will, on average, fail. The insider is privy to this knowledge in cases when the uninformed agent is not. It is precisely this knowledge which leads the insider to hold back from being aggressive.

Conclusions

- In the face of parameter uncertainty, stock price information is more valuable than information on a driving Brownian motion.
- The stock price information contrbutes to a better estimate of the unknown drift.
- The insider with stock price knowledge takes more aggressive positions in the stock due to her better certainty of the drift value, except for when the utility function can lead to a St Petersburg-style paradox, a previously unseen effect that has arisen from a combination of inside information and drift uncertainty.

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