Minimizing Probability of Lifetime Ruin Under Stochastic Volatility

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Joint work with
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Outline

1. Lifetime Ruin Problem
2. Stochastic Volatility Model
3. Mathematical Tools
4. Main Results with Numerical Examples
5. Conclusion

Introduction

An optimal investment problem:

- Individual can invest in a market with
  - Risk less account: $dB_t = rB_t dt$;
  - Risky asset /stock: $dS_t = \mu S_t dt + \sigma_t S_t dB_t$
- She earns income A and has a minimal consumption c;
- Her future lifetime is random.

Question: How should she invest in order to minimize the probability of outliving her wealth, i.e, the probability of lifetime ruin?

Wealth dynamic: $dW_t = [\mu\pi_t + r(W_t - \pi_t) + A - c]dt + \sigma_t\pi_t dB_t$, with investing strategy π_t denoting the amount of money invested in risky asset at time t.

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Lifetime Ruin Problem

- Ruin level: $w_a = 0$
- Safe level: $w_s = \frac{c-A}{r}$
- Ruin time: $\tau_0 = \inf\{t : W_t \le w_a\}$
- Death time: τ_d, random, depends on the hazard rate λ
- Minimum probability of ruin:

$$\begin{aligned}\psi(w,t) &= \inf_{\pi_t \in \mathcal{A}} \mathbb{P}(\tau_0 < \tau_d | W_t = w, t < \tau_d) \\ &= \inf_{\pi_t \in \mathcal{A}} \mathbb{E}^w[e^{-\int_t^{\tau_0} \lambda ds}]\end{aligned} \tag{1}$$

- Boundary conditions:

$$\begin{aligned}\psi(w,t) &= 1, \quad \text{for } w \le w_a, \\ \psi(w,t) &= 0, \quad \text{for } w \ge w_s,\end{aligned}$$

Motivation

- Young [2004] obtained explicit formula for the optimal strategy and minimum lifetime ruin probability, when the stock price follows standard Black-Scholes model;
- Bayraktar et al. [2008] modeled the consumption as an increasing function of wealth, and considers random consumption;
- Young and Moore [2006] considered varying hazard rate.
- However, all the above work model the stock as a geometric Brownian motion with constant volatility.

More realistic stock model: Stochastic Volatility Model.

Stochastic Volatility Model

- Stock price:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma_t S_t dB_t^0 \\
\sigma_t &= f(Y_t, Z_t), \\
dY_t &= \frac{1}{\epsilon}(m - Y_t)dt + \frac{\sqrt{2}\nu}{\sqrt{\epsilon}} dB_t^1, \quad 0 < \epsilon << 1, \\
dZ_t &= \delta c(Z_t)dt + \sqrt{\delta} d(Z_t) dB_t^2, \quad 0 < \delta << 1.
\end{aligned} \tag{2}
$$

- Two volatility factors
  - Y_t: fast volatility factor
  - Z_t: slow volatility factor
- Reasons to use this model
  - can fit the implied volatility smile well;
  - we can obtain analytical approximation using multi-scale analysis (Fouque et al. [2000])).

Minimizing Probability of Ruin Under Stochastic Volatility

- Minimum probability of ruin: $\psi(w,y,z) = \inf_{\pi_t} \mathbb{E}^{w,y,z}[e^{-\lambda\tau_0}]$.
- Ito's formula and Dynamic Programming Principle give HJB equation for ψ:

$$\inf_{\pi\in\mathcal{A}} \mathcal{D}^\pi\psi = 0$$

where

$$\mathcal{D}^\pi\psi = -\lambda\psi + (rw-c)\psi_w + \frac{1}{\epsilon}(m-y)\psi_y + \delta c(z)\psi_z$$
$$+\frac{1}{\epsilon}\nu^2\psi_{yy} + \frac{1}{2}\delta g^2(z)\psi_{zz} + \rho_{23}\sqrt{2}\nu\frac{\sqrt{\delta}g(z)}{\sqrt{\epsilon}}\psi_{yz}$$
$$+\left[\pi(\mu-r)\psi_w + \frac{1}{2}f^2(y,z)\pi^2\psi_{ww} + \frac{\rho_{12}f(y,z)\pi\nu\sqrt{2}}{\sqrt{\epsilon}}\psi_{wy} + \sqrt{\delta}\rho_{13}\pi f(y,z)g(z)\psi_{wz}\right]$$

Mathematical Tools

- Verification Theorem: to validate a candidate solution.
- Legendre Transform: to obtain duality relationship between ψ and a concave function $\hat{\psi}$ satisfying a PDE with free boundary condition.
- Asymptotic Analysis (Fouque et al. [2000]): to asymptotically expand $\hat{\psi}$ in power of $\sqrt{\epsilon}$ and $\sqrt{\delta}$, then compute explicit formula for each component.
- Markov Chain Approximation Method (MCAM) (Kushner and Dupuis [2001]): alternatively, we can approach the original problem directly and obtain numerical approximation.

Verification Theorem

Theorem 3.1. Suppose $v : \mathbf{D} \to \mathbb{R}$ is a bounded, continuous function that satisfies the following conditions:

1. $v(\cdot, y, z) \in C^2$ is a non-increasing, convex function;
2. $v(w, \cdot, \cdot) \in C^{2,2}$;
3. $v(0, y, z) = 1$;
4. $v(c/r, y, z) = 0$;
5. $\mathcal{D}^\beta v \geq 0$ for all $\beta \in \mathbf{R}$.

Then, $v \leq \psi$ on $\mathbf{D}$.

Dual of ψ

- A related optimal controller-stopper problem:

$$dX_t^\gamma = -(r-\lambda)X_t^\gamma dt - \frac{\mu - r}{f(Y_t, Z_t)} X_t^\gamma d\tilde{B}_t^{(0)} + \gamma_t^{(1)} d\tilde{B}_t^{(1)} + \gamma_t^{(2)} d\tilde{B}_t^{(2)}$$

- Define

$$\hat{\psi}(x,y,z) = \inf_\tau \sup_\gamma \mathbf{E}^{x,y,z} \left[\int_0^\tau e^{-\lambda t} c\, X_t^\gamma \, dt + e^{-\lambda \tau} \min\left((c/r) X_\tau^\gamma, 1\right) \right]. \quad (3)$$

- Convex Legendre Dual:

$$\Psi(w,y,z) = \max_x \left(\hat{\psi}(x,y,z) - wx \right). \quad (4)$$

- Theorem 4.1. Ψ equals the minimum prob of lifetime ruin ψ, and the optimal strategy π^* is given by the first order condition.

Asymptotic Approximation

- Asymptotic approximation result for $\hat{\psi}$:

$$\begin{aligned}\hat{\psi}(x,z) &= \hat{\psi}_{0,0}(x,z) + \sqrt{\epsilon}\,\hat{\psi}_{0,1}(x,z) + \sqrt{\delta}\,\hat{\psi}_{1,0}(x,z) + \mathcal{O}(\epsilon,\delta,\sqrt{\epsilon\,\delta}) \\ &= -\frac{1}{B_1(z)-1}\left(\frac{c}{r}\cdot\frac{B_1(z)-1}{B_1(z)}\cdot x\right)^{B_1(z)} + \frac{c}{r}x \\ &\quad + \sqrt{\epsilon}\,A(z)\,x^{B_1(z)}\,\log\left(x\cdot\frac{B_1(z)-1}{B_1(z)}\cdot\frac{c}{r}\right) \\ &\quad + \sqrt{\delta}\,x^{B_1(z)}\,\log\left(x\cdot\frac{B_1(z)-1}{B_1(z)}\cdot\frac{c}{r}\right)\left[A_1(z)+A_2(z)\log\left(x\cdot\frac{B_1(z)}{B_1(z)-1}\cdot\frac{r}{c}\right)\right] \\ &\quad + \mathcal{O}(\epsilon,\delta,\sqrt{\epsilon\,\delta}),\end{aligned} \tag{5}$$

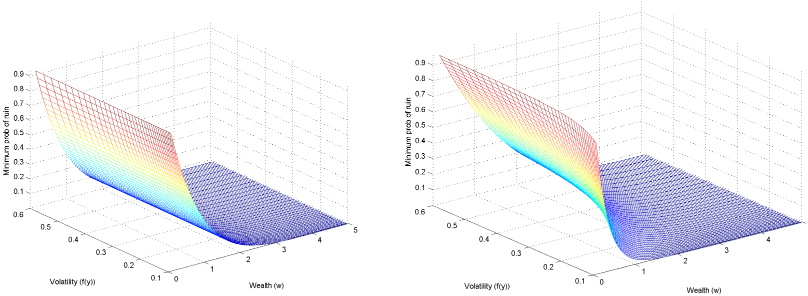
- Asymptotic approximation result for optimal strategy

$$\begin{aligned}\hat{\pi^*}(x,y,z) = &-\frac{\mu-r}{f^2(y,z)}\,x\,\hat{\psi}_{0,0,xx} + \sqrt{\epsilon}\left(-\frac{\mu-r}{f^2(y,z)}\,x\,\hat{\psi}_{0,1,xx} + \rho_{12}\,\frac{\nu\sqrt{2}}{f(y,z)}\,\hat{\psi}_{0,2,xy}\right) \\ &+ \sqrt{\delta}\left(-\frac{\mu-r}{f^2(y,z)}\,x\,\hat{\psi}_{1,0,xx} + \rho_{13}\,\frac{h(z)}{f(y,z)}\,\hat{\psi}_{0,0,xz}\right) + \mathcal{O}(\epsilon,\delta,\sqrt{\epsilon\,\delta}).\end{aligned} \tag{6}$$

Numerical Example

- Question 1: What do the optimal solutions look like?
- Question 2: How does the stochastic environment affect our strategy?
- Question 3: How do different strategies perform in stochastic environment?

Q1: Minimum Ruin Probability



(a) Fast volatility factor $\epsilon = 0.004$ (reverting speed $= 1/\epsilon = 250$)

(b) Slow volatility factor $\delta = 0.02$ (reverting speed $= \delta = 0.02$)

FIGURE 1: Minimum Probability of Ruin

Q1: Optimal Strategy

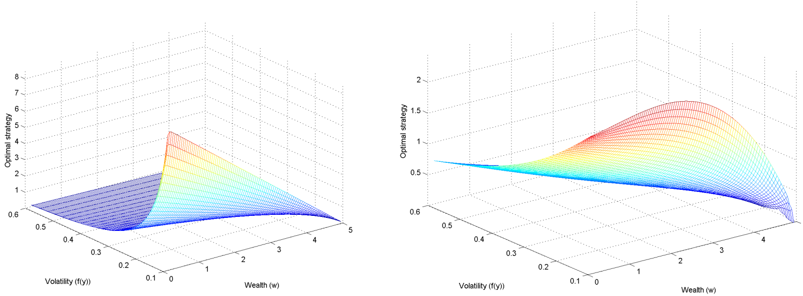


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FIGURE 2: Optimal Strategy

Q2: Stochastic Environment

Question 2: How does the stochastic environment affect our strategy?

- Recall that in constant volatility environment, (Young[2004]),
  - Optimal strategy $\tilde{\pi}(w;\sigma) = \frac{\mu - r}{\sigma^2} \frac{c - wr}{(p-1)r}$;
  - Minimum ruin probability $\tilde{\psi}(w) = (1 - \frac{rw}{c})^p$.

  where $p = \frac{1}{2r}[(r + \lambda + s) + \sqrt{(r + \lambda + s)^2 - 4r\lambda}]$,
  and $s = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2$.

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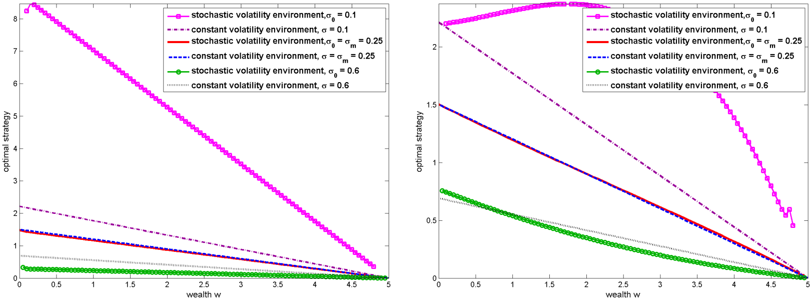

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FIGURE 3: Minimum Probability of Ruin

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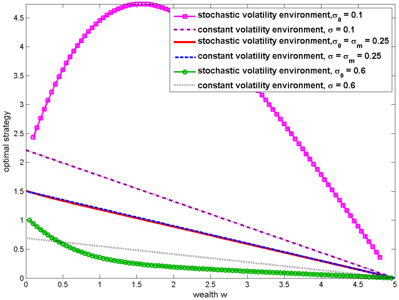
Optimal Strategy(Cont):



(a) Medium reverting speed = 0.2

FIGURE 4: Minimum Probability of Ruin

Q3: Performance of Different Strategies

Question 3: How do different strategies perform in stochastic environment?

- Consider the following strategies:
  - π^a: $\pi^a(w) = \tilde{\pi}(w; \sigma_0)$;
  - π^b: $\pi^a(w) = \tilde{\pi}(w; \sigma_m)$;
  - π^c: $\pi^a(w) = \tilde{\pi}(w; f(y,z))$;
  - π^M: invest only in money market.
  - π^ϵ,π^δ : strategy obtained by asymptotic approximation.
  - π^*: optimal strategy;

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Ruin Probability: (volatility factor reverts with medium speed 0.2)

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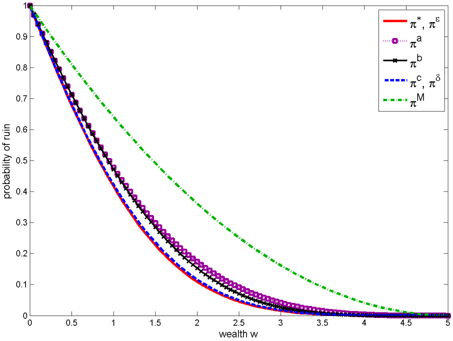
- Figure 3.2: $\sigma_0 = \sigma_m$=0.25;



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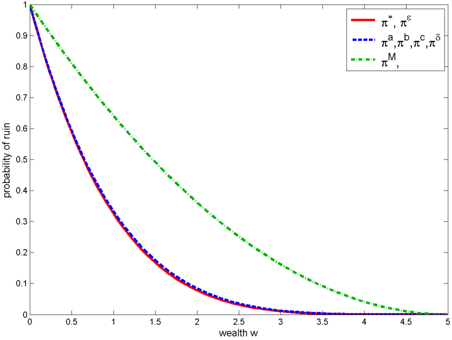

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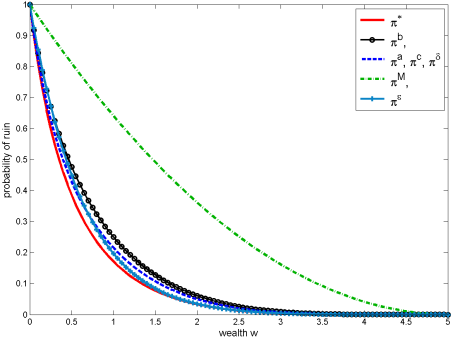

Conclusion

We obtained

- closed-form asymptotic approximation to optimal investment strategy and minimum probability of ruin;
- effects of the stochastic volatility environment;
- an easy-to-implement rule for nearly optimal lifetime ruin probability.

Thanks for your attention!

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