Affine processes on positive semidefinite matrices

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1 Introduction
- Definition of affine processes on $S_d^+$

2 Applications of $S_d^+$-valued affine processes in finance
- Multivariate affine stochastic volatility models
- Affine term structure models
- Literature

3 Characterization of affine processes on $S_d^+$
- Feller property, regularity and related ODEs
- Main theorem
- Admissible parameters

4 Implications for financial modeling
Setting and notation

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- $S_d$, the vector space of symmetric $d \times d$-matrices equipped with scalar product $\langle x, y \rangle = \text{Tr}(xy)$ (isomorphic to $\mathbb{R}^{d(d+1)/2}$).
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- $S_d$, the vector space of symmetric $d \times d$-matrices equipped with scalar product $\langle x, y \rangle = \text{Tr}(xy)$ (isomorphic to $\mathbb{R}^{d(d+1)/2}$).
- $(P_t)_{t \geq 0}$: semigroup associated to the Markov process which acts on bounded measurable functions $f : S_d^+ \to \mathbb{R}$,

$$P_t f(x) := \mathbb{E}_x[f(X_t)] = \int_{S_d^+} f(\xi) p_t(x, d\xi), \quad x \in S_d^+. $$
Definition

An $S_d^+$-valued Markov process $X$ is called affine if

1. it is stochastically continuous, that is,
   \[
   \lim_{s \to t} p_s(x, \cdot) = p_t(x, \cdot)\text{ weakly on } S_d^+ \text{ for every } t \text{ and } x \in S_d^+, \text{ and}
   \]

2. its Laplace transform has exponential-affine dependence on the initial state,
   \[
   \mathbb{E}_x \left[ e^{-\langle u, X_t \rangle} \right] = e^{-\phi(t,u)-\langle \psi(t,u), x \rangle},
   \]

for all $t$ and $u, x \in S_d^+$ and some functions

$\phi : \mathbb{R}_+ \times S_d^+ \to \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \times S_d^+ \to S_d^+$.
Aim of today’s talk

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- **Understanding of this class of processes:**
  - Necessary admissibility conditions on the parameters of the infinitesimal generator.
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- **Understanding of this class of processes:**
  - Necessary admissibility conditions on the parameters of the infinitesimal generator.
  - Sufficient conditions for the existence of affine processes on $S_d^+$. 
One-dimensional affine stochastic volatility models

- **Examples:** Heston [17], Barndorff-Nielsen Shepard model [2], Bates [4], etc.
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- Risk neutral dynamics for the log-price process $Y_t$ and the $\mathbb{R}_+^\times$-valued variance process $X_t$:

  \[
  dX_t = (b + \beta X_t) \, dt + \sigma \sqrt{X_t} \, dW_t, \quad X_0 = x,
  \]

  \[
  dY_t = \left( r - \frac{X_t}{2} \right) \, dt + \sqrt{X_t} \, dB_t, \quad Y_0 = y.
  \]

- $B, W$: correlated Brownian motions,
- $r$: constant interest rate.
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- **Risk neutral dynamics for the log-price process** $Y_t$ **and the** $\mathbb{R}_+$-valued variance process $X_t$:

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- **Efficient valuation of European options via Fourier methods** since the moment generating function is explicitly known (up to the solution of an ODE) and of the following form

\[
\mathbb{E}_{x,y} \left[ e^{-uX_t + vY_t} \right] = e^{\Phi(t,u,v) + \psi(t,u,v)x + vy}, \quad (u, v) \in \mathbb{C}^2.
\]
Multivariate affine stochastic volatility models

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Extension to **multivariate stochastic volatility models** with the aim to...

- ...capture the **dependence structure** between different assets,
- ...obtain a consistent pricing framework for **multi-asset options** such as basket options,
- ...use them as a basis for financial decision-making in the area of **portfolio optimization** and hedging of **correlation risk**.
Model specification

- Multivariate stochastic volatility models consist of a \(d\)-dimensional logarithmic price process with risk-neutral dynamics

\[
dY_t = \left( r1 - \frac{1}{2} X_t^{\text{diag}} \right) dt + \sqrt{X_t} dB_t, \quad Y_0 = y,
\]

and stochastic covariation process \(X = \langle Y, Y \rangle\).

- \(B\): \(d\)-dimensional Brownian motion,
- \(r\): constant interest rate,
- \(1\): the vector whose entries are all equal to one,
- \(X^{\text{diag}}\): the vector containing the diagonal entries of \(X\).
Model specification

- Multivariate stochastic volatility models consist of a \( d \)-dimensional logarithmic price process with risk-neutral dynamics

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and stochastic covariation process \( X = \langle Y, Y \rangle \).

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- \( r \): constant interest rate,
- \( \mathbf{1} \): the vector whose entries are all equal to one,
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- In order to qualify for a covariation process, \( X \) must be specified as a process in \( S_d^+ \). Affine dynamics for \( X \) guarantee tractability of the model.
Prototype equation of an affine process in $S_d^+$

- The following affine dynamics for $X$ have been proposed in the literature:

$$dX_t = (b + HX_t + X_t H^\top)dt + \sqrt{X_t}dW_t \Sigma + \Sigma^\top dW_t^\top \sqrt{X_t} + dJ_t,$$

$X_0 = x \in S_d^+.$

- $b$: suitably chosen matrix in $S_d^+$,
- $H, \Sigma$: invertible matrices,
- $W$ a standard $d \times d$-matrix of Brownian motions possibly correlated with $B$,
- $J$ a pure jump process whose compensator is an affine function of $X$. 
Trajectory of a $2 \times 2$ positive semidefinite valued affine process
Term structure models based on affine processes with canonical state space

Here, $X$ is $\mathbb{R}_+^m \times \mathbb{R}^n$-valued, $N = m + n$.

- **Examples**: Vasiček [21], Cox, Ingersoll, Ross model [7], etc.
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- If the short rate $r_t$ is specified as an affine function of an affine process, that is
  \[ r_t = l + \lambda^\top X_t, \quad l \in \mathbb{R}, \lambda \in \mathbb{R}^N, \]
  
  then the zero coupon bond prices have exponential affine form
  \[ B_{t,T} = \mathbb{E} \left[ e^{-\int_t^T r_s ds} \bigg| X_t \right] = e^{G(t,T) + H(t,T)^\top X_t}. \]
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- The functions $G$ and $H$ solve a system of a generalized Riccati ODEs.
Term structure models based on $S_d^+$ valued affine processes

- Shortcomings of affine term structure models on $\mathbb{R}_+^m \times \mathbb{R}^n$: 

- For nonnegative short rates the state space has to be chosen to be $\mathbb{R}_+^m$. Due to admissibility conditions, this implies mutually independent positive factors. The introduction of correlated factors induces a positive probability of negative yields. The use of $S_d^+$-valued affine processes allows for nonnegative affine term structure models with stochastically correlated risk factors while preserving tractability. By specifying the short rate like before as $r_t = l + T\lambda^X_t$, $l \in \mathbb{R}_+^m$, $\lambda \in S_d^+$, where $X$ is now an affine process on $S_d^+$, the exponential affine form of the zero coupon prices is maintained.
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- The use of $S_d^+$-valued affine processes allows for nonnegative affine term structure models with stochastically correlated risk factors while preserving tractability. By specifying the short rate like before as

$$r_t = l + \text{Tr}(\lambda X_t), \quad l \in \mathbb{R}_+, \; \lambda \in S_d^+,$$

where $X$ is now an affine process on $S_d^+$, the exponential affine form of the zero coupon prices is maintained.
Related Literature

- **Theory of affine processes on** $\mathbb{R}_+^m \times \mathbb{R}^n$:
  - Duffie, Filipović and Schachermayer [13]: Characterization of affine processes $\mathbb{R}_+^m \times \mathbb{R}^n$.
  - Keller-Ressel, Schachermayer and Teichmann [19]: Regularity.
  - etc.

- Theory of affine processes on $\mathbb{S}_d^+$: 
  - Bru [5]: Existence and uniqueness (in law) of Wishart processes of type $dX_t = (\delta I_d)dt + \sqrt{X_t}dW_t + dW_t^\top \sqrt{X_t}$, $X_0 \in \mathbb{S}_d^+$, for $\delta > d - 1$.
  - Barndorff-Nielsen and Stelzer [3]: Matrix-valued Lévy driven Ornstein-Uhlenbeck processes.
Related Literature

- **Theory of affine processes on $\mathbb{R}^m_+ \times \mathbb{R}^n$:**
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  - Barndorff-Nielsen and Stelzer [3]: Matrix-valued Lévy driven Ornstein-Uhlenbeck processes.
Related Literature

- Affine processes on $S^+_d$ - Applications in mathematical finance:
  - Buraschi et al. [6],
  - Da Fonseca et al. [9, 10, 11, 12],
  - Gourieroux and Sufana [15, 16],
  - Leippold and Trojani [20],
  - etc.
Related Literature

- **Affine processes on $S_d^+$ - Applications in mathematical finance:**
  - Buraschi et al. [6],
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  - Gourieroux and Sufana [15, 16],
  - Leippold and Trojani [20],
  - etc.

- **Numerics and simulation of affine processes on $S_d^+$:**
  - Ahdida and Alfonsi [1],
  - Gauthier and Possamai [14],
  - etc.
Feller property, regularity and related ODEs

**Theorem**

Let $X$ be an affine process with state space $S^+_d$. Then, $X$ is a **Feller process** and it is **regular**, that is the derivatives

$$F(u) = \partial_t \phi(t, u)|_{t=0+}, \quad R(u) = \partial_t \psi(t, u)|_{t=0+}$$

exist and are continuous at $u = 0$. 
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exist and are continuous at $u = 0$.

- By the regularity of $X$, it follows that the function $\phi$ and $\psi$ are solutions of ODEs:

  $$\frac{\partial \phi(t, u)}{\partial t} = F(\psi(t, u)), \quad \phi(0, u) = 0, \quad (1)$$

  $$\frac{\partial \psi(t, u)}{\partial t} = R(\psi(t, u)), \quad \psi(0, u) = u \in S_d^+, \quad (2)$$

  which we call generalized Riccati equations due to the particular form of $F$ and $R$. 
Infinitesimal generator

**Theorem**

If $X$ is an affine process on $S_d^+$, then its infinitesimal generator is affine:

$$Af(x) = 2\left\langle \left( \frac{\partial}{\partial x} \right) \alpha \left( \frac{\partial}{\partial x} \right), x \right\rangle f|_x + \langle b + B(x), \nabla f(x) \rangle$$

$$- (c + \langle \gamma, x \rangle) f(x) + \int_{S_d^+ \setminus \{0\}} (f(x + \xi) - f(x)) \, m(d\xi)$$

$$+ \int_{S_d^+ \setminus \{0\}} (f(x + \xi) - f(x) - \langle \chi(\xi), \nabla f(x) \rangle) \frac{\langle x, \mu(d\xi) \rangle}{\|\xi\|^2 \wedge 1},$$

for some truncation function $\chi$ and admissible parameters

$$\left( \alpha, b, B(x) = \sum_{i,j} \beta^i_j x^i_j, c, \gamma, m(d\xi), M(x, d\xi) = \frac{\langle x, \mu \rangle}{\|\xi\|^2 \wedge 1} \right).$$
The functions $F$ and $R$ and existence of affine processes

**Theorem**

Moreover, $\phi(t, u)$ and $\psi(t, u)$ solve the differential equations (1) and (2), where $F$ and $R$ have the following form

$$F(u) = \langle b, u \rangle + c - \int_{S_d^+ \setminus \{0\}} (e^{-\langle u, \xi \rangle} - 1) m(d\xi),$$

$$R(u) = -2u\alpha u + B^\top(u) + \gamma - \int_{S_d^+ \setminus \{0\}} \left( e^{-\langle u, \xi \rangle} - 1 + \langle \chi(\xi), u \rangle \right) \frac{\mu(d\xi)}{\|\xi\|^2 \wedge 1}.$$ 

Conversely, let $(\alpha, b, \beta^{ij}, c, \gamma, m, \mu)$ be an admissible parameter set. Then there exists a unique affine process on $S_d^+$ with infinitesimal generator (3).
Relation to semimartingales

**Corollary**

Let $X$ be a conservative affine process on $S_d^+$. Then $X$ is a semimartingale. Furthermore, there exists, possibly on an enlargement of the probability space, a $d \times d$-matrix of standard Brownian motions $W$ such that $X$ admits the following representation

$$
X_t = x + \int_0^t \left( b + \int_{S_d^+ \setminus \{0\}} \chi(\xi) m(d\xi) + B(X_s) \right) ds,
$$

$$
+ \int_0^t \left( \sqrt{X_s} dW_s \Sigma + \Sigma^T dW_s \sqrt{X_s} \right)
$$

$$
+ \int_0^t \int_{S_d^+ \setminus \{0\}} \chi(\xi) \left( \mu^X(ds, d\xi) - (m(d\xi) + M(X_s, d\xi)) ds \right)
$$

$$
+ \int_0^t \int_{S_d^+ \setminus \{0\}} (\xi - \chi(\xi)) \mu^X(ds, d\xi),
$$

where $\Sigma$ is a $d \times d$ matrix satisfying $\Sigma^T \Sigma = \alpha$ and $\mu^X$ denotes the random measure associated with the jumps of $X$. 
Admissible parameters

- linear diffusion coefficient: \( \alpha \in S^+_d \),
Admissible parameters

- **linear diffusion coefficient**: $\alpha \in S^+_d$,
- **linear jump coefficient**: $d \times d$-matrix $\mu = (\mu_{ij})$ of finite signed measures on $S^+_d \setminus \{0\}$ with
  - $\mu(E) \in S^+_d$ for all $E \in \mathcal{B}(S^+_d \setminus \{0\})$,
  - $M(x, d\xi) := \langle x, \mu(d\xi) \rangle / \|\xi\|^2 \wedge 1$, 

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- **linear killing rate coefficient**: \( \gamma \in \mathbb{S}_d^+ \).
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- **linear killing rate** coefficient: \( \gamma \in S^+_d \),
- **constant drift** term: \( b - (d - 1)\alpha \in S^+_d \),
- **constant jump** term: Borel measure \( m \) on \( S^+_d \setminus \{0\} \),
- **constant killing rate** term: \( c \in \mathbb{R}^+ \).
Remark on the admissible parameters

- No constant diffusion part, linear part is of very specific form

\[ \langle v, A(x)v \rangle = 4\langle x, v\alpha v \rangle \text{ for all } v \in S_d^+. \]

This is a consequence of the fact that there is no diffusion in directions orthogonal to the boundary, i.e. \( \langle u, A(x)u \rangle = 0 \) for \( u \in S_d^+ \) with \( \langle u, x \rangle = 0. \)
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- Jumps described by \( m \) are of finite variation, for the linear jump part we have finite variation for the directions orthogonal to the boundary while parallel to the boundary general jump behavior is allowed. Thus,
  \[ \int_{S^+_d \setminus \{0\}} \langle \chi(\xi), u \rangle M(x, d\xi) < \infty, \quad u \in S^+_d \text{ with } \langle u, x \rangle = 0. \]
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- The linear drift part has to be inward pointing, that is
  \[ \langle B(x), u \rangle - \int_{S_d^+ \setminus \{0\}} \langle \chi(\xi), u \rangle M(x, d\xi) \geq 0 \quad u \in S_d^+ \text{ with } \langle u, x \rangle = 0. \]
Remark on the admissible parameters

Very remarkable admissibility condition between the constant drift $b$ and the linear diffusion coefficient $\alpha$ due to

$$\langle b, \nabla \det(x) \rangle + 2 \left\langle \left( \frac{\partial}{\partial x} \right) \alpha \left( \frac{\partial}{\partial x} \right), x \right\rangle \det |x| \geq 0$$

for all $x \in \partial S^+_d$. 
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$$\Rightarrow b - (d - 1)\alpha \in S^+_d$$
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$$\Rightarrow b - (d - 1)\alpha \in S^+_d$$

For $d \geq 2$ the boundary of $S^+_d$ is curved and implies this relation between linear diffusion coefficient $\alpha$ and drift part $b$. 
What are the new results

- **Full characterization** and exact assumptions under which affine processes on $S_d^+$ actually exist.
What are the new results

- **Full characterization** and exact assumptions under which affine processes on $S_d^+$ actually exist.
  - Necessity and sufficiency of the **drift condition**
    \[ b - (d - 1)\alpha \in S_d^+. \]
What are the new results

- **Full characterization** and exact assumptions under which affine processes on $S_d^+$ actually exist.
  - Necessity and sufficiency of the drift condition $b - (d - 1)\alpha \in S_d^+$.
- Extension of the model class.
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- **Extension of the model class.**
  - **General linear drift** part $B(x) = \sum_{ij} \beta_{ij} x_{ij}$. This allows dependency of the volatility of one asset on the other ones which is not possible for $B(x) = Hx + xH^\top$. Example: $d = 2$ and
    $$B(x) = \begin{pmatrix} x_{22} & x_{12} \\ x_{12} & x_{11} \end{pmatrix}.$$
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- **Full generality of jumps** (quadratic variation jumps parallel to the boundary).
Multivariate affine stochastic volatility models - analytic tractability

- Option pricing and model calibration can be reduced to the solutions of the generalized Riccati equations for $\phi$ and $\psi$. 
Introduction
Applications of $S_d^+$-valued affine processes in finance
Characterization of affine processes on $S_d^+$
Implications for financial modeling

Multivariate affine stochastic volatility models - analytic tractability

- Option pricing and model calibration can be reduced to the solutions of the generalized Riccati equations for $\phi$ and $\psi$.
- Consider a multivariate stochastic volatility model:

$$d Y_t = \left( r \mathbf{1} - \frac{1}{2} X_t^{\text{diag}} \right) dt + \sqrt{X_t} dB_t,$$

$$d X_t = (b + B(X_t)) dt + \sqrt{X_t} dW_t \Sigma + \Sigma^T dW_t^T \sqrt{X_t} + d J_t.$$
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\end{align*}
$$

- Then, the moment generating function of the process $(X, Y)$ has the following form:

$$
\mathbb{E}_{x, y} \left[ e^{-\text{Tr}(uX_t) + v^T Y_t} \right] = e^{\Phi(t, u, v) + \text{Tr}(\Psi(t, u, v)x) + v^T y}
$$

for appropriate arguments $u \in S_d \times iS_d$ and $v \in \mathbb{C}^d$. The functions $\Phi$ and $\Psi$ solve a system of generalized Riccati ODEs.
Option pricing

- Computation of the price $\pi_0$ of a European claim with payoff function $f(Y_T)$

$$\pi_0 = e^{-rT} \mathbb{E}_{x,y}[f(Y_T)]$$

via Fourier methods.
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- Assume $f(y) = \int_{\mathbb{R}^d} e^{(c+i\lambda)^\top y} \tilde{f}(\lambda) d\lambda$ for some integrable function $\tilde{f}$ and some constant $c \in \mathbb{R}^d$.
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- **Efficient valuation of European options** since
  \[ \pi_0 = e^{-rT} E_{x,y} \left[ \left( \int_{\mathbb{R}^d} e^{(c+i\lambda)^T Y_T} \tilde{f}(\lambda) d\lambda \right) \right] \]
  \[ = \int_{\mathbb{R}} e^{\Phi(t,0,c+i\lambda)+\text{Tr}(\Psi(t,0,c+i\lambda)x)+(c+i\lambda)^T y} \tilde{f}(\lambda) d\lambda. \]
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- Example Spread option: $c_2 < 0, c_1 + c_2 > 1$
  \[ (e^{y_1} - e^{y_2} - 1)^+ = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{(c+i\lambda)^\top y} \frac{\Gamma(c_1 + c_2 - 1 + i(\lambda_1 + \lambda_2))\Gamma(-c_2 - i\lambda_2)}{\Gamma(c_1 + 1 + i\lambda_1)} d\lambda_1 d\lambda_2. \]
  This representation is due to Hurd and Zhou [18].
Discounting - affine transform formula

- $X$ affine process on $S_d^+$. 

Bond prices $B_t$, $T$ can be obtained by setting $u = 0$ in the above formula. Bond option prices (e.g. caps) are computed efficiently by Fourier pricing methods.
Introduction
Applications of $S_d^+$-valued affine processes in finance
Characterization of affine processes on $S_d^+$
Implications for financial modeling

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- Under some technical conditions, we have

$$\mathbb{E}_x \left[ e^{-\int_0^t r_s ds} e^{-\langle u, X_t \rangle} \right] = e^{-\tilde{\phi}(t, u) - \langle \tilde{\psi}(t, u), x \rangle},$$

where $\tilde{\phi}$ and $\tilde{\psi}$ satisfy the extended generalized Riccati equations

$$\partial_t \tilde{\phi} = \tilde{F}(\tilde{\psi}) = F(\tilde{\psi}) + l, \quad \tilde{\phi}(0, u) = 0,$$
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Conclusion and Outlook

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**Conclusion**
- Full characterization and exact assumptions under which affine processes on $S_d^+$ actually exist.
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**Outlook**
- Affine processes on other state spaces (symmetric cones).
- Calibration of affine term structure models and multivariate stochastic volatility models.
Introduction
Applications of $S^+_d$-valued affine processes in finance
Characterization of affine processes on $S^+_d$
Implications for financial modeling

Exact and high order discretization schemes for wishart processes and their affine extension.

Modeling by Lévy processes for financial econometrics.

Positive-definite matrix processes of finite variation.

Post-'87 crash fears in the S&P 500 futures option market.

Wishart processes.

Correlation risk and optimal portfolio choice.

A theory of the term structure of interest rates.
Applications of $S_+^d$-valued affine processes in finance
Characterization of affine processes on $S_+^d$
Implications for financial modeling

Affine processes on positive semidefinite matrices.

Hedging (co)variance risk with variance swaps.

Estimating the Wishart affine stochastic correlation model using the empirical characteristic function.

Option pricing when correlations are stochastic: an analytical framework.

A multifactor volatility Heston model.

Affine processes and applications in finance.

Efficient simulation of the Wishart model.
Introduction
Applications of $S_d^+$-valued affine processes in finance
Characterization of affine processes on $S_d^+$
Implications for financial modeling

Wishart quadratic term structure models.

Derivative pricing with Wishart multivariate stochastic volatility: application to credit risk.

A closed-form solution for options with stochastic volatility with applications to bond and currency options.

A Fourier transform method for spread option pricing.

Affine processes are regular.

Asset pricing with matrix jump diffusions.

An equilibrium characterization of the term structure.
Thank you for your attention!