Convex duality in constrained mean-variance portfolio optimization under a regime-switching model

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Motivation

Investor

- Wishes to have \$100 in 1 year's time.
- Starts with \$90.
- Invests money in stockmarket and bank account.
- No short-selling.
- How to invest to minimize:

$$\mathbb{E}\left(\{\text{Investor's wealth in 1 year}\} - 100\right)^2$$

subject to satisfying the investment restrictions and

$$\mathbb{E}\left(\{\text{Investor's wealth in 1 year}\}\right) = 100.$$

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Outline

Market model

- $(\Omega, \mathcal{F}, \mathbb{P})$ and finite time horizon $[0, T]$.
- Market consists of N traded assets and a risk-free asset.
- Risk-free asset price process obeys

$$\frac{\mathrm{d}S_0(t)}{S_0(t)} = r(t)\,\mathrm{d}t.$$

- Price processes of stocks obey

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu(t)\,\mathrm{d}t + \sigma^\top(t)\,\mathrm{d}W(t).$$

Market model

Market subject to regime-switching modeled by Markov chain α.

- Finite-state-space $I = \{1, \dots, D\}$.
- Generator matrix $G = (g_{ij})$.
- Jump processes for $i \neq j$

$$N_{ij}(t) = \sum_{0<s\leq t} \mathbf{1}[\alpha(s_-) = i]\mathbf{1}[\alpha(s) = j]$$

- Martingales for $i \neq j$

$$M_{ij}(t) = N_{ij}(t) - \int_0^t g_{ij}\mathbf{1}[\alpha(s_-) = i]\,\mathrm{d}s.$$

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The investor's wealth process

- Portfolio process $\pi(t) = \{\pi_1(t), \ldots, \pi_N(t)\}$ at time t.
- Wealth process $X^\pi(t)$ at time t, given by

$$X^\pi(t) = \pi_0(t) + \sum_{n=1}^{N} \pi_n(t).$$

- Wealth equation: $X^\pi(0) = x_0$, a.s. and

$$\mathrm{d}X^\pi(t) = \left(r(t)X^\pi(t) + \pi^\top(t)\sigma(t)\theta(t)\right)\mathrm{d}t + \sigma^\top(t)\pi(t)\,\mathrm{d}W(t),$$

where the market price of risk is

$$\theta(t) := \sigma^{-1}(t)\left(\mu(t) - r(t)\mathbf{1}\right).$$

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The investor's portfolio constraints

- $K \subset \mathbb{R}^N$ closed, convex set with $0 \in K$.

- Example: no short-selling

$$K := \{\pi = (\pi_1, \ldots, \pi_N) \in \mathbb{R}^N : \pi_1 \geq 0, \ldots, \pi_N \geq 0\}.$$

- Set of admissible portfolios

$$\mathcal{A} := \{\pi \in L^2(W) \,|\, \pi(t) \in K, \text{ a.e.}\}.$$

The investor's risk measure

- Risk measure J

$$J(x) = \frac{1}{2}Ax^2 + Bx + C, \quad \forall x \in \mathbb{R},$$

where A, B and C are random variables.

- Example: $J(x) = (x - 100)^2$.

The investor's problem

- Does there exist $\bar{\pi} \in \mathcal{A}$ such that

$$\mathbb{E}(J(X^{\bar{\pi}}(T))) = \inf_{\pi \in \mathcal{A}} \mathbb{E}(J(X^{\pi}(T)))?$$

Can we characterize $\bar{\pi}$? Can we find $\bar{\pi}$?

- Example:

$$\mathcal{A} := \{\pi \in L^2(W) \,|\, \pi(t) \geq 0 \text{ a.e.}\}$$

and

$$\mathbb{E}(X^{\bar{\pi}}(T) - 100)^2 = \inf_{\pi \in \mathcal{A}} \mathbb{E}(X^{\pi}(T) - 100)^2$$

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Convex duality approach

- Restate MVO problem as primal problem.
- Construct dual problem.
- Necessary and sufficient conditions.
- Existence of a solution to the dual problem.
- Construct candidate primal solution.
- Verification.

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Restate MVO problem as primal problem

- Risk measure is minimized over portfolio processes.

$$\inf_{\pi \in \mathcal{A}} \mathbb{E}(J(X^{\pi}(T)))$$

- Move this minimization to one over a space of processes.

$$\inf_{X \in \mathbb{A}} \mathbb{E}(\Phi(X))$$

- Key is the wealth equation.
- Wealth processes X^{π} embedded in space $\mathbb{A}$.

Restate MVO problem as primal problem

- Space $\mathbb{A}$ consists of square-integrable, continuous processes.
- If $X \in \mathbb{A}$ then a.s. $X(0) = X_0$ and

$$\mathrm{d}X(t) = \Upsilon^X(t)\,\mathrm{d}t + \left(\Lambda^X\right)^\top(t)\,\mathrm{d}W(t).$$

Restate MVO problem as primal problem

- Encode constraints as penalty functions l_0, l_1.
- The initial wealth requirement $X(0) = x_0$ motivates

$$l_0(x) := \begin{cases} 0 & \text{if } x = x_0 \\ \infty & \text{otherwise,} \end{cases}$$

for all $x \in \mathbb{R}$.
- The wealth equation and portfolio constraints motivate:

$$l_1(\omega, t, x, \nu, \lambda) := \begin{cases} 0 & \text{if } \nu = r(\omega, t)x + \lambda^\top \theta(\omega, t) \\ & \text{and } (\sigma^\top(\omega, t))^{-1}\lambda \in K \\ \infty & \text{otherwise,} \end{cases}$$

for all $(\omega, t, x, \nu, \lambda) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$.

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Restate MVO problem as primal problem

- Primal cost functional $\Phi : \mathbb{A} \to \mathbb{R} \cup \{\infty\}$,

$$\Phi(X) := l_0(X_0) + \mathbb{E} \int_0^T l_1(t, X(t), \Upsilon^X(t), \Lambda^X(t))\, \mathrm{d}t + \mathbb{E}(J(X(T))).$$

- Primal problem: find $\bar{X} \in \mathbb{A}$ such that

$$\Phi(\bar{X}) = \inf_{X \in \mathbb{A}} \Phi(X).$$

- Use wealth equation to recover $\bar{\pi}$.

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Construct dual problem

- Look for solutions to dual problem in the space $\mathbb{B}$.
- Space $\mathbb{B}$ consists of square-integrable, right-continuous processes.
- If $Y \in \mathbb{B}$ then a.s. $Y(0) = Y_0$ and

$$\mathrm{d}Y(t) = \Upsilon^Y(t)\,\mathrm{d}t + \left(\Lambda^Y\right)^\top(t)\,\mathrm{d}W(t) + \sum_{i \neq j} \Gamma_{ij}^Y(t)\,\mathrm{d}M_{ij}(t).$$

Construct dual problem

- Take convex conjugates of l_0, l_1 and J.

$$m_0(y) := \sup_{x\in\mathbb{R}}\{xy - l_0(x)\}, \qquad \forall y \in \mathbb{R}.$$

- Dual cost functional $\Psi : \mathbb{B} \to \mathbb{R} \cup \{\infty\}$,

$$\Psi(Y) := m_0(Y_0) + \mathbb{E}\int_0^T m_1(t, Y(t), \Upsilon^Y(t), \Lambda^Y(t))\,\mathrm{d}t + \mathbb{E}(m_J(-Y(T))).$$

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Necessary and sufficient conditions

For $(\bar{X}, \bar{Y}) \in \mathbb{A} \times \mathbb{B}$,

$\bar{X} \in \mathbb{A}$ solves the primal problem and $\bar{Y} \in \mathbb{B}$ solves the dual problem

if and only if

$(\bar{X}, \bar{Y}) \in \mathbb{A} \times \mathbb{B}$ satisfy necessary and sufficient conditions, eg

$$\bar{X}(T) = -\frac{\bar{Y}(T) + B}{A}, \quad \text{a.s.},$$
$$\Upsilon^{\bar{Y}}(t) = -r(t)\bar{Y}(t), \quad \text{a.e.}$$

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Existence of a solution to the dual problem

- There exists $\bar{Y} \in \mathbb{B}$ such that

$$\Psi(\bar{Y}) = \inf_{Y \in \mathbb{B}} \Psi(Y).$$

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Construct candidate primal solution

- State price density process

$$H(t) = \exp\{-\int_0^t r(s)\,\mathrm{d}s\}\mathcal{E}(-\theta \bullet W)(t)$$

- $X^{\pi}(t)H(t) = \mathbb{E}\left(X^{\pi}(T)H(T)\,|\,\mathcal{F}_t\right)$
- From necessary and sufficient conditions,

$$\bar{X}(T) = -\frac{\bar{Y}(T)+B}{A}.$$

- Candidate primal solution $\tilde{X} \in \mathbb{B}$

$$\tilde{X}(t) := -\frac{1}{H(t)}\mathbb{E}\left(\left(\frac{\bar{Y}(T)+B}{A}\right)H(T)\,\middle|\,\mathcal{F}_t\right).$$

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- From necessary and sufficient conditions,

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Examples

- Market parameters $\{\mathcal{F}_t^\alpha\}$-previsible.
- $J(x) = (x - d)^2$, some $d \in \mathbb{R}$.
- Use necessary and sufficient conditions.

No portfolio constraints

$K := \mathbb{R}^N$

$$\bar{\pi}(t) = -\left(X^{\bar{\pi}}(t) - d\frac{R(t)}{S(t)}\right)\left(\sigma^\top(t)\right)^{-1}\theta(t),$$

for

$$R(t) = \mathbb{E}\left[\exp\left\{\int_t^T \left(r(u) - \|\theta(u)\|^2\right)\,\mathrm{d}u\right\}\Big|\alpha(t)\right],$$

$$S(t) = \mathbb{E}\left[\exp\left\{\int_t^T \left(2r(u) - \|\theta(u)\|^2\right)\,\mathrm{d}u\right\}\Big|\alpha(t)\right].$$

Restricted investment portfolio constraints

Example: $K = \{\pi = (\pi_1, \ldots, \pi_N) \in \mathbb{R}^N : \pi_1 = 0, \ldots, \pi_M = 0\}$.

$$\bar{\pi}(t) = -\left(X^{\bar{\pi}}(t) - d\frac{R(t)}{S(t)} \right) \left(\sigma^\top(t) \right)^{-1} \xi(t),$$

for

$$\xi(t) = \theta(t) - \text{proj}\left[\theta(t) \middle| \sigma^{-1}(t)\tilde{K} \right],$$

$$R(t) = \mathbb{E}\left[\exp\left\{ \int_t^T \left(r(u) - \theta^\top(u)\xi(u) \right) \mathrm{d}u \right\} \middle| \alpha(t) \right],$$

$$S(t) = \mathbb{E}\left[\exp\left\{ \int_t^T \left(2r(u) - \theta^\top(u)\xi(u) \right) \mathrm{d}u \right\} \middle| \alpha(t) \right]$$

Convex conic portfolio constraints

K closed convex cone containing the origin.

Further assume: r deterministic and $x_0 \leq d \exp\{-\int_0^T r(u)\,\mathrm{d}u\}$.

$$\bar{\pi}(t) = -\left(X^{\bar{\pi}}(t) - d\exp\{-\int_t^T r(u)\,\mathrm{d}u\}\right)\left(\sigma^\top(t)\right)^{-1}\xi(t),$$

for

$$\xi(t) = \theta(t) - \text{proj}\left[\theta(t)\middle|\sigma^{-1}(t)\tilde{K}\right].$$

Summary

Showed existence and characterized the solution for MVO problem with

- general convex portfolio constraints; and
- random market coefficients

in a regime-switching model.

Solutions in feedback form.

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