An Equity-Interest Rate Hybrid Model with Stochastic Volatility and the Interest Rate Smile

Lech A. Grzelak & Cornelis W. Oosterlee

Bachelier Conference- 6th World Congress
June 22-26, 2010 Toronto
The Objectives of the Research

To build an Equity-Interest Rate Hybrid model which:

⇒ generates a smile on the equity side;
⇒ includes stochastic interest rate with interest rate smile;
⇒ enables non-zero correlations between the underlying processes;
⇒ allows efficient calibration;
⇒ First, the Heston-Hull-White Hybrid model:

\[
\begin{align*}
\frac{dS}{S} &= r \, dt + \sqrt{\sigma} \, dW^Q_x, \\
\, d\sigma &= \kappa (\bar{\sigma} - \sigma) \, dt + \gamma \sqrt{\sigma} \, dW^Q_\sigma, \\
\, dr &= \lambda (\theta - r) \, dt + \eta \, dW^Q_r,
\end{align*}
\]

with correlations: \(\rho_{x,\sigma} \neq 0\), \(\rho_{x,r} \neq 0\) and \(\rho_{\sigma,r} \neq 0\).

⇒ With the Feynman-Kac theorem, for \(x = \log S\) the corresponding PDE is given by:

\[
\begin{align*}
r \phi &= \phi_t + (r - 1/2\sigma) \phi_x + \kappa (\bar{\sigma} - \sigma) \phi_\sigma + \lambda (\theta_t - r) \phi_r \\
&\quad + 1/2 \sigma \phi_{x,x} + 1/2 \gamma^2 \sigma \phi_{\sigma,\sigma} + 1/2 \eta^2 \phi_{r,r} \\
&\quad + \rho_{x,\sigma} \gamma \sigma \phi_{x,\sigma} + \rho_{x,r} \eta \sqrt{\sigma} \phi_{x,r} + \rho_{\sigma,r} \eta \gamma \sqrt{\sigma} \phi_{\sigma,r}.
\end{align*}
\]

⇒ In the present form the model is not affine \([\text{Duffie et al. 2000}]\).
By linearization of the non-affine terms in the covariance matrix we find an approximation:

\[
\begin{pmatrix}
\sigma & \rho_{\sigma,\gamma} \sqrt{\sigma} & \rho_{\sigma,\eta}\Psi \\
\rho_{\sigma,\gamma} \sqrt{\sigma} & \gamma_2 \sigma & \rho_{\sigma,\eta}\Psi \\
\rho_{\sigma,\eta}\Psi & \gamma_2 \sigma & \eta^2
\end{pmatrix}
\approx
\begin{pmatrix}
\sigma & \rho_{\sigma,\gamma} \sqrt{\sigma} & \rho_{\sigma,\eta}\Psi \\
\rho_{\sigma,\gamma} \sqrt{\sigma} & \gamma_2 \sigma & \rho_{\sigma,\eta}\Psi \\
\rho_{\sigma,\eta}\Psi & \gamma_2 \sigma & \eta^2
\end{pmatrix}.
\]

We linearize the non-affine term $\sqrt{\sigma}$ by $\Psi$:

\[\Psi = \mathbb{E}(\sqrt{\sigma}) \quad \text{or} \quad \Psi = \mathcal{N}(\mathbb{E}(\sqrt{\sigma}), \text{Var}(\sqrt{\sigma}))\].

The expectation for the CIR-type process is known analytically:

\[\mathbb{E}(\sqrt{\sigma}) = \sqrt{2}c e^{-\lambda/2} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\lambda}{2}\right)^k \frac{\Gamma\left(\frac{1+d}{2} + k\right)}{\Gamma\left(\frac{d}{2} + k\right)},\]

with $c$, $d$ and $\lambda$ being known deterministic functions.

Affine approximation $\Rightarrow$ efficient pricing!
We set: $\kappa = 0.5$, $\gamma = 0.1$, $\lambda = 1$, $\eta = 0.01$, $\theta = 0.04$ and $\rho_{x,\sigma} = -0.5$, $\rho_{x,r} = 0.6$.

Figure: Comparison of implied Black-Scholes volatilities from Monte Carlo (40.000 paths and 500 steps) and Fourier inversion.
Intermediate Summary

⇒ The linearization method provides a high quality approximation;
⇒ The projection procedure can be simply extended to high dimensions;
⇒ The method is straightforward, and does not involve complex techniques;
⇒ Alternative methods for approximating the hybrid models are:
  ● Markovian projection based methods [Antonov-2008].
  ● Models with indirect correlation structure [Giese-2004, Andreasen-2006];
We now consider the Stochastic Volatility Libor Market Model [Andersen, Brotherton-Ratcliffe-2005], [Andersen, Andreasen-2000]. For $L_k := L(t, T_{k-1}, T_k)$ we define

$$L(t, T_{k-1}, T_k) \equiv \frac{1}{\tau_k} \left( \frac{P(t, T_{k-1})}{P(t, T_k)} - 1 \right), \text{ for } t < T_{k-1}. $$

with the dynamics under their natural measure given by:

$$\begin{cases}
    dL_k = \sigma_k (\beta_k L_k + (1 - \beta_k)L_k(0)) \sqrt{V} dW^k_k, \\
    dV = \lambda (V(0) - V) dt + \eta \sqrt{V} dW^k_V,
\end{cases}$$

with $dW^k_i dW^k_j = \rho_{i,j} dt$, for $i \neq j$ and $dW^k_V dW^k_i = 0$.

Efficient calibration with Markovian Projection Method [Piterbarg-2005].
Fast pricing of European-style equity options:

\[ \Pi(t) = B(t)\mathbb{E}^Q \left( \frac{(S(T_N) - K)^+}{B(T_N)} \middle| \mathcal{F}_t \right) \], with \( t < T_N \),

with \( K \) the strike, \( S(T_N) \) the stock price at time \( T_N \), filtration \( \mathcal{F}_t \) and a numéraire \( B(T_N) \).

The money-savings account \( B(T_N) \) is assumed to be correlated with stock \( S(T_N) \).

We switch between the measures: From risk neutral \( Q \) to the \( T_N \)-forward \( Q^{T_N} \):

\[ \Pi(t) = P(t, T_N)\mathbb{E}^{T_N} \left( (F^{T_N}(T_N) - K)^+ \middle| \mathcal{F}_t \right) \], with \( t < T_N \),

with \( F^{T_N}(t) \) the forward of the stock \( S(t) \), defined as:

\[ F^{T_N}(t) = \frac{S(t)}{P(t, T_N)}. \]

The ZCB \( P(t, T_N) \) is not well-defined for all \( t \)!
Since $P(T_{k-1}, T_{k-1}) = 1$ we find for the ZCB $P(t, T_k)$:

$$P(t, T_k) = (1 + \tau_k L(t, T_{k-1}, T_k))^{-1}.$$ 

For $t \neq T_{k-1}$ we use the interpolation from [Schlögl-2002]:

$$P(t, T_k) \approx (1 + (T_k - t)L(t, T_{k-1}, T_k))^{-1}, \text{ for } T_{k-1} \leq t \leq T_k.$$ 

This ZCB interpolation is sufficient for calibration purposes but for pricing callable exotics more attention is needed [Piterbarg-2004, Davis et al.-2009, Beveridge & Joshi-2009].
Under the $T_N$-forward measure we have:

$\Rightarrow$ An equity part is driven by the Heston model:

$$
\frac{dS}{S} = (\ldots)dt + \sqrt{\xi}dW^N_x,
$$
$$
d\xi = \kappa(\bar{\xi} - \xi)dt + \gamma\sqrt{\xi}dW^N_{\xi}.
$$

$\Rightarrow$ The SV Libor Market Model under the $T_N$-measure is given by:

$$
dL_k = -\phi_k\sigma_k V \sum_{j=k+1}^{N} \frac{\tau_j\phi_j\sigma_j}{1 + \tau_jL_j} \rho_{k,j}dt + \sigma_k\phi_k\sqrt{V}dW^N_k,
$$
$$
dV = \lambda(V(0) - V)dt + \eta\sqrt{V}dW^N_V,
$$

with $\phi_k = \beta_kL_k + (1 - \beta_k)L_j(0)$.
We define the following correlation structure:
Deriving the Forward Dynamics

\[ F_{TN} = \frac{S}{P(t, TN)} \] is a tradable, so \( F_{TN} \) is a martingale under the \( T_N \)-forward measure:

\[
dF_{TN}(t) = \frac{1}{P(t, T_N)} dS(t) - \frac{S(t)}{P^2(t, T_N)} dP(t, T_N).
\]

⇒ Dynamics for \( S(t) \) are known (the Heston model), for ZCB \( P(t, T_N) \) we find:

\[
\frac{1}{P(t, T_N)} = (1 + (T_{m(t)} - t)L_{m(t)}(T_{m(t)} - 1)) \prod_{j=m(t)+1}^{N} (1 + \tau_j L(t, T_{j-1}, T_j)).
\]

with \( m(t) = \min\{k : t \leq T_k\} \).
For the ZCB $P(t, T_N)$ we are only interested in diffusion coefficients:

$$\frac{dP(t, T_N)}{P(t, T_N)} = (...) dt - \sqrt{V} \sum_{j=m(t)+1}^{N} \frac{\tau_j \sigma_j \phi_j}{1 + \tau_j L_j} dW_j^N.$$ 

The forward $F_{T_N}(t)$ dynamics are now given by:

$$\frac{dF_{T_N}}{F_{T_N}} = \sqrt{\xi} dW_x^N \underbrace{+ \sqrt{V} \sum_{j=m(t)+1}^{N} \frac{\tau_j \sigma_j \phi_j}{1 + \tau_j L_j} dW_j^N}_{\text{asset}} \underbrace{+ \sqrt{V} \sum_{j=m(t)+1}^{N} \frac{\tau_j \sigma_j \phi_j}{1 + \tau_j L_j} dW_j^N}_{\text{interest rate}}.$$ 

The model is not affine!
We freeze the Libor rates [Glasserman, Zhao-1999], [Hull, White-1996], [Jäckel, Rebonato-2000], i.e.:

\[ L_j(t) \approx L_j(0) \quad \Rightarrow \quad \phi_j(t) \approx L_j(0). \]

Now, the linearized dynamics are given by:

\[ \frac{dF^T_N}{F^T_N} \approx \sqrt{\xi} dW^N_x + \sqrt{V} \sum_{j=m(t)+1}^{N} \frac{\tau_j \sigma_j L_j(0)}{1 + \tau_j L_j(0)} dW^N_j. \]

The model does not depend on the Libor processes! It is fully described by the volatility structure.
The model is now given by:

\[
\frac{dF^T_N}{F^T_N} \approx \sqrt{\xi} dW^N_x + \sqrt{V} \Sigma^T dW^N, \\
d\xi = \kappa(\bar{\xi} - \xi)dt + \gamma \sqrt{\xi} dW^N_\xi, \\
dV = \lambda(V(0) - V)dt + \eta \sqrt{V} dW^N_V,
\]

with appropriate column vectors $\Sigma$ and $dW^N$.

Under the log-transform, $x = \log F^T_N$, we find:

\[
dx \approx -\frac{1}{2} \left( \sqrt{\xi} dW^N_x + \sqrt{V} \Sigma^T dW^N \right)^2 + \sqrt{\xi} dW^N_x + \sqrt{V} \Sigma^T dW^N.
\]

Since $dW^N_x$ is correlated with $dW^N$ cross terms are still not affine!
We set: \( A = m(t) + 1, \ldots, N \) and \( \psi_j = \frac{\tau_j \sigma_j L_j(0)}{1 + \tau_j L_j(0)} \).

The dynamics for \( x = \log F^T_N \) are given by:

\[
\begin{align*}
\text{d}x & \approx -\frac{1}{2} \left( \xi + A_1(t) V + 2\sqrt{V}\sqrt{\xi} A_2(t) \right) \text{d}t + \sqrt{\xi} \text{d}W^N_x + \sqrt{V} \Sigma^T \text{d}W^N,
\end{align*}
\]

with

\[
A_1(t) := \sum_{j \in A} \psi_j^2 + \sum_{i,j \in A, i \neq j} \psi_i \psi_j \rho_{i,j}, \quad \text{and} \quad A_2(t) := \sum_{j \in A} \psi_j \rho_{x,j}.
\]

The drift and covariance matrix include the non-affine term \( \sqrt{V}\sqrt{\xi} \), we linearize it by:

\[
\sqrt{\xi} \sqrt{V} \approx \mathbb{E}(\sqrt{\xi} \sqrt{V})
\]

\[
\Downarrow \quad \mathbb{E}(\sqrt{\xi}) \mathbb{E}(\sqrt{V}) =: \vartheta(t).
\]
With Feynman-Kac theorem we find the corresponding PDE:

\[
0 = \phi_t + \frac{1}{2} \left( \xi + A_1 V + 2A_2 \vartheta(t) \right) (\phi_{xx} - \phi_x) \\
+ \kappa (\bar{\xi} - \xi) \phi_x + \lambda (V(0) - V) \phi_V + \frac{1}{2} \eta^2 V \phi_{VV} \\
+ \frac{1}{2} \gamma^2 \phi_{\xi,\xi} + \rho \phi_{x,\xi} \gamma \phi_{x,\xi} 
\]

subject to \( \phi(u, X(T), 0) = \exp(iux(T_N)) \).

The corresponding characteristic function is given by:

\[
\phi(u, X(t), \tau) = \exp(A(u, \tau) + iux(t) + B(u, \tau) \xi(t) + C(u, \tau) V(t)),
\]

with \( \tau = T_N - t \).

The ODEs for \( A(u, \tau), B(u, \tau), C(u, \tau) \) are of Heston-type and can be solved recursively [Andersen, Andreasen-2000].
We price an equity call option and investigate the accuracy of the approximation.

For equity we take:

\( \kappa = 1.2, \quad \bar{\xi} = 0.1, \quad \gamma = 0.5, \quad S(0) = 1, \quad \xi(0) = 0.1. \)

For the interest rate model we take term structure:

\[ P(0, T) = \exp(-0.05 T), \quad \beta_k = 0.5, \quad \sigma_k = 0.25, \quad \lambda = 1, \quad V(0) = 1, \quad \eta = 0.1. \]

The correlation structure is given by:

\[
\begin{pmatrix}
1 & \rho_{x,\xi} & \rho_{x,1} & \cdots & \rho_{x,N} \\
\rho_{\xi,x} & 1 & \rho_{\xi,1} & \cdots & \rho_{\xi,N} \\
\rho_{1,x} & \rho_{1,\xi} & 1 & \cdots & \rho_{1,N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{N,x} & \rho_{N,\xi} & \rho_{N,1} & \cdots & 1
\end{pmatrix}
= \begin{pmatrix}
1 & -0.3 & 0.5 & \cdots & 0.5 \\
-0.3 & 1 & 0 & \cdots & 0 \\
0.5 & 0 & 1 & \cdots & 0.98 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0.5 & 0 & 0.98 & \cdots & 1
\end{pmatrix}.
\]
Figure: Comparison of implied Black-Scholes volatilities for the European equity option, obtained by Fourier inversion of approximation and by Monte Carlo simulation.
Conclusion

⇒ We have developed an efficient approximation method projecting non-affine models on affine versions;
⇒ We have presented an extension of the Heston model with stochastic interest rates:
   ● Short-rate processes;
   ● SV LMM;
⇒ The model can be easily generalized to FX options;
References

We investigate the effect of $\beta$ on equity implied vol. with Monte Carlo simulation of the full-scale model:

Figure: The effect of the interest rate skew, controlled by $\beta_k$, on the equity implied volatilities. The Monte Carlo simulation was performed with for maturity $T = 10$.

The prices of the European style options are rather insensitive to skew parameter $\beta$!
We consider an investor who is willing to take some risk in one asset class in order to obtain a participation in a different asset class.

An example of such hybrid product is *minimum of several assets* [Hunter-2005] with payoff defined as:

\[
\text{Payoff} = \max \left( 0, \min \left( C_n(T), k\% \times \frac{S(T)}{S(t)} \right) \right),
\]

where \( C_n(T) \) is an n-years CMS, and \( S(T) \) is a stock.

By taking \( T = \{1, 2, \ldots, 10\} \) and the payment date \( T_N = 5 \) we get:

\[
\Pi_H(t) = P(t, T_5) \mathbb{E}^{T_5} \left[ \max \left( 0, \min \left( \frac{1 - P(T_5, T_{10})}{\sum_{k=6}^{10} P(T_5, T_k)}, k\% \times \frac{S(T_5)}{S(t)} \right) \right) \bigg| \mathcal{F}_t \right].
\]
Figure: The value for a minimum of several assets hybrid product. The prices are obtained by Monte Carlo simulation with 20,000 paths and 20 intermediate points. Left: Influence of $\beta$; Right: Influence of $\rho_{x,L}$. 
Now, we compare the results with Heston-Hull-White model

⇒ From calibration routine we have: \( \lambda = 0.0614, \eta = 0.0133, \)
\( r_0 = 0.05 \) and \( \kappa = 0.65, \gamma = 0.469, \bar{\xi} = 0.090, \rho_{x,\xi} = -0.222 \) and
\( \xi_0 = 0.114. \)

⇒ Calibration ensures that prices on the equities are the same, so the hybrid price differences can only result from the interest rate component!

**Figure:** Hybrid prices obtained by two different hybrid models, H-LMM and HHW. The models were calibrated to the same data set.
The SV LMM model provides much fatter tails for CMS rate than the Hull-White model.