Commodity Derivatives Valuation with Autoregressive and Moving Average Components in the Price Dynamics

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joint work with
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Motivation: Oil Price Development

![Graph of Oil Price Development from Dec 1996 to Dec 2008](image-url)
Motivation: Term Structure of Futures Prices

![Graph showing the term structure of futures prices for 08-Jan-1997. The graph plots USD per Barrel against Maturity [months]. The data points form a downward-sloping curve, indicating a negative yield curve.](image-url)
Motivation: Term Structure of Futures Prices

13–May–1998

USD per Barrel vs. Maturity [months]
Motivation: Term Structure of Futures Prices

06–Jun–2001

USD per Barrel

Maturity [months]

29
28
27
26
25
24

0 5 10 15 20 25
Motivation: Term Structure of Futures Prices

31-May-2006

USD per Barrel

Maturity [months]

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Motivation: Term Structure of Futures Prices

30–Apr–2008

USD per Barrel

Maturity [months]

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Futures contracts can be priced by no-arbitrage arguments:

- **Financial contracts: Cost-of-carry**
  \[ F_t(T) = S_t e^{r(T-t)} \]

- With storage costs:
  \[ F_t(T) = S_t e^{(r+s)(T-t)} \]

- No explanation for backwardation

- Inferior empirical performance

- Equity futures can show backwardation due to dividends
  \[ F_t(T) = S_t e^{(r-q)(T-t)} \]
Pricing of Commodity Futures Contracts

• Commodities differ from pure financial assets as they are held for consumption or production

• Similar to the dividend yield of a stock, the holder of the commodity receives a convenience yield from holding stocks of commodities

• Kaldor (1939): ”... stocks of goods... also have a yield..., by enabling the producer to lay hands on them the moment they are wanted, and thus saving the cost and trouble of ordering frequent deliveries, or waiting for deliveries.”
The Convenience Yield

**Convenience yield deterministic function of price:**

- Brennan/Schwartz (1985)
- Brennan (1991)

→ Poor empirical performance

**Stochastic convenience yield:**

- Gibson/Schwartz (1990)
- Schwartz (1997)
- Schwartz/Smith (2000)
- Cassasus/Collin-Dufresne (2005)

→ All models assume explicitly or implicitly that the **convenience yield** follows an **Ornstein-Uhlenbeck process**
Modelling the Convenience Yield

- The assumed Ornstein-Uhlenbeck process is the continuous limit of an AR(1) process

- An analysis of the approximated (net) convenience yield

\[ \delta_{t,T-1,T} = \ln \left( \frac{F(t,T)}{F(t,T-1)} \right) \]

shows that an AR(1) is not able to capture the dynamics appropriately

- An ARMA(1,1) or higher order AR(q) model yield much better fit to the data
Modelling Idea

• Model the convenience yield as continuous autoregressive moving average process: \textbf{CARMA}(p,q)

• CARMA\((p,q)\) processes have a long history in the statistics literature: Doob (1944), ..., Brockwell (2001)

• No usage in the finance literature

• One exception for interest rates: Benth, Koekebakker, and Zakamouline (2008)
Contribution

Our contribution to the literature:

1. Formulation of a commodity pricing model in continuous time allowing for higher order auto-regression and moving average components:

   \[ \text{ABM-CARMA}(p,q) \]

2. Derivation of closed-form solutions for futures and options prices

3. Application to the crude oil futures market, demonstrating the model’s superior empirical performance
Model Description: **ABM-CARMA(2,1)**

**Latent factor** spot price model in continuous time:

- One non-stationary factor $Z_t$: **long-term equilibrium** modelled by an Arithmetic Brownian Motion
- One stationary factor $Y_t$: **short-term deviations** from the equilibrium modelled by a CARMA(1,0) process (Schwartz/Smith 2000)

\[
\ln S_t = Z_t + Y_t
\]

\[
dZ_t = \mu dt + \sigma_Z dW_t^Z
\]

\[
dY_t = -kY_t dt + \sigma_Y dW_t^Y
\]
Model Description: \textit{ABM-CARMA}(2,1)

\textbf{Latent factor} spot price model in continuous time:

- One non-stationary factor $Z_t$: \textit{long-term equilibrium} modelled by an Arithmetic Brownian Motion
- One stationary factor $Y_t$: \textit{short-term deviations} from the equilibrium modelled by a CARMA(2,0) process

\[
\ln S_t = Z_t + Y_t
\]

\[
dZ_t = \mu dt + \sigma_Z dW_t^Z
\]

\[
d\dot{Y}_t = -k \dot{Y}_t dt + \sigma_Y dW_t^Y
\]

\[
dY_t = \dot{Y}_t dt
\]
Model Description: *ABM-CARMA(2,1)*

**Latent factor** spot price model in continuous time:

- One non-stationary factor $Z_t$: *long-term equilibrium* modelled by an Arithmetic Brownian Motion
- One stationary factor $Y_t$: *short-term deviations* from the equilibrium modelled by a CARMA($2,1$) process

\[
\ln S_t = Z_t + Y_t
\]

\[
dZ_t = \mu dt + \sigma_Z dW_t^Z
\]

\[
d\dot{Y}_t = -k \dot{Y}_t dt + \sigma_Y dW_t^Y
\]

\[
dY_t = \dot{Y}_t dt
\]
Model Description: **ABM-CARMA(2,1)**

**Latent factor** spot price model in continuous time:

- One non-stationary factor $Z_t$: **long-term equilibrium** modelled by an Arithmetic Brownian Motion
- One stationary factor $Y_t$: **short-term deviations** from the equilibrium modelled by a CARMA(2,1) process

\[
\ln S_t = Z_t + Y_t + \beta \dot{Y}_t
\]

\[
dZ_t = \mu dt + \sigma_Z dW^Z_t
\]

\[
d\dot{Y}_t = -k \dot{Y}_t dt + \sigma_Y dW^Y_t
\]

\[
dY_t = \dot{Y}_t dt
\]
Model Discussion

Model is formulated **directly under the equivalent martingale measure**

Closed form (affine) solutions for the **futures price**:

$$\ln F(Y_t, \dot{Y}_t, Z_t, t; T) = Z_t + A + B\dot{Y}_t + CY_t + D$$

Difference to the standard Schwartz/Smith 2000 model:

- **Term structure:**
  Much **more flexible**, especially at the **short end**

- **Volatilities:**
  Non-monotonous structure and **higher curvature**
Model Implementation: Data

Data used:

- **Crude oil futures** traded at the New York Mercantile Exchange (NYMEX)
- Sample period: January 1996 to December 2008
- Weekly observations (Wednesday)
- Maturities 1 to 24 months
- Data source: Bloomberg

→ Panel data set of 676 x 24 observations
Model Implementation: Estimation

Implementation of the \textbf{ABM-CARMA(2,1)} model:

- Write discretized version in \textit{state space} form
- Dynamics of latent factors:
  \textit{Translation equation}
- Add measurement error to the pricing formula:
  \textit{Measurement equation}
- \textbf{Kalman filter maximum likelihood} estimation of parameters

## In-Sample Pricing Errors

<table>
<thead>
<tr>
<th></th>
<th>Root Mean Squared Error</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Absolute</td>
<td>% Decrease</td>
</tr>
<tr>
<td>F01</td>
<td>0.0409</td>
<td>0.0486</td>
</tr>
<tr>
<td>F02</td>
<td>0.0283</td>
<td>0.0330</td>
</tr>
<tr>
<td>F03</td>
<td>0.0207</td>
<td>0.0230</td>
</tr>
<tr>
<td>All</td>
<td>0.0122</td>
<td>0.0141</td>
</tr>
</tbody>
</table>

\[
AIC_{ABM-CARMA} = -157,285, \quad AIC_{SS2000} = -152,935, \\
SIC_{ABM-CARMA} = -157,131, \quad SIC_{SS2000} = -152,795.
\]
Out-of-Sample Pricing Errors: Time-Series

Split Data Sample into two periods

- Estimation: First half
- Prediction: Second half

<table>
<thead>
<tr>
<th></th>
<th>Absolute</th>
<th>% Decrease</th>
<th>Relative</th>
</tr>
</thead>
<tbody>
<tr>
<td>F01</td>
<td>0.0564</td>
<td>0.0627</td>
<td>10.1%</td>
</tr>
<tr>
<td></td>
<td>1.48%</td>
<td>1.59%</td>
<td></td>
</tr>
<tr>
<td>F02</td>
<td>0.0510</td>
<td>0.0543</td>
<td>5.9%</td>
</tr>
<tr>
<td></td>
<td>1.33%</td>
<td>1.38%</td>
<td></td>
</tr>
<tr>
<td>F03</td>
<td>0.0472</td>
<td>0.0488</td>
<td>3.2%</td>
</tr>
<tr>
<td></td>
<td>1.23%</td>
<td>1.24%</td>
<td></td>
</tr>
<tr>
<td>All</td>
<td>0.0375</td>
<td>0.0381</td>
<td>1.6%</td>
</tr>
<tr>
<td></td>
<td>0.94%</td>
<td>0.95%</td>
<td></td>
</tr>
</tbody>
</table>
### Out-of-Sample Pricing Errors: Cross-Section

**Split Data Sample into two parts**

- **Estimation:** F01 - F12
- **Prediction:** F13 - F24

<table>
<thead>
<tr>
<th></th>
<th>Absolute</th>
<th>% Decrease</th>
<th>Relative</th>
</tr>
</thead>
<tbody>
<tr>
<td>F15</td>
<td>0.0068</td>
<td>0.0090</td>
<td>24.4%</td>
</tr>
<tr>
<td>F18</td>
<td>0.0115</td>
<td>0.0149</td>
<td>22.8%</td>
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<tr>
<td>F21</td>
<td>0.0173</td>
<td>0.0216</td>
<td>19.9%</td>
</tr>
<tr>
<td>F24</td>
<td>0.0237</td>
<td>0.0284</td>
<td>16.5%</td>
</tr>
<tr>
<td><strong>All</strong></td>
<td>0.0144</td>
<td>0.0179</td>
<td>19.6%</td>
</tr>
</tbody>
</table>
Conclusion

- AR(1) poor description of the convenience yield
- **Extension** of Schwartz/Smith model using **continuous time limit of ARMA** processes to describe the convenience yield
- Results in:
  - More flexible futures curves
  - **Without** the use of **additional risk factors**
- Applied to crude oil futures:
  - Better fit/prediction at the short end
  - Better prediction of long maturity contracts from short maturity contracts