Term structure models driven by Wiener processes and Poisson measures: Existence and positivity

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Introduction

- Zero Coupon Bonds \( P(t, T) \).

- The Heath-Jarrow-Morton-Musiela (HJMM) equation:

\[
dr_t = \left( \frac{d}{d\xi} r_t + \alpha_{\text{HJM}}(r_t) \right) dt + \sigma(r_t) dW_t + \int_E \gamma(r_t, x)(\mu(dt, dx) - F(dx)dt).
\]

- Establish existence and positivity.

- The Brody-Hughston equation:

\[
d\rho_t = \left( \frac{d}{d\xi} \rho_t + \rho_t(0)\rho_t \right) dt + \sigma(\rho_t) dW_t + \int_E \gamma(\rho_t, x)(\mu(dt, dx) - F(dx)dt).
\]
Zero Coupon Bonds

- Zero Coupon Bonds $P(t, T)$.

- Financial assets paying the holder one unit of cash at $T$.

Figure 1: Price process of a $T$-bond with date $T = 10$. 
The HJM model with jumps

- Björk, Kabanov, Runggaldier, Di Masi 1997 [1]: For $T \geq 0$ we have

$$f(t, T) = f^*(0, T) + \int_0^t \alpha(s, T)ds + \int_0^t \sigma(s, T)dW_s$$

$$+ \int_0^t \int_E \gamma(s, x, T)(\mu(ds, dx) - F(dx)ds), \quad t \in [0, T].$$

- Implied bond market:

$$P(t, T) = \exp \left( - \int_t^T f(t, s)ds \right).$$
From HJM to Stochastic Equations

- Drift and volatilities depend on the current forward curve:

\[
\alpha(t, T, \omega) = \alpha(t, T, f(t, \cdot, \omega)), \\
\sigma(t, T, \omega) = \sigma(t, T, f(t, \cdot, \omega)), \\
\gamma(t, x, T, \omega) = \gamma(t, x, T, f(t, \cdot, \omega)).
\]

- Infinite dimensional stochastic equation:

\[
\begin{align*}
\{ \quad df(t, T) &= \alpha(t, T, f(t, \cdot))dt + \sigma(t, T, f(t, \cdot))dW_t \\
&\quad + \int_E \gamma(t, x, T, f(t, \cdot))(\mu(dt, dx) - F(dx)dt) \\
\quad f(0, T) &= f^*(0, T). \}
\end{align*}
\]
The transformed equation

- **Musiela parametrization** of forward rates:

  \[ r_t(\xi) := f(t, t + \xi), \quad \xi \geq 0. \]

- Making the transformation \( f(t, T) \mapsto r_t(\xi) \) we obtain

  \[
  r_t = S_t h_0 + \int_0^t S_{t-s} \alpha(r_s) ds + \int_0^t S_{t-s} \sigma(r_s) dW_s \\
  + \int_0^t \int_E S_{t-s} \gamma(r_{s-}, x)(\mu(ds, dx) - F(dx)ds), \quad t \geq 0
  \]

- where \((S_t)_{t \geq 0}\) denotes the shift-semigroup \(S_t h := h(t + \cdot)\) on \(H\).
From HJMM to SPDEs

• Thus, \((r_t)_{t \geq 0}\) is a mild solution of the SPDE

\[
    dr_t = \left( \frac{d}{d\xi} r_t + \alpha(r_t) \right) dt + \sigma(r_t) dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt),
\]

• with given vector fields

\[
    \alpha : H \to H, \quad \sigma : H \to L^0_2(H), \quad \gamma : H \times E \to H,
\]

• where \(\frac{d}{d\xi}\) is the infinitesimal generator of \((S_t)_{t \geq 0}\).
The HJMM equation

- The bond market $P(t, T)$ should be free of arbitrage.

- Under a martingale measure $\mathbb{Q} \sim \mathbb{P}$ we have

$$\alpha_{\text{HJM}}(h) = \sum_j \sigma_j^j(h) \int_0^\in \sigma^j(h)(\eta)d\eta - \int_E \gamma(h, x) \left( e^{-\int_0^\in \gamma(h, x)(\eta)d\eta - 1} \right) F(dx).$$

- This leads to the HJMM equation

$$dr_t = \left( \frac{d}{d\xi} r_t + \alpha_{\text{HJM}}(r_t) \right) dt + \sigma(r_t) dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx) dt).$$
Stochastic partial differential equations

- Consider the SPDE

\[
\begin{aligned}
\frac{dr_t}{dt} &= (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_t, x)(\mu(dt, dx) - F(dx)dt) \\
\end{aligned}
\]

\[
\begin{aligned}
r_0 &= h_0,
\end{aligned}
\]

- with given vector fields

\[
\alpha : H \rightarrow H, \quad \sigma : H \rightarrow L^0_2(H), \quad \gamma : H \times E \rightarrow H,
\]

- where \( A : \mathcal{D}(A) \subset H \rightarrow H \) is the generator of a \( C_0 \)-semigroup on \( H \).
Assumptions for the existence result

• **Lipschitz continuity:** For all $h_1, h_2 \in H$ we have

\[
\|\alpha(h_1) - \alpha(h_2)\| + \|\sigma(h_1) - \sigma(h_2)\|_{L^2_0(H)} \\
+ \left( \int_E \|\gamma(h_1, x) - \gamma(h_2, x)\|^2 F(dx) \right)^{1/2} \leq L\|h_1 - h_2\|.
\]

• **Linear growth:** We have $\int_E \|\gamma(0, x)\|^2 F(dx) < \infty$.

• We assume that $(S_t)_{t \geq 0}$ is pseudo-contractive, that is

\[
\|S_t\| \leq e^{\omega t}, \quad t \geq 0.
\]
Existence- and uniqueness result

• Unique mild solutions for the SPDE

\[
\begin{aligned}
   dr_t &= (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_t, x)(\mu(dt, dx) - F(dx)dt) \\
   r_0 &= h_0,
\end{aligned}
\]

• i.e., the "Variation of constants formula" is satisfied:

\[
  r_t = S_th_0 + \int_0^t S_{t-s} \alpha(r_s)ds + \int_0^t S_{t-s} \sigma(r_s)dW_s \\
  &\quad + \int_0^t \int_E S_{t-s} \gamma(r_s-, x)(\mu(ds, dx) - F(dx)ds), \quad t \geq 0.
\]
The HJMM equation

- The HJMM equation is an SPDE

\[ dr_t = (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_t, x)(\mu(dt, dx) - F(dx)dt), \]

- for which we have

\[ H = H_\beta, \quad A = \frac{d}{d\xi}, \quad \alpha = \alpha_{\text{HJM}}, \]

- where \( \alpha_{\text{HJM}} \) is given by

\[ \alpha_{\text{HJM}}(h) = \sum_j \sigma^j(h) \int_0^\bullet \sigma^j(h)(\eta)d\eta - \int_E \gamma(h, x) \left( e^{-\int_0^\bullet \gamma(h, x)(\eta)d\eta} - 1 \right) F(dx). \]
The space of forward curves

- For $\beta > 0$ we define the space
  
  $$H_\beta := \{ h : \mathbb{R}_+ \to \mathbb{R} : h \text{ is absolutely continuous with } \|h\|_\beta < \infty \},$$

- where the norm is defined by
  
  $$\|h\|_\beta := \left( |h(0)|^2 + \int_{\mathbb{R}_+} |h'(\xi)|^2 e^{\beta \xi} d\xi \right)^{1/2}.$$  

- The shift semigroup $(S_t)_{t \geq 0}$ on $H_\beta$ has the generator

  $$A = \frac{d}{d\xi}, \quad \mathcal{D}\left(\frac{d}{d\xi}\right) = \{ h \in H_\beta : h' \in H_\beta \}. $$
Assumptions on the vector fields

• **Lipschitz continuity:** For all $h_1, h_2 \in H_{\beta}$ we have

\[
\|\sigma(h_1) - \sigma(h_2)\|_{L^0_2(H_{\beta})} \leq L\|h_1 - h_2\|_{\beta},
\]

\[
\left( \int_E e^{\Phi(x)} \|\gamma(h_1, x) - \gamma(h_2, x)\|_{\beta}^2 F(dx) \right)^{1/2} \leq L\|h_1 - h_2\|_{\beta}.
\]

• **Boundedness:** For all $h \in H_{\beta}$ we have

\[
\|\sigma(h)\|_{L^0_2(H_{\beta})} \leq M,
\]

\[
\int_E e^{\Phi(x)} (\|\gamma(h, x)\|_{\beta}^2 \vee \|\gamma(h, x)\|_{\beta}^4) F(dx) \leq M.
\]
Solution of the HJMM equation

- The HJM drift term $\alpha_{HJM} : H_\beta \to H_\beta$ is Lipschitz continuous:

$$\|\alpha_{HJM}(h_1) - \alpha_{HJM}(h_2)\|_\beta \leq K\|h_1 - h_2\|_\beta.$$ 

- Unique mild solutions for the HJMM equation

$$
\begin{align*}
\left\{ 
    dr_t &= \left( \frac{d}{d\xi} r_t + \alpha_{HJM}(r_t) \right) dt + \sigma(r_t) dW_t + \int_E \gamma(r_{t-}, x) (\mu(dt, dx) - F(dx)dt) \\
    r_0 &= h_0.
\end{align*}
$$

- The interest rates $r_t(\xi)$ should not be negative.
Positivity preserving models

• Let $P \subset H_\beta$ be the convex cone

$$P = \{h \in H_\beta : h \geq 0\} = \bigcap_{\xi \in \mathbb{R}^+} \{h \in H_\beta : h(\xi) \geq 0\}.$$

• The HJMM equation is \textit{positivity preserving} if for all $h_0 \in P$ we have

$$\mathbb{P}(r_t \in P) = 1, \quad t \geq 0.$$

• Stochastic invariance problem.
A general invariance result

- Consider an SPDE on the space $H_\beta$ of forward curves

$$dr_t = \left(\frac{d}{d\xi}r_t + \alpha(r_t)\right)dt + \sigma(r_t)dW_t + \int_E \gamma(r_t-, x)(\mu(dt, dx) - F(dx)dt),$$

- with given vector fields

$$\alpha : H_\beta \rightarrow H_\beta, \quad \sigma : H_\beta \rightarrow L^0_2(H_\beta), \quad \gamma : H_\beta \times E \rightarrow H_\beta.$$

- This SPDE is positivity preserving if and only if we have (1)–(4).
The volatility and the jumps

• At the boundary, the volatility $\sigma$ is parallel to the edge:

$$\sigma^j(h)(\xi) = 0, \quad h \geq 0 \text{ with } h(\xi) = 0.$$  \hspace{1cm} (1)

• The convex cone $P$ captures all jumps:

$$h + \gamma(h, x) \in P, \quad h \in P \text{ and } F\text{-almost all } x \in E.$$  \hspace{1cm} (2)
Small jumps at the boundary

- In general, we have:
  \[ \int_E \|\gamma(h, x)\|_{\beta}^2 F(dx) < \infty, \quad \text{but} \quad \int_E \|\gamma(h, x)\|_{\beta} F(dx) = \infty. \]

- Small jumps, which are not parallel to boundary, are of finite variation:
  \[ \int_E |\gamma(h, x)(\xi)| F(dx) < \infty, \quad h \geq 0 \text{ with } h(\xi) = 0. \quad (3) \]
The drift vector field

• Subtract the $F(dx)dt$-part of the stochastic integral to the drift:

$$dr_t = \left( \frac{d}{d\xi} r_t + \alpha(r_t) \right) dt + \sigma(r_t) dW_t + \int_E \gamma(r_t-, x) (\mu(dt, dx) - F(dx)dt).$$

• At the boundary, the corrected drift term is inward pointing:

$$\alpha(h)(\xi) - \int_E \gamma(h, x)(\xi) F(dx) \geq 0, \quad h \geq 0 \text{ with } h(\xi) = 0. \quad (4)$$
Remarks concerning the drift vector field

- The convex cone $P$ has particular properties.
- The shift semigroup $(S_t)_{t \geq 0}$ leaves $P$ invariant:

$$S_t P \subset P \quad \text{for all } t \geq 0.$$ 

- No Stratonovich correction term, because

$$\left(D \sigma^j (h) \sigma^j (h)\right)(\xi) = 0, \quad h \geq 0 \text{ with } h(\xi) = 0.$$
Invariance conditions for the HJMM equation

- This SPDE is positivity preserving if and only if we have (1)–(4).

- **Consequence:** The HJMM equation

\[
\frac{dr_t}{dt} = \left( \frac{d}{d\xi} r_t + \alpha_{\text{HJM}}(r_t) \right) dt + \sigma(r_t) dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt)
\]

- is positivity preserving if and only if

\[
\begin{align*}
\sigma^j(h)(\xi) &= 0, \quad h \geq 0 \text{ with } h(\xi) = 0 \quad (5) \\
\gamma(h, x)(\xi) &= 0, \quad h \geq 0 \text{ with } h(\xi) = 0 \text{ and } F\text{-almost all } x \in E \quad (6) \\
h + \gamma(h, x) &\in P, \quad h \in P \text{ and } F\text{-almost all } x \in E. \quad (7)
\end{align*}
\]
Another approach to bond price markets

- Following Brody, Hughston 2001 [2] we define the bond prices

\[ P(t, T) = \int_{T-t}^{\infty} \rho_t(\xi) d\xi, \]

- where \((\rho_t)_{t \geq 0}\) is a process of probability densities on \(\mathbb{R}_+\).

- Then we have \(P(T, T) = 1\) for all \(T \geq 0\)

- and \(T \mapsto P(t, T)\) is non-increasing with limit 0 for \(T \to \infty\).
The Brody-Hughston equation

- Consider the Brody-Hughston equation

\[ d\rho_t = \left( \frac{d}{d\xi} \rho_t + \rho_t(0) \rho_t \right) dt + \sigma(\rho_t) dW_t + \int_E \gamma(\rho_t-, x)(\mu(dt, dx) - F(dx)dt), \]

- on the state space \( H^0_\beta \) with vector fields

\[ \sigma : H^0_\beta \to L^0_2(H^0_\beta), \quad \gamma : H^0_\beta \times E \to H^0_\beta, \]

- where \( H^0_\beta = \{ h \in H_\beta : \lim_{\xi \to \infty} h(\xi) = 0 \} \).
Stochastic invariance problem

• We observe that $H_\beta^0 \subset L^1(\mathbb{R}_+)$. 

• Stochastic invariance of the convex set $\mathcal{P} \subset H_\beta^0$ of probability densities

$$\mathcal{P} = \left\{ h \in H_\beta^0 : h \geq 0 \text{ and } \int_{\mathbb{R}_+} h(\xi) d\xi = 1 \right\}$$

$$= \left\{ h \in H_\beta^0 : h \geq 0 \right\} \cap \left\{ h \in H_\beta^0 : \int_{\mathbb{R}_+} h(\xi) d\xi = 1 \right\}$$

Use our previous results Invariance conditions are known

• Unique mild solutions for the Brody-Hughston equation.
Conclusion

- Zero Coupon Bonds $P(t, T)$.

- The Heath-Jarrow-Morton-Musiela (HJMM) equation:

$$dr_t = \left( \frac{d}{d\xi} r_t + \alpha_{HJM}(r_t) \right) dt + \sigma(r_t) dW_t + \int_E \gamma(r_t, x) (\mu(dt, dx) - F(dx) dt).$$

- We have established existence and positivity.

- The Brody-Hughston equation:

$$d\rho_t = \left( \frac{d}{d\xi} \rho_t + \rho_t(0) \rho_t \right) dt + \sigma(\rho_t) dW_t + \int_E \gamma(\rho_t, x) (\mu(dt, dx) - F(dx) dt).$$
References


