Ambiguity Aversion in Real Options:

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The Real Option Problem

- Classical work of **McDonald & Siegel (86)** assigns the value

$$f_t = \mathbb{E}_t \left[e^{-\rho(T-t)} \ (\mathbf{P_T} - I)_+ \right]$$

to the **option to invest** in a project at T

  - $\mathbf{P_t}$ – value of a project if invested in at time t
  - I – the cost of the investment
  - ρ – discount rate

- If **early investment** is allowed (e.g. qrtly or mthly), then

$$f_t = \sup_{\tau \in \mathscr{T}} \mathbb{E}_t \left[e^{-\rho(\tau-t)} \ (\mathbf{P}_\tau - I)_+ \right]$$

  - $\mathscr{T}$ – a set of admissible stopping times

The Real Option Problem

- $\mathbf{P_t}$ often assumed **spanned** by a traded asset – mostly **unrealistic**
  - Spanning allows the project to effectively be traded and therefore valued using discounted expectations
- Instead view $\mathbf{P_t}$ as **strongly correlated** to a **tradable asset** $\mathbf{S_t}$
- Two key questions addressed here:
  - How to value the option on P_t by trading in S_t?
    - Will use **Utility indifference pricing**
    - Henderson & Hobson (02) and Henderson (07) for perpetual version
  - An agent may have a good model for S_t but not P_t...
    how to account for this **ambiguity**?
    - **Knightian Uncertainty / ambiguity aversion**
    - *Robustness Approach:* **Anderson, Hansen, & Sargent (99); Uppal & Wang (03); Maenhout (04); and J. & Sigloch (09)**
    - *Recursive multiple priors:* Epstein & Wang (94) Chen & Epstein (02) extension of Gilboa & Schmeidler (89)

Utility Indifference Pricing

- Consider:
  - Suppose want to value the risk Y received at T
  - Agent's utility is exponential $u(x) = -\frac{1}{\gamma} e^{-\gamma x}$
  - Agent's initial wealth is x and risk-free rate is r
- Basic utility indifference valuation:
  1. Invest all of x in bank account:

     $$V(x) = -\frac{1}{\gamma} e^{-\gamma\, x\, e^{rT}}$$

  2. Invest $x - v$ in bank account and receive Y at T:

     $$U(x) = \mathbb{E}[u((x-v)e^{rT} + \mathbf{Y})] = V(x-v)\mathbb{E}[e^{-\gamma \mathbf{Y}}]$$

  3. Indifference value v solves

     $$V(x) = U(x) \quad \Rightarrow \quad v = -\frac{1}{\gamma} e^{-rT} \ln \mathbb{E}[e^{-\gamma \mathbf{Y}}]$$

Utility Indifference Pricing

- Invest optimally in S_t **without option** to invest in project

$$U(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[u(\mathbf{X_T})\right]$$

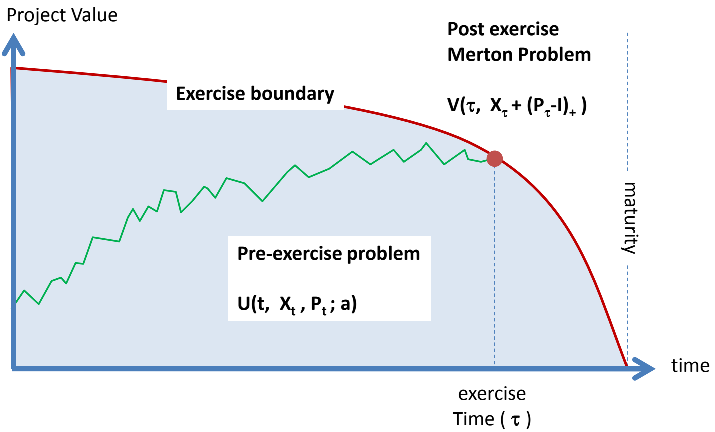
  - classical Merton (69) problem, admits explicit solution
- Invest optimally in S_t **with option** to invest in project
  - Upon exercise, receive option value, and revert to Merton:

$$U(x, P; a) = \sup_{\tau \in \mathscr{T}} \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[V(\tau, \mathbf{X}_\tau + a\,(\mathbf{P}_\tau - \mathbf{I})_+)\right]$$

$$V(t, x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[u(\mathbf{X_T})|X_t = x\right]$$

  - Henderson (07) solved the perpetual version of this problem

Utility Indifference Pricing



Indifference value v of option to invest in project defined as

$$U(x,\, \mathbf{P};\, 0) = U(x - \mathbf{v},\, \mathbf{P};\, 1)$$

Utility Indifference Pricing

- Non-traded project value $\mathbf{P_t}$ and correlated traded equity $\mathbf{S_t}$ satisfy

$$d\mathbf{P_t} = \mathbf{P_t}\left(\nu dt + \eta dW_t^P\right) \ , \quad d\mathbf{S_t} = \mathbf{S_t}\left(\mu dt + \sigma dW_t^S\right)$$

  with $d[W^P, W^S]_t = \rho dt$.
- For risk-neutral valuation can use the **minimal entropy martingale measure**:

$$d\mathbf{P_t} = \mathbf{P_t}\left(\hat{\nu} dt + \eta d\hat{W}_t^P\right) \ , \quad d\mathbf{S_t} = \mathbf{S_t}\left(r dt + \sigma d\hat{W}_t^S\right)$$

  with $\hat{\nu} = \nu - \rho\eta\frac{\mu - r}{\sigma}$ and $d[\hat{W}^P, \hat{W}^S]_t = \rho dt$
- The **MEMM** appears in indifference valuation as well
- **Ambiguity adjusted MEMM** appears for ambiguity-averse agents

Utility Indifference Pricing

- Let $\mathbf{X_t}$ denote the **investor's wealth**
- Let $\pi_{\mathbf{t}}$ denote the **dollar amount invested** in the tradable asset S_t
- Let $\mathcal{A}$ denote the set of **admissible strategies**

$$\mathcal{A} = \left\{ \pi_{\mathbf{t}.} | \text{self financing and } \int_0^T \pi_{\mathbf{t}_s}^2 ds < +\infty \right\}$$

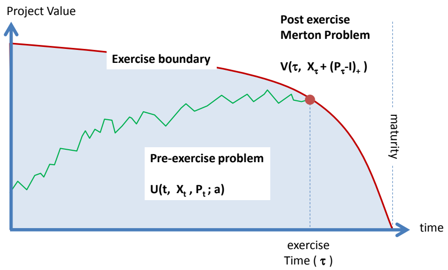
- **Self-financing strategies** imply

$$d\mathbf{X_t} = ((\mu - r)\pi_{\mathbf{t}} + r\,\mathbf{X_t})dt + \sigma \pi_{\mathbf{t}}\, dW_t^S$$

Utility Indifference Pricing

- **Dynamic programming principle** leads to the HJB eqn

$$\begin{cases} \partial_t U + \max_{\pi} \mathcal{L}_\pi U = 0 \\ U(t, b(x), P; a) = V\left(t, x + a(P - I)_+\right) \end{cases}$$



Utility Indifference Pricing

- Assume **exp. utility**: $u(x) = -\frac{1}{\gamma} e^{-\gamma x}$ then **wealth factors**:

$$V(t,x) = u\left(x\, e^{r(T-t)}\right)\, e^{-\frac{1}{2}\lambda^2\,(T-t)}$$
$$U(t,x,e^y) = V(t,x)\, G^\beta(t,y)$$

  where $\lambda = (\mu - r)/\sigma$ is the **market price of risk** and $\beta = (1-\rho^2)^{-1}$ is the power transform coefficient

- G solves a **linear complementarity problem**

$$\begin{cases} \partial_t G + \mathcal{L} G & \leq \; 0, \\ \ln G(t,y) & \geq h(t,y), \\ (\partial_t G + \mathcal{L} G)\cdot(\ln G(t,y) - h(t,y)) & = \; 0, \end{cases}$$

  where

$$h(t,y) = a\frac{\gamma}{\beta}(e^y - K)_+ e^{r(T-t)}\,, \qquad \text{and,} \quad \mathcal{L} = \hat{\nu}\partial_y + \frac{1}{2}\eta^2\partial_{yy}$$

Utility Indifference Pricing

- Since wealth factors, the **indifference value** is simply:

$$\mathbf{v(t,y)} = \frac{\beta}{\gamma} e^{-r(T-t)} \ln G(t,y)$$

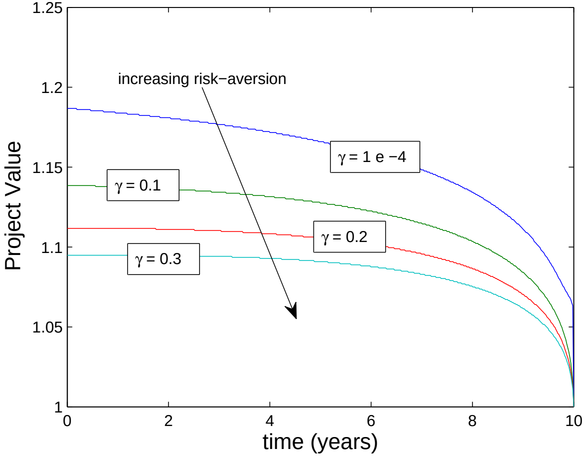
- $\mathbf{v(t,y)}$ then satisfies a **non-linear complementarity problem**:

$$\begin{cases} \partial_t v + \mathcal{L} v - \frac{1}{2}\eta^2 \frac{\gamma}{\beta} e^{r(T-t)} (\partial_{\mathbf{y}} \mathbf{v})^{\mathbf{2}} & \le r\, v, \\ v(t,y) & \ge (e^y - K)_+, \\ \left(\partial_t v + \mathcal{L} v - \frac{1}{2}\eta^2 \frac{\gamma}{\beta} e^{r(T-t)} (\partial_y v)^2 - r\, v\right) & \\ \cdot (v(t,y) - (e^y - K)_+) & = 0. \end{cases}$$

- As $\gamma \downarrow 0$, the non-linearity disappears
- Recovers the risk-neutral American option price

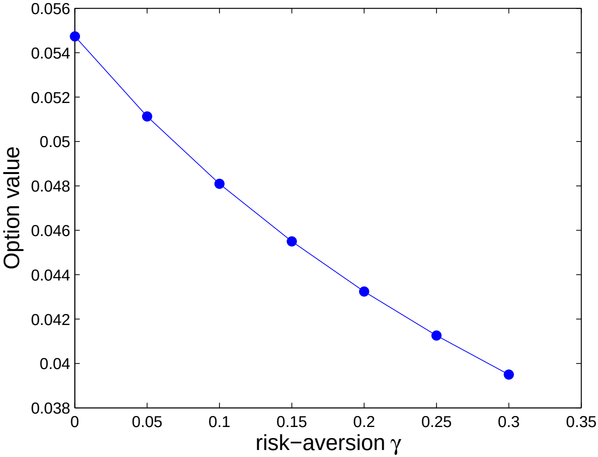
Utility Indifference Pricing

The effect of **risk-aversion** on **exercise policy**



Utility Indifference Pricing

The effect of **risk-aversion** on **option value**



Robust Utility Indifference

- Agent's may **lack confidence** in their model and this uncertainty affects decisions
- As illustrated in the classical **Ellsberg paradox**
  - You are given 40 **red** marbles; and a total of 60 **black** and **green** marbles
  - Mix all marbles, 1 chosen at random
  - Most investors prefer A to B

A	B
receive $100 if **red**	receive $100 if **black**

  - Most investors prefer D to C

C	D
receive $100 if **red** or **green**	receive $100 if **black** or **green**

  - Inconsistent with maximizing expected utility
  - Resolved through including ambiguity aversion

Robust Utility Indifference

- Agent's may **lack confidence** in their model
  - **Knightian Uncertainty** viewed as **ambiguity aversion**
- Use ideas from **Robust Portfolio Optimization**
  - Agent has some confidence in a **reference measure** $\mathbb{P}$
  - Agent is willing to consider a class of **candidate measures** $\mathcal{Q}$
  - Agent then solves the problem

$$V(x,P,S) = \sup_{\pi\in\mathcal{A}} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}_{x,P,S}\left[u(X^{\pi}_{T}) + \frac{1}{\varepsilon}\, h(\mathbb{Q}|\mathbb{P})\right].$$

  - $h(\mathbb{Q}|\mathbb{P})$is a **penalty function**... e.g. relative entropy
  - The parameter ε acts as a measure of ambiguity aversion
    - As $\varepsilon \downarrow 0$ reference measure is picked out
    - $\varepsilon \uparrow +\infty$ all candidates measures are equal

Robust Utility Indifference

- For relative entropy: $h(\mathbb{Q}|\mathbb{P}) = \mathbb{E}^{\mathbb{Q}}[\ln \frac{d\mathbb{Q}}{d\mathbb{P}}] = \mathbb{E}^{\mathbb{Q}}[\int_0^T \mu_s' \Sigma^{-1} \mu_s ds]$
- Instead use scaled relative entropy similar to in J. & Sigloch (09):

$$\mathbf{U^a(t, x, P, S)} = \sup_{\tau \in \mathcal{T}_t} \sup_{\pi \in \mathcal{A}} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}\Bigg[V(\widehat{\tau}, X^{\pi}_{\widehat{\tau}} + a(P_{\widehat{\tau}} - I)_+, P_{\widehat{\tau}}, S_{\widehat{\tau}}) - \frac{\mathbf{1}}{\epsilon} \int_t^{\widehat{\tau}} \mathbf{U^a(s, X^{\pi}_s, P_s, S_s)} \mu^{\mathbb{Q}\prime}_{\mathbf{s}} \mathbf{\Sigma^{-1}} \mu^{\mathbb{Q}}_{\mathbf{s}} \, ds \Bigg],$$

where, $\widehat{\tau} = \tau \wedge T$ and

$$\mathbf{V(t, x, P, S)} = \sup_{\pi \in \mathcal{A}} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}\Bigg[u(X^{\pi}_T) - \frac{\mathbf{1}}{\epsilon} \int_t^{T} \mathbf{V(s, X^{\pi}_s, P_s, S_s)} \mathbf{v}^{\mathbb{Q}\prime}_{\mathbf{s}} \mathbf{\Sigma^{-1}} \mu^{\mathbb{Q}}_{\mathbf{s}} \, ds \Bigg].$$

Robust Utility Indifference

- The **Dynamic programming principle** leads to the HJB eqn

$$\begin{cases} \partial_t U + \max_{\pi,\mu}\left(\mathcal{L}_{\pi,\mu} U - \frac{1}{\varepsilon}\mu'\Sigma^{-1}\mu U\right) = 0 \\ \qquad U(t, b(x), P; a) = V\left(t, x + a(P - I)_+\right) \\ \partial_t V + \max_{\pi,\mu}\left(\mathcal{L}_{\pi,\mu} V - \frac{1}{\varepsilon}\mu'\Sigma^{-1}\mu V\right) = 0 \\ \qquad V(T, x) = u(x) \end{cases}$$

- The scaling of relative entropy allows explicit solutions the DPE
- Equations are similar to previous case with modified parameters

Robust Utility Indifference

- The ansatz

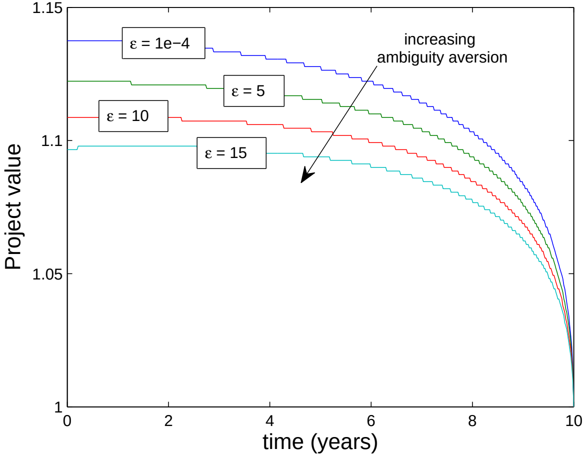
$$V(t,x) = u\left(x\, e^{r(T-t)}\right)\, e^{-\frac{1}{2}\lambda^2\,(T-t)}, \quad U(t,x,e^y) = V(t,x)\, G^\beta(t,y)$$

  solves the resulting dynamic programming equations
- $\lambda^2 = \frac{1}{1+\varepsilon}\left(\frac{\mu-r}{\sigma}\right)$ is **ambiguity adjusted market price of risk**
- The power transform coefficient β also depends on the ambiguity aversion parameter
- **indifference value** $\mathbf{v(t,y)} = \frac{\beta}{\gamma} e^{-r(T-t)} \ln G(t,y)$ solves a non-linear complimentary problem

$$\begin{cases} \partial_t v + \mathcal{L}_\varepsilon \mathbf{v} - \frac{1}{2}\eta^2 \frac{\gamma}{\beta} e^{r(T-t)} (\partial_\mathbf{y} \mathbf{v})^2 & \le \; r\, v, \\ v(t,y) & \ge \; (e^y - K)_+, \\ \left(\partial_t v + \mathcal{L}_\varepsilon \mathbf{v} - \frac{1}{2}\eta^2 \frac{\gamma}{\beta} e^{r(T-t)}\, (\partial_y v)^2 - r\, v\right) & \\ \cdot (v(t,y) - (e^y - K)_+) & = \; 0. \end{cases}$$

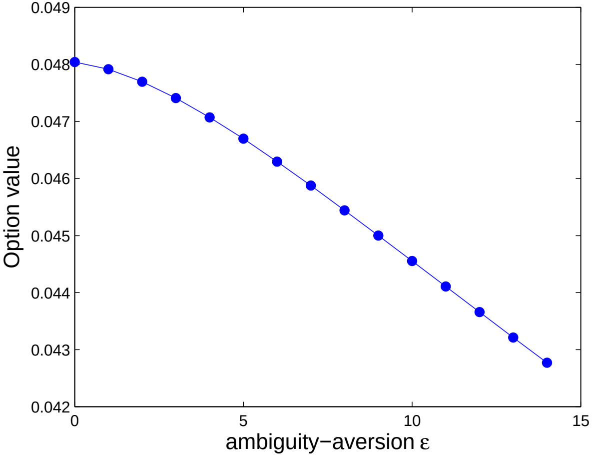
Robust Utility Indifference

The effect of **ambiguity-aversion** on **exercise boundary**



Robust Utility Indifference

The effect of **ambiguity-aversion** on **option price**



Robust Utility Indifference

- **Ambiguity and Risk aversion** are similar **but distinct**
- **As $\gamma \downarrow 0$ non-linearity in LC problem is removed but dependence on ε remains** through the **ambiguity adjusted MEMM drift**

$$\hat{\nu} = \nu - \frac{1}{1+\varepsilon}\rho\eta\frac{\mu - r}{\sigma}$$

  - As $\varepsilon \downarrow 0$, $\hat{\nu}$ decreases to **MEMM drift**
  - As $\varepsilon \uparrow +\infty$, $\hat{\nu}$ increases to ν – **reference measure drift**
- An agent may be risk-neutral but severely ambiguity averse

Conclusions

- Project value modeled as non-traded asset
- Correlated traded asset provides partial hedge
- Use utility indifference to value option
- Risk-aversion affects option value and exercise strategy in non-linear way
- Ambiguity aversion can be incorporated trough a scaled entropic penalty
- Ambiguity also affects option value and exercise strategy in non-linear way
- Ambiguity and risk aversion are similar but distinct factors in explaining agent's behavior

Conclusion

Thank you for your attention!!

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