# Credit Risk, Market Sentiment and Randomly-Timed Default

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Bachelier Finance Society
6th World Congress

Toronto

22-26 June 2010

**This presentation is based on:**

D. C. Brody, L. P. Hughston & A. Macrina (2010) “Credit Risk, Market Sentiment and Randomly-Timed Default”. In: Stochastic Analysis in 2010, edited by D. Crisan, Springer Verlag (to appear).

# Credit Modelling

We consider a simple model for defaultable securities and, more generally, for a class of financial instruments for which the cash flows depend on the default times of a set of defaultable securities.

For example, if $\tau$ is the time of default of a zero-coupon bond that matures at time $T$, then the bond delivers a single cash-flow $H_T$ at time $T$ given by

$$H_T = N\,\mathbb{1}\{\tau > T\}, \tag{1}$$

where $N$ is the principal.

As another example, let $\tau_1, \tau_2, \ldots, \tau_n$ denote the default times of a set of $n$ discount bonds, each with maturity after some time $T$.

Write $\bar{\tau}_1, \bar{\tau}_2, \ldots, \bar{\tau}_n$ for the "order statistics" of these default times.

So $\bar{\tau}_1$ denotes the time of the first default (among $\tau_1, \tau_2, \ldots, \tau_n$), $\bar{\tau}_2$ denotes the time of the second default, and so on.

Then a structured product that pays

$$H_T = K\,\mathbb{1}\{\bar{\tau}_k \le T\} \tag{2}$$

is a kind of "insurance" policy that pays $K$ at time $T$ if there have been $k$ or more defaults by time $T$.

Problem:

What counts in the valuation of such products is not necessarily the "actual" or "objective" probability of default (even if this can be meaningfully determined), but rather the "perceived probability of default".

This can change over time, depending on shifts in market sentiment based on the flow of relevant market information.

How can we incorporate the notion of "perceived probability of default" in credit risk modelling?

# Asset pricing

We introduce a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with a probability measure $\mathbb{Q}$ which we take to be the risk-neutral measure.

For simplicity we assume that the default-free interest-rate term structure is deterministic.

We write $P_{tT}$ for the price at time $t$ of a default-free discount bond that matures at time $T$, $0 \leq t \leq T$.

The price $S_t$ at time $t$ of a non-dividend-paying asset with payoff $H_T$ at time $T$ is

$$S_t = P_{tT}\mathbb{E}\left[H_T \,|\, \mathcal{G}_t\right], \tag{3}$$

where $\{\mathcal{G}_t\}$ represents the market filtration.

We shall construct $\{\mathcal{G}_t\}$ in such a way that the notion of "market sentiments" concerning default times can be included in the pricing method.

# Modelling the market filtration

We let $\tau_1$, $\tau_2$,. . . ,$\tau_n$ be a collection of random times, and set

$$\begin{aligned} \tau_1 &= f_1(X_1, X_2, \dots, X_n), \\ \tau_2 &= f_2(X_1, X_2, \dots, X_n), \\ &\vdots \\ \tau_n &= f_n(X_1, X_2, \dots, X_n). \end{aligned} \tag{4}$$

Here $X_1, X_2, \dots, X_n$ are $n$ independent "market factors" that determine the default times.

We emphasize that if two default times share an $X$-factor in common, then they are dependent random variables.

We write $\tau_\alpha$ $(\alpha = 1, \dots, m)$ for the default times, and $X_k$ $(k = 1, \dots, n)$ for the market factors.

With each $\tau_\alpha$ we associate a "default indicator process" $\mathbb{1}\{\tau_\alpha > t\}$, $t \geq 0$, which takes the value unity until default occurs, at which time it drops to zero.

We also introduce a set of $n$ "information processes" $\{\xi_t^k\}_{t\geq 0}$, one for each $X_k$.

These are defined by

$$\xi_t^k = \sigma_k \, t \, X_k + B_t^k. \tag{5}$$

Here, $\sigma_k$ denotes a constant information flow rate, and $\{B_t^k\}_{t\geq 0}$ is an independent Brownian motion.

We take the market filtration $\{\mathcal{G}_t\}_{t\geq 0}$ to be generated jointly by the information processes and the default indicator processes:

$$\mathcal{G}_t = \boldsymbol{\sigma}\left[\mathbb{1}\{\tau^\alpha > s\}_{0\leq s\leq t}^{\alpha=1,\dots,m}, \{\xi_s^k\}_{0\leq s\leq t}^{k=1,\dots,n}\right]. \tag{6}$$

Hence at time $t$ market participants have partial information about $X_k$ generated up to time $t$, and they can also check whether default has occurred or not.

It is useful to introduce the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated solely by the information processes:

$$\mathcal{F}_t = \boldsymbol{\sigma}\left[\{\xi_s^k\}_{0\leq s\leq t}^{k=1,\dots,n}\right]. \tag{7}$$

Clearly $\mathcal{F}_t \subset \mathcal{G}_t$.

We do not as such require the notion of a “background filtration” for our theory.

We can think of the information processes and the default indicator processes as providing two related but different types of information about the economic factors.

# Credit-risky discount bond

Let us focus on the special case $m = 1$, $n = 1$.

We have a single market factor $X$ and a single default time $\tau = f(X)$, where we assume that $f(X)$ is invertible.

We consider

$$\xi_t = \sigma\, t\, X + B_t \qquad \text{and} \qquad \mathbb{1}\{\tau > t\}. \tag{8}$$

The market filtration $\{\mathcal{G}_t\}$ is defined by

$$\mathcal{G}_t = \boldsymbol{\sigma}\left( \mathbb{1}\{\tau > s\}_{0 \le s \le t}, \{\xi_s\}_{0 \le s \le t} \right). \tag{9}$$

The price at time $t$ of a defaultable discount bond (no recovery) with maturity $T$ is

$$B_{tT} = P_{tT}\, \mathbb{E}\left[ \mathbb{1}\{\tau > T\} \,|\, \mathcal{G}_t \right]. \tag{10}$$

It follows then that

$$B_{tT} = P_{tT}\,\mathbb{1}\{\tau > t\}\frac{\mathbb{E}\left[\mathbb{1}\{\tau > T\}|\mathcal{F}_t\right]}{\mathbb{E}\left[\mathbb{1}\{\tau > t\}|\mathcal{F}_t\right]}, \tag{11}$$

where $\mathcal{F}_t$ is generated by $\{\xi_s\}_{0\le s\le t}$.

One can show that $\{\xi_t\}$ has the Markov property with respect to its own filtration $\{\mathcal{F}_t\}$. Thus we have:

$$B_{tT} = P_{tT}\,\mathbb{1}\{\tau > t\}\,\frac{\mathbb{E}\left[\mathbb{1}\{\tau > T\}\,|\,\xi_t\right]}{\mathbb{E}\left[\mathbb{1}\{\tau > t\}\,|\,\xi_t\right]}. \tag{12}$$

Let $\rho_t(x)\mathrm{d}x = \mathbb{Q}\left[X \in \mathrm{d}x\,|\,\xi_t\right]$. Then we can write

$$B_{tT} = P_{tT}\,\mathbb{1}\{\tau > t\}\,\frac{\int_{-\infty}^{\infty}\mathbb{1}\{f(x) > T\}\rho_t(x)\mathrm{d}x}{\int_{-\infty}^{\infty}\mathbb{1}\{f(x) > t\}\rho_t(x)\mathrm{d}x}. \tag{13}$$

By use of the Bayes rule, we obtain

$$\rho_t(x) = \frac{\rho_0(x)\exp\left[\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t\right]}{\int_{-\infty}^{\infty}\rho_0(x)\exp\left[\sigma x\xi_t - \frac{1}{2}\sigma^2x^2t\right]\mathrm{d}x}, \tag{14}$$

where $\rho_0(x)$ is the *a priori* density of $X$.

Thus the bond price is

$$B_{tT} = P_{tT}\, \mathbb{1}\{\tau > t\}\, \frac{\int_{-\infty}^{\infty} \rho_0(x) \mathbb{1}\{f(x) > T\} \exp\left[\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t\right] \mathrm{d}x}{\int_{-\infty}^{\infty} \rho_0(x) \mathbb{1}\{f(x) > t\} \exp\left[\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t\right] \mathrm{d}x}. \quad (15)$$

We observe that the bond price fluctuates as $\xi_t$ changes.

This reflects changes in market sentiments concerning the possibility of default.

The bond price dynamics can be simulated as follows:

1. We simulate a value for $X$, then we deduce the corresponding value of $\tau$.
2. We simulate an independent Brownian motion, and thereby also the information process.
3. The simulation of the bond price is thus obtained by applying (15).

# Discount bond dynamics

By use of Ito calculus we can work out the SDE of the bond price process:

$$\mathrm{d}B_{tT} = (r_t + h_t)B_{tT}\,\mathrm{d}t + \Sigma_{tT}\,B_{tT}\,\mathrm{d}W_t + B_{tT}^{(-)}\mathrm{d}\mathbb{1}\{\tau > t\}. \tag{16}$$

Here $r_t = -\partial_t \ln(P_{0t})$. We denote by $\{h_t\}_{t\geq 0}$ the hazard rate process, which is defined by

$$h_t = \frac{\mathbb{E}\left[\delta(f(X) - t)\,|\,\xi_t\right]}{\mathbb{E}\left[\mathbb{1}\{f(X) > t\}\,|\,\xi_t\right]}. \tag{17}$$

The bond volatility $\{\Sigma_{tT}\}$ is defined by

$$\Sigma_{tT} = \frac{\mathbb{E}\left[\mathbb{1}\{f(X) > T\}X\,|\,\xi_t\right]}{\mathbb{E}\left[\mathbb{1}\{f(X) > T\}\,|\,\xi_t\right]} - \frac{\mathbb{E}\left[\mathbb{1}\{f(X) > t\}X\,|\,\xi_t\right]}{\mathbb{E}\left[\mathbb{1}\{f(X) > t\}\,|\,\xi_t\right]}, \tag{18}$$

and the process $\{W_t\}$ is defined by

$$W_t = \int_0^t \mathbb{1}\{f(X) > s\}\left(\mathrm{d}\xi_s - \sigma\mathbb{E}\left[X\,|\,\mathcal{G}_s\right]\mathrm{d}s\right). \tag{19}$$

It can be shown by Lévy's characterisation theorem that $\{W_t\}_{0\leq t\leq\tau}$ is a $\{\mathcal{G}_t\}$-stopped Brownian motion.

The process $\{W_t\}$ is the innovations process associated with the information flow $\{\mathcal{G}_t\}$.

# Hazard rate process

We recall that $\tau = f(X)$ is an invertible relation, so that we can write

$$\phi(\tau) := f^{-1}(\tau) = X, \tag{20}$$

and thus

$$\xi_t = \sigma\, t\, \phi(\tau) + B_t. \tag{21}$$

Let $p(u)$ be the initial density of the random variable $f(X)$. Then the hazard rate process can be expressed by

$$h_t = \frac{\mathbb{E}\left[\delta(f(X)-t)\,|\,\xi_t\right]}{\mathbb{E}\left[\mathbb{1}\{f(X)>t\}\,|\,\xi_t\right]}, \tag{22}$$

$$= \frac{p(t)\,\exp\left[\sigma\phi(t)\,\xi_t - \frac{1}{2}\sigma^2\phi^2(t)\,t\right]}{\int_t^\infty p(u)\,\exp\left[\sigma\phi(u)\,\xi_t - \frac{1}{2}\sigma^2\phi^2(u)\,t\right]\mathrm{d}u} \tag{23}$$

This shows that the default intensity at time $t$ is determined by "market perceptions" based on the information $\xi_t$. In the following plots we show a simulation of the bond price and the associated hazard rate process.

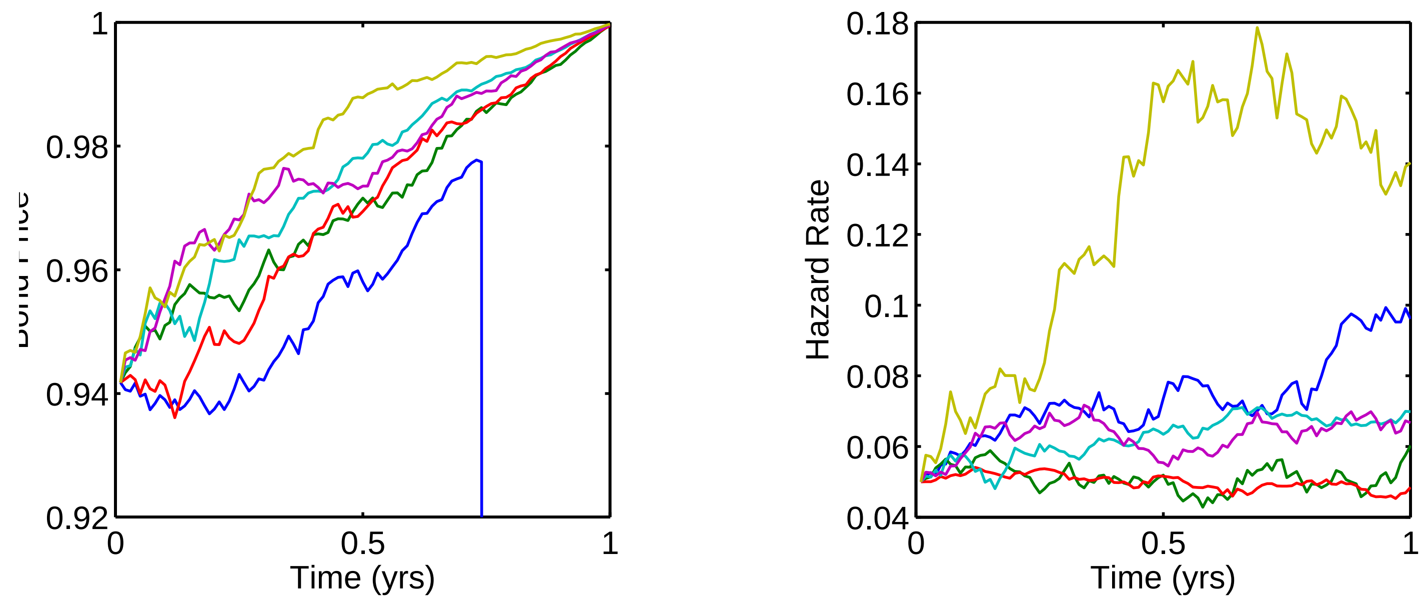


Figure 1: Sample paths of the defaultable discount bond and the associated hazard rate. We choose $\phi(t) = \mathrm{e}^{-0.025t}$, and the initial term structure is assumed to be flat so that $P_{0t} = \mathrm{e}^{-0.02t}$. The information flow rate is set to be $\sigma = 0.3$, and the bond maturity is five years.

# Bond options

We now consider the problem of pricing an option on a defaultable bond.

Let $K$ be the strike, and $t < T$ be the option maturity. The payoff of a European-style call bond option is the random variable $H_t = (B_{tT} - K)^+$.

Let us write $B_{tT} = \mathbb{1}\{\tau > t\}\tilde{B}_{tT}$, where

$$\tilde{B}_{tT} = P_{tT}\,\frac{\int_T^\infty p(u)\exp\left[\sigma\phi(u)\xi_t - \frac{1}{2}\sigma^2\phi^2(u)t\right]\mathrm{d}u}{\int_t^\infty p(u)\exp\left[\sigma\phi(u)\xi_t - \frac{1}{2}\sigma^2\phi^2(u)t\right]\mathrm{d}u}. \tag{24}$$

Then the option payoff can be written in the form $\mathbb{1}\{\tau > t\}(\tilde{B}_{tT} - K)^+$, and the option price is

$$C_0 = P_{0t}\mathbb{E}\Big[\mathbb{1}\{\tau > t\}(\tilde{B}_{tT} - K)^+\Big]. \tag{25}$$

Note that $\tilde{B}_{tT} = \tilde{B}(t, \xi_t)$. Hence the option payoff $H_t$ is a function of the two random variables $\tau$ and $\xi_t$.

We thus need to work out the joint density function:

$$\rho(u,y)=\mathbb{E}\left[\delta(\tau-u)\delta(\xi_t-y)\right]=-\frac{\mathrm{d}}{\mathrm{d}u}\mathbb{E}\left[\mathbb{1}\{\tau>u\}\delta(\xi_t-y)\right]. \qquad (26)$$

One can regard

$$\mathcal{A}_t(u,y)=P_{0t}\,\mathbb{E}\left[\mathbb{1}\{\tau>u\}\delta(\xi_t-y)\right] \qquad (27)$$

as the price of a "defaultable Arrow-Debreu security" based on the value of the market information process at time $t$.

A calculation shows that

$$\rho(u,y)=\frac{1}{\sqrt{2\pi t}}p(u)\exp\left[-\frac{1}{2t}(y-\sigma\phi(u)t)^2\right]. \qquad (28)$$

Therefore the price of the option is given by

$$C_0 = P_{0t} \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}y\, \rho(u,y) \mathbb{1}\{u > t\} (\tilde{B}(t,y) - K)^+ \tag{29}$$

$$= \frac{P_{0t}}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \mathrm{d}y\, \exp\left(-\frac{1}{2t}\, y^2\right) (\tilde{B}(t,y) - K)^+ \times \int_t^{\infty} p(u) \exp\left[\sigma\phi(u)y - \frac{1}{2}\sigma^2\phi^2(u)t\right] \mathrm{d}u. \tag{30}$$

We notice, however, that the term in the third line is identical to the denominator of the expression for $\tilde{B}(t,y)$. Therefore, we have:

$$C_0 = \frac{P_{0t}}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \mathrm{d}y\, \exp\left(-\frac{1}{2t}\, y^2\right) \left(P_{tT} \int_T^{\infty} p(u) \exp\left[\sigma\phi(u)y - \frac{1}{2}\sigma^2\phi^2(u)t\right] \mathrm{d}u - K \int_t^{\infty} p(u) \exp\left[\sigma\phi(u)y - \frac{1}{2}\sigma^2\phi^2(u)t\right] \mathrm{d}u\right)^+. \tag{31}$$

In the case that $\phi(u)$ is a decreasing function one can show that there exists a unique $y^* > y$ such that

$$P_{tT}\int_{T}^{\infty} p(u)\,\exp\left[\sigma\phi(u)y-\frac{1}{2}\sigma^2\phi^2(u)t\right]\mathrm{d}u$$

$$-\,K\int_{t}^{\infty} p(u)\,\exp\left[\sigma\phi(u)y-\frac{1}{2}\sigma^2\phi^2(u)t\right]\mathrm{d}u>0. \qquad (32)$$

Therefore, we can perform the $y$-integration in (31) to obtain

$$C_0 \;=\; P_{0T}\int_{T}^{\infty} p(u)N\left(\frac{y^*-\sigma\phi(u)t}{\sqrt{t}}\right)\mathrm{d}u$$

$$-\,P_{0t}\,K\int_{t}^{\infty} p(u)N\left(\frac{y^*-\sigma\phi(u)t}{\sqrt{t}}\right)\mathrm{d}u, \qquad (33)$$

where $\phi(u)$ is a decreasing function, and $N(x)$ is the cumulative normal distribution function.

The critical value $y^*_\phi(t,T,K,\sigma)$ can be determined numerically in order to calculate the call option price.

An example is shown in the following Figure 2.

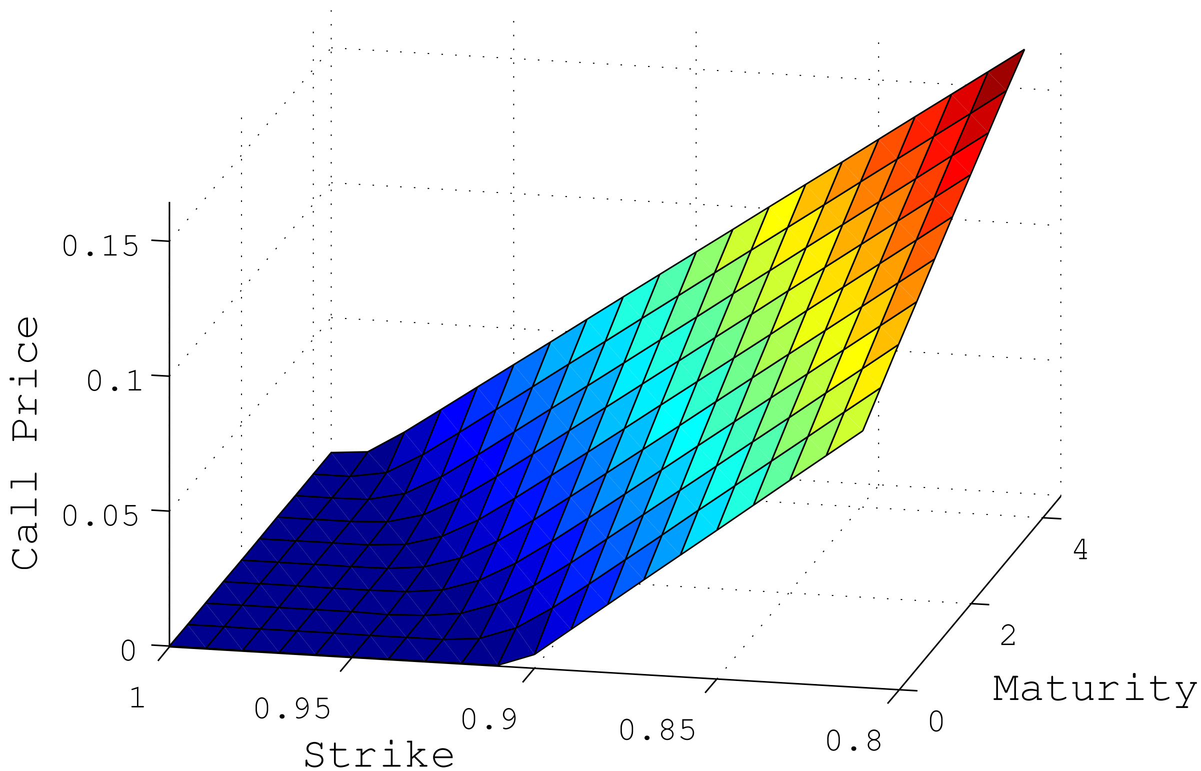


Figure 2: Price of a call option on a defaultable discount bond. The bond maturity is five years, $\phi(t) = \mathrm{e}^{-0.05t}$, and the initial term structure is assumed to be flat so that $P_{0t} = \mathrm{e}^{-0.02t}$. The information flow rate is set to be $\sigma = 0.25$.

# Conclusions

The Brownian motion that "drives" the price process of the defaultable bond is not prespecified (as it is in most theories) and associated with some kind of "background market filtration".

Rather, it is directly associated with information about the economic factors determining default.

In this respect, the information-based approach is closer in spirit to a structural model, even though it retains the features of a reduced form model.

The theory gives rise to a tractable expression for credit risky bonds:

$$B_{tT} = \mathbb{1}\{\tau > t\}\, P_{tT}\, \frac{\int_T^\infty p(u) \exp\left[\sigma\phi(u)\xi_t - \frac{1}{2}\sigma^2\phi^2(u)t\right] \mathrm{d}u}{\int_t^\infty p(u) \exp\left[\sigma\phi(u)\xi_t - \frac{1}{2}\sigma^2\phi^2(u)t\right] \mathrm{d}u}. \tag{34}$$

Analogous expressions can be obtained for the hazard processes.

The dynamics of the hazard rate process reflects the market perceptions of the timing of the default.

The volatility structure is deduced endogenously from the specification of the market information flow associated with the timing of default.

The pricing of an option on a defaultable bond can be carried out explicitly.

When the default times of two or more bonds depend on a common market factor, the default times are correlated and so are the bond price dynamics.

In particular, we are able to characterize in a rather precise way the sense in which "correlations" in credit portfolios can be regarded as deriving from "market perceptions" concerning correlation as much as "actual" correlation.

This allows one to resolve an otherwise enigmatic situation.

In most theories of credit, correlations are "objective" in the sense that they measure what can be regarded as the "actual" correlations between bond price movements.

In an information-based setting we are no longer required to take such a strong view.

Correlations are conditional on imperfect information, and as that information shifts (as it indeed will) so does the correlation.

This may leave one with an uncomfortable feeling — with a kind of "credit-correlation sea-sickness".

But in fact this kind of shiftiness in the correlation is in our view exactly what characterizes real markets.

It is better to recognize that correlation is indeed a dynamic, information-driven quantity, and needs to be modelled as such.

The framework we have presented here allows for this possibility.

# Related work

D. C. Brody, L. P. Hughston & A. Macrina (2007) Beyond hazard rates: a new framework for credit-risk modelling. In *Advances in Mathematical Finance, Festschrift Volume in Honour of Dilip Madan*, edited by R. Elliott, M. Fu, R. Jarrow & J.-Y. Yen. Birkhäuser, Basel and Springer, Berlin.

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