Optimal Stock Selling Based on the Global Maximum

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(joint work with Dr. M. Dai and Z. Yang)
If I were an Innocent Investor...

- I just bought a stock and must sell it in one year
- Need to decide when to sell?
- Obviously, sell it at the maximum price of the whole year. However, this is an impossible mission.
- So, what about selling at the price ”closest” to the maximum?
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- So, what about selling at the price ”closest” to the maximum?
- This talk is using square error to measure ”closeness” and studying the optimal selling strategy under this criterion.
The Model

- A Black-Scholes market with one stock and one saving account
- The *discounted* stock price follows, on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\),
  \[dS_t = \mu S_t dt + \sigma S_t dW_t,\]
  where \(\mu \in (-\infty, \infty)\) and \(\sigma > 0\) are constants
- Let \(M_s = \max_{0 \leq t \leq s} S_t, 0 \leq s \leq T\) be the running maximum of stock price
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- Let \(M_s = \max_{0 \leq t \leq s} S_t, 0 \leq s \leq T\) be the running maximum of stock price
- Consider the following optimal stopping problem

\[
\inf_{0 \leq \nu \leq T} \mathbb{E}[(S_\nu - M_T)^2],
\]

where \(\mathbb{E}\) stands for the expectation, \(\nu\) is an \(\mathcal{F}_t\)-stopping time.
Graversen, Peskir and Shiryaev (2000), Theory Prob Appl, studied
\[
\inf_{0 \leq \nu \leq T} \mathbb{E}[(S^0_\nu - M^0_T)^2],
\]
where \( S^0_t = W_t \), \( M^0_T = \max_{0 \leq t \leq T} W_t \) and obtained explicit optimal solution
\[
\nu^* = \inf\{t : M^0_t - S^0_t \geq z^* \sqrt{T - t}\}, z^* = 1.12\ldots
\]

du Toit and Peskir (2007), Ann Prob, considered
\[
\inf_{0 \leq \nu \leq T} \mathbb{E}[(S^\mu_\nu - M^0_T)^2],
\]
where \( \mu \neq 0 \).
Related (Probabilistic) Literature

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where \( \mu \neq 0. \)
Related (Financial) Literature

- Shiryaev, Xu and Zhou (2008), Quant Fin, studied the relative error between the selling price and global maximum,

\[ \inf_{0 \leq \nu \leq T} \mathbb{E} \left[ \frac{M_T - S_\nu}{M_T} \right] \]

- "Bang-bang" strategy:
  - Sell at time \( T \): \( \mu > \frac{\sigma^2}{2} \)
  - Sell at time \( 0 \): \( \mu \leq \frac{\sigma^2}{2} \)
The problem is
\[
\inf_{0 \leq \nu \leq T} \mathbb{E}[(S_\nu - M_T)^2]
\]
Not a standard optimal stopping problem, since \(M_T\) is not \(\mathcal{F}_t\)-adapted
One more step:
\[
\inf_{0 \leq \nu \leq T} \mathbb{E}[(S_\nu - M_T)^2] = \inf_{0 \leq \nu \leq T} \mathbb{E}\left\{ \mathbb{E}[(S_\nu - M_T)^2 \mid \mathcal{F}_\nu] \right\} \\
= \inf_{0 \leq \nu \leq T} \mathbb{E}\left\{ \phi(\nu, S_\nu, M_\nu) \right\},
\]
where \(\phi(t, S_t, M_t) = \mathbb{E}[(S_t - M_T)^2 \mid \mathcal{F}_t]\)
PDE Formulation (Con’t)

- Denote the value function

\[ \psi(t, S_t, M_t) = \inf_{t \leq \nu \leq T} \mathbb{E} \left\{ \phi(\nu, S_{\nu}, M_{\nu}) \mid \mathcal{F}_t \right\} \]

- Dynamic programming equation (Variational Inequalities)

\[
\begin{align*}
\max \{-\partial_t \psi - \mathcal{L}^0 \psi, \psi - \phi\} &= 0, \quad (t, S, M) \in D, \\
\partial_M \psi(t, M, M) &= 0, \quad \psi(T, S, M) = (S - M)^2,
\end{align*}
\]

where \( \mathcal{L}^0 = \frac{\sigma^2}{2} \partial_{SS} + \mu \partial_S \) and

\[ D = \{(t, S, M) : 0 < S < M, 0 \leq t < T \} \].
The Obstacle Function $\phi(t, S, M)$

- Recall

\[
\phi(t, S_t, M_t) = \mathbb{E}[(S_t - M_T)^2 | \mathcal{F}_t] \\
= S_t^2 - 2S_t \mathbb{E}[M_T | \mathcal{F}_t] + \mathbb{E}[M_T^2 | \mathcal{F}_t] \\
=: S_t^2 - 2S_t \phi_1(t, S_t, M_t) + \phi_2(t, S_t, M_t),
\]

where $\phi_i(t, S_t, M_t) = \mathbb{E}[M_T^i | \mathcal{F}_t]$.

- Then, $\phi_i(t, S, M)$ satisfies

\[
\begin{cases}
-\partial_t \phi_i - \mathcal{L}^0 \phi_i = 0, & (t, S, M) \in D, \\
\partial_M \phi_i(t, M, M) = 0, & \phi_i(T, S, M) = M^i.
\end{cases}
\]
Change of Variables

- Denote $\tau = T - t$, $x = \ln \frac{M}{S}$, $u_i(\tau, x) = \frac{\phi_i(t,S,M)}{S^i}$, $u(\tau, x) = \frac{\phi(t,S,M)}{S^2}$.

- Then, $u_1$ and $u_2$ satisfy

\[
\begin{align*}
\partial_\tau u_1 - L^1_x u_1 &= 0 \quad \text{in } \Omega, \\
\partial_x u_1(\tau, 0) &= 0, \quad u_1(0, x) = e^x,
\end{align*}
\]

\[
\begin{align*}
\partial_\tau u_2 - L^2_x u_2 &= 0 \quad \text{in } \Omega, \\
\partial_x u_2(\tau, 0) &= 0, \quad u_2(0, x) = e^{2x},
\end{align*}
\]

where $L^1_x = \frac{\sigma^2}{2} \partial_{xx} - \left( \mu + \frac{\sigma^2}{2} \right) \partial_x + \mu$,

$L^2_x = \frac{\sigma^2}{2} \partial_{xx} - \left( \mu + \frac{3\sigma^2}{2} \right) \partial_x + (2\mu + \sigma^2)$,

$\Omega = (0, T] \times (0, \infty)$. 

Change of Variables (con’t)

- Denote \( \nu(\tau, x) = \frac{\psi(t,S,M)-\phi(t,S,M)}{S^2} \)

- Then, \( \nu \) satisfies

\[
\begin{align*}
\max \left\{ \partial_\tau \nu - \mathcal{L}_x^2 \nu - H, \nu \right\} &= 0 \quad \text{in } \Omega, \\
\partial_x \nu(\tau, 0) &= 0, \quad \nu(0, x) = 0,
\end{align*}
\]

where \( H = \mathcal{L}_x^2 u - \partial_\tau u = 2\mu + \sigma^2 + 2\left(\sigma^2 \partial_x u_1 - (\mu + \sigma^2)u_1\right), \)

\( \mathcal{L}_x^2 = \frac{\sigma^2}{2} \partial_{xx} - \left(\mu + \frac{3\sigma^2}{2}\right) \partial_x + (2\mu + \sigma^2). \)

- Define the selling region (the stopping region) as follows:

\[
SR = \{ (\tau, x) \in [0, \infty) \times (0, T) : \nu(\tau, x) = 0 \}.
\]
The Optimal Selling Strategy: Good Stock ($\mu > 0$)

Figure: Two optimal selling boundaries. Parameter values used:
$\mu = 0.045$, $\sigma = 0.3$, $T = 1$. 
The Optimal Selling Strategy: Bad Stock \((-\sigma^2 \leq \mu \leq 0)\)

Figure: The monotonically increasing optimal selling boundary. Parameter values used: \(\mu = -0.010, \sigma = 0.3, T = 1\).
The Optimal Selling Strategy: Very Bad Stock ($\mu < -\sigma^2$)

Figure: The nonmonotone optimal selling boundary. Parameter values used: $\mu = -0.032$, $\sigma = 0.4$, $T = 3$. 
The Proof

Recall:
\[
\begin{align*}
\max \left\{ \partial_\tau v - L^2_x v - H, v \right\} &= 0 \text{ in } \Omega, \\
\partial_x v(\tau, 0) &= 0, \quad v(0, x) = 0,
\end{align*}
\]

So, 
\[
SR = \left\{ (\tau, x) : v = 0 \right\}
\subseteq \left\{ (\tau, x) : \partial_\tau 0 - L^2_x 0 - H \leq 0 \right\}
= \left\{ (\tau, x) : H \geq 0 \right\}
\]
The Set \( \{(\tau, x) : H \geq 0\} \)

**Lemma:** Recall \( H(\tau, x) = 2\mu + \sigma^2 + 2\left(\sigma^2 \partial_x u_1 - (\mu + \sigma^2)u_1\right) \).

- If \( \mu \leq 0 \), \( \partial_x H > 0 \);
- If \( \mu \geq -\sigma^2 \), \( \partial_\tau H < 0 \);
- If \( \mu > 0 \), \( \partial_x H(\tau, x) = 0 \) has at most one solution for any given \( \tau > 0 \);
The Main Results: $\mu \leq 0$

With the help of previous lemma, we have

- $\partial_x v \geq 0$ if $\mu \leq 0$;
- $\partial_\tau v \leq 0$ if $\mu \geq -\sigma^2$;
- These are due to

$$\partial_\tau v - \mathcal{L}_x^2 v = H, \text{ in } \{ (\tau, x) : v < 0 \}.$$
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With the help of previous lemma, we have

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- $\partial_{\tau} \nu \leq 0$ if $\mu \geq -\sigma^2$;
- These are due to

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- Define $x_s^*(\tau) = \inf\{x \in (0, +\infty) : \nu(\tau, x) = 0, \forall \tau \in (0, T]\}$. 

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- Define $x_s^*(\tau) = \inf\{x \in (0, +\infty) : v(\tau, x) = 0, \forall \tau \in (0, T]\}$.
- Thanks to $\partial_x v \geq 0$, we can show

$$SR = \{(\tau, x) : v(\tau, x) = 0\}$$

$$= \{(\tau, x) : x \geq x_s^*(\tau), 0 < \tau \leq T\}.$$
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With the help of previous lemma, we have

- $\partial_x v \geq 0$ if $\mu \leq 0$;
- $\partial_\tau v \leq 0$ if $\mu \geq -\sigma^2$;
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  \[ \partial_\tau v - \mathcal{L}_x^2 v = H, \text{ in } \{(\tau, x) : v < 0\}. \]
- Define $x_s^*(\tau) = \inf\{x \in (0, +\infty) : v(\tau, x) = 0, \forall \tau \in (0, T]\}$.
- Thanks to $\partial_x v \geq 0$, we can show
  \[ SR = \{(\tau, x) : v(\tau, x) = 0\} = \{(\tau, x) : x \geq x_s^*(\tau), 0 < \tau \leq T\}. \]
- $\partial_\tau v \leq 0$ gives the monotonicity of the free boundary.
The Main Results: \( \mu > 0 \)

- With \( \mu > 0 \), we have \( \partial_{\tau} \nu \leq 0 \), which implies that \( (\tau_2, x) \in SR \) if \( (\tau_1, x) \in SR \) and \( \tau_2 < \tau_1 \).
The Main Results: $\mu > 0$

- With $\mu > 0$, we have $\partial_\tau v \leq 0$, which implies that $(\tau_2, x) \in SR$, if $(\tau_1, x) \in SR$ and $\tau_2 < \tau_1$. 

![Diagram showing the relationship between $\tau$, $x$, and $v$ for different regions of $H$.]
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![Diagram showing the regions for $v > 0$ and $v < 0$ with the boundaries and points indicating $H > 0$ and $H < 0$.]
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![Graph showing the regions where $H < 0$ and $H > 0$.]
The Main Results: $\mu > 0$

- The sell region $SR$ is connected;
- We can define
  
  \[
  x_{1s}^*(\tau) = \inf \{ x \in [0, +\infty) : v(\tau, x) = 0 \} \\
  x_{2s}^*(\tau) = \sup \{ x \in [0, +\infty) : v(\tau, x) = 0 \}
  \]

- It is easy to show
  
  \[
  SR = \{ (\tau, x) : x_{1s}^*(\tau) \leq x \leq x_{2s}^*(\tau), 0 < \tau \leq \tau^* \}.
  \]

- The monotonicity of $x_{is}^*(\tau)$ follows by $\partial_\tau v \leq 0$. 
Smoothness of the Free Boundary

- For $\mu \geq -\sigma^2$, we have $\partial_\tau v \leq 0$. So, one can easily establish the smoothness of $x^*_s(\tau)$ following Friedman (1975).
  - First, show $x^*_s(\tau) \in C^{3/4}((0, T])$
  - Then, show $x^*_s(\tau) \in C^1((0, T])$
  - By a bootstrap argument, show $x^*_s(\tau) \in C^\infty((0, T])$
Smoothness of the Free Boundary

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- For $\mu < -\sigma^2$, we change of variables. Let $y = x - \mu/\sigma^2\tau$, and $V(\tau, y) = \nu(\tau, x)$.
  - Show $\partial_\tau V(\tau, y) \leq 0$ and $\partial_y V(\tau, y) \geq 0$
  - Apply Friedman (1975) to show smoothness of the corresponding $y_s^*(\tau)$, which gives the desired result
Conclusion

- We examine the optimal decision to sell a stock with the criteria of minimizing the square error between the selling price and the global maximum.
- For good stock, i.e. $\mu > 0$, the optimal selling boundary has two branches and only exists when time to maturity is not long enough.
- For bad stock, i.e. $\mu \leq 0$, the optimal selling boundary only has one branch and always exists.
- The smoothness of the free boundary is also established.
Thank you !