Portfolio optimization under partial information
with expert opinions

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Introduction

Classical **Merton problem** in dynamic portfolio optimization

- Stock returns \( \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \)
  - risk-free interest rate \( r \)

- Maximize \( E [U(X_T)] \)
  - for power utility \( U(x) = \frac{x^\theta}{\theta}, \theta < 1, \theta \neq 0 \)

- Optimal proportion of wealth invested in risky asset
  \[
  h_t^{(0)} = \frac{1}{1 - \theta} \frac{\mu - r}{\sigma^2} = \text{const}
  \]

\( h^{(0)} \) is a key building block of optimal strategies in more complicated models
Portfolio Optimization and Drift

- Sensitive dependence of investment strategies on drift of assets
- Drifts are hard to estimate empirically
  need data over long time horizons
  (other than volatility estimation)
- Problems with stationarity: drift is not constant
Implications

- **Academic literature:** drift is driven by unobservable factors
  Models with partial information, apply filtering techniques
  Björk, Davis, Landén (2010)
  - Linear Gaussian models
    Lakner (1998), Nagai, Peng (2002), Brendle (2006), ...
  - Hidden Markov models

- Practitioners use static Black-Litterman model
  Apply Bayesian updating to combine subjective views (such as "asset 1 will grow by 5%")
  with empirical or implied drift estimates

- Present paper combines the two approaches
  consider dynamic models with partial observation including expert opinions
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Financial Market Model

\((\Omega, \mathcal{G} = (\mathcal{G}_t)_{t \in [0, T]}, P)\) filtered probability space (full information)

**Bond**
\[ S^0_t = 1 \]

**Stocks**
prices \( S_t = (S^1_t, \ldots, S^n_t)\), returns \( dR^i_t = \frac{dS^i_t}{S^i_t} \)

\[ dR_t = \mu(Y_t) \, dt + \sigma \, dW_t \]
\( \mu(Y_t) \in \mathbb{R}^n \) drift, \( \sigma \in \mathbb{R}^{n \times n} \) volatility
\( W_t \) \( n \)-dimensional \( \mathcal{G} \)-Brownian motion

**Factor process**
\( Y_t \) finite-state Markov chain, independent of \( W_t \)
Financial Market Model

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\[ S_t^0 = 1 \]

**Stocks**  
prices  
\[ S_t = (S_t^1, \ldots, S_t^n)\top, \]  
returns  
\[ dR_t^i = \frac{dS_t^i}{S_t^i} \]

\[ dR_t = \mu(Y_t) \, dt + \sigma \, dW_t \]

\[ \mu(Y_t) \in \mathbb{R}^n \]  drift,  
\[ \sigma \in \mathbb{R}^{n \times n} \]  volatility

\[ W_t \quad n\text{-dimensional } \mathcal{G}\text{-Brownian motion} \]

**Factor process**  
\( Y_t \)  finite-state Markov chain, independent of \( W_t \)

state space  
\[ \{e_1, \ldots, e_d\}, \]  unit vectors in \( \mathbb{R}^d \)

states of drift  
\[ \mu(Y_t) = MY_t \]  where  
\[ M = (\mu_1, \ldots, \mu_d) \]

generator matrix  \( Q \)

initial distribution  
\[ (\pi^1, \ldots, \pi^d)\top \]
Investor Information

Investor is not informed about factor process $Y_t$, he only observes

**Stock prices** $S_t$ or equivalently stock returns $R_t$

**Expert opinions** own view about future performance
news, recommendations of analysts or rating agencies

$\Rightarrow$ Model with *partial information*.

Investor needs to “learn” the drift from observable quantities.
Modelled by marked point process \( I = (T_n, Z_n) \sim I(dt, dz) \)

- At random points in time \( T_n \sim \text{Poi}(\lambda) \) investor observes r.v. \( Z_n \in \mathcal{Z} \)
- \( Z_n \) depends on current state \( Y_{T_n} \), density \( f(Y_{T_n}, z) \)
  \((Z_n)\) cond. independent given \( \mathcal{F}_T^Y = \sigma(Y_s : s \in [0, T]) \)
Expert Opinions

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Examples

- Absolute view: $Z_n = \mu(Y_{T_n}) + \sigma \varepsilon_n$, $(\varepsilon_n)$ i.i.d. $\mathcal{N}(0, 1)$
  The view “S will grow by 5%” is modelled by $Z_n = 0.05$
  $\sigma_\varepsilon$ models confidence of investor

- Relative view (2 assets): $Z_n = \mu_1(Y_{T_n}) - \mu_2(Y_{T_n}) + \tilde{\sigma}_\varepsilon \varepsilon_n$
Expert Opinions

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Examples

- Absolute view: \( Z_n = \mu(Y_{T_n}) + \sigma \varepsilon \varepsilon_n, \quad (\varepsilon_n) \text{ i.i.d. } N(0, 1) \)
  The view “S will grow by 5%” is modelled by \( Z_n = 0.05 \)
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- Relative view (2 assets): \( Z_n = \mu_1(Y_{T_n}) - \mu_2(Y_{T_n}) + \tilde{\sigma} \varepsilon \varepsilon_n \)

Investor filtration \( \mathbb{F} = (\mathcal{F}_t) \) with \( \mathcal{F}_t = \sigma(R_u: u \leq t; (T_n, Z_n): T_n \leq t) \)
Admissible Strategies described via portfolio weights $h^1_t, \ldots, h^n_t$

$$\mathcal{H} = \{(h_t)_{t \in [0, T]} \mid h_t \in \mathbb{R}^n, \int_0^T \|h_t\|^2 < \infty, \quad h \text{ is } \mathbb{F}\text{-adapted} \}$$
Optimization Problem

**Admissible Strategies** described via portfolio weights \( h^1_t, \ldots, h^n_t \)

\[
\mathcal{H} = \{(h_t)_{t \in [0,T]} \mid h_t \in \mathbb{R}^n, \int_0^T ||h_t||^2 < \infty, \\
h \text{ is } \mathbb{F}-\text{adapted}\}\]

**Wealth**

\[
dX_t^h = X_t^h h_t^\top (\mu(Y_t) \, dt + \sigma \, dW_t), \quad X_0^h = x_0
\]

**Utility function**

\[
U(x) = \frac{x^\theta}{\theta}, \quad \text{power utility}, \quad \theta \in (-\infty, 1) \setminus \{0\}
\]

\[
U(x) = \log(x) \quad \text{logarithmic utility} \quad (\theta = 0)
\]
Optimization Problem

Admissible Strategies: described via portfolio weights $h^1_t, \ldots, h^n_t$

$$\mathcal{H} = \{(h_t)_{t \in [0,T]} \mid h_t \in \mathbb{R}^n, \int_0^T \|h_t\|^2 < \infty, \quad h \text{ is } \mathbb{F}-\text{adapted}\}$$

Wealth:

$$dX^h_t = X^h_t h^\top_t (\mu(Y_t) dt + \sigma dW_t), \quad X^h_0 = x_0$$

Utility function:

$$U(x) = \frac{x^\theta}{\theta}, \quad \text{power utility,} \quad \theta \in (-\infty, 1) \setminus \{0\}$$

$$U(x) = \log(x) \quad \text{logarithmic utility} \quad (\theta = 0)$$

Reward function:

$$v(t, x, h) = E_{t,x}[U(X^h_T)] \quad \text{for } h \in \mathcal{H}$$

Value function:

$$V(t, x) = \sup_{h \in \mathcal{H}} v(t, x, h)$$

Find optimal strategy $h^* \in \mathcal{H}$ such that $V(0, x_0) = v(0, x_0, h^*)$
Filtering and Reduction to Full Information

HMM Filtering - only return observation

**Filter**

\[ p_t^k := P(Y_t = e_k | \mathcal{F}_t) \]

\[ \hat{\mu}(Y_t) := E[\mu(Y_t) | \mathcal{F}_t] = \mu(p_t) = \sum_{j=1}^{d} p_t^j \mu_j \]
Filtering and Reduction to Full Information

HMM Filtering - only return observation

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Innovation process

\[ \tilde{W}_t := \sigma^{-1}(R_t - \int_0^t \hat{\mu}(Y_s) ds) \] is an \( \mathbb{F} \)-BM

HMM filter


\[ p_0^k = \pi^k \]

\[ dp_t^k = \sum_{j=1}^{d} Q_{jk} p_t^j dt + a_k(p_t)^\top d\tilde{W}_t \]

where

\[ a_k(p) = p^k \sigma^{-1}(\mu_k - \sum_{j=1}^{d} p^j \mu_j) \]
HMM Filtering - including expert opinions

Extra information has no impact on filter $p_t$ between ‘information dates’ $T_n$
HMM Filtering - including expert opinions

Extra information has no impact on filter $p_t$ between ‘information dates’ $T_n$

**Bayesian updating** at $t = T_n$:

$$p^k_{T_n} \propto p^k_{T_{n-}} f(e_k, Z_n)$$

recall: $f(Y_{T_n}, z)$ is density of $Z_n$ given $Y_{T_n}$

with normalizer  

$$\sum_{j=1}^{d} p^j_{T_{n-}} f(e_j, Z_n) =: \bar{f}(p_{T_{n-}}, Z_n)$$
HMM Filtering - including expert opinions

Extra information has no impact on filter $p_t$ between ‘information dates’ $T_n$

**Bayesian updating** at $t = T_n$:

$$p^k_{T_n} \propto p^k_{T_n^-} f(e_k, Z_n) \quad \text{recall: } f(Y_{T_n}, z) \text{ is density of } Z_n \text{ given } Y_{T_n}$$

with normalizer $\sum_{j=1}^d p^j_{T_n^-} f(e_j, Z_n) =: \tilde{f}(p_{T_n^-}, Z_n)$

HMM filter

$$p^k_0 = \pi^k$$

$$dp^k_t = \sum_{j=1}^d Q^{jk} p^j_t dt + a_k(p_t)^\top d\tilde{W}_t + p^k_{t^-} \int_Z \left( \frac{f(e_k, z)}{f(p_{t^-}, z)} - 1 \right) \gamma(dt \times dz)$$

**Compensated measure**  

$$\gamma(dt \times dz) := l(dt \times dz) - \lambda dt \sum_{k=1}^d p^k_{t^-} f(e_k, z) \, dz \quad \text{compensator}$$
Filtering: Example

Drift

Stock Price

\[ \exp \left( \int_0^t \mu(Y_s) \, ds \right) \]
Filtering: Example

**Drift**

- Drift

**Stock Price**

- $\exp(\int_0^t \mu(Y_s) \, ds)$
- Stock Price $S_t$
Filtering: Example

Drift

drift

Stock Price

exp(∫₀ᵗ μ(Yₛ) ds )

Stock Price Sₜ

time t
Consider augmented state process \((X_t, p_t)\)

**Wealth**

\[
dX_t^h = X_t^h h_t^\top \left( \mu(Y_t) \right) dt + \sigma d\tilde{W}_t, \quad X_0^h = x_0
\]

**Filter**

\[
dp_t^k = \sum_{j=1}^{d} Q^{jk} p_t^j dt + a_k(p_t)^\top d\tilde{W}_t
\]

\[
+p_t^k \int_{\mathcal{Z}} \left( \frac{f(e_k, z)}{f(p_{t-}, z)} - 1 \right) \gamma(dt \times dz), \quad p_0^k = \pi^k
\]
Consider augmented state process \((X_t, p_t)\)

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dX^h_t = X^h_t h^\top_t \left( \mathbf{\mu}(Y_t) \right) dt + \sigma d\tilde{W}_t, \quad X^0_t = x_0
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**Filter**
\[
dp^k_t = \sum_{j=1}^d Q^k j p^j_t dt + a_k(p_t)^\top d\tilde{W}_t
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**Reward function**
\[
v(t, x, p, h) = E_{t,x,p}[ U(X^h_T) ] \quad \text{for} \quad h \in \mathcal{H}
\]

**Value function**
\[
V(t, x, p) = \sup_{h \in \mathcal{H}} v(t, x, p, h)
\]

Find \(h^* \in \mathcal{H}(0)\) such that \(V(0, x_0, \pi) = v(0, x_0, \pi, h^*)\)
Solution for Power Utility

Risk-sensitive control problem  (Nagai & Runggaldier (2008))

Let  \( Z^h := \exp \left\{ \theta \int_0^T h_s^\top \sigma d\tilde{W}_s - \frac{\theta^2}{2} \int_0^T h_s^\top \sigma \sigma^\top h_s ds \right\} \), assume  \( E[Z^h] = 1 \)

Change of measure:  \( P^h(A) = E[Z^h 1_A] \)  for  \( A \in \mathcal{F}_T \)
Solution for Power Utility

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Change of measure: \(P^h(A) = E[Z^h 1_A] \) for \(A \in \mathcal{F}_T\)

Reward function

\[
E_{t,x,p}[U(X^h_T)] = \frac{x^\theta}{\theta} E_{t,p}^h \exp \left\{ - \int_t^T b^{(\theta)}(p_s, h_s) ds \right\}
\]

\[=: v(t, p, h) \text{ independent of } x\]

where \(b^{(\theta)}(p, h) := -\theta \left( h^\top Mp - \frac{1 - \theta}{2} h^\top \sigma \sigma^\top h \right)\)
Solution for Power Utility

Risk-sensitive control problem  
(Nagai & Runggaldier (2008))

Let  
\[ Z^h := \exp \left\{ \theta \int_0^T h_s^\top \sigma d\tilde{W}_s - \frac{\theta^2}{2} \int_0^T h_s^\top \sigma \sigma^\top h_s ds \right\}, \]  
assume  \( E[Z^h] = 1 \)

Change of measure:  
\[ P^h(A) = E[Z^h 1_A] \quad \text{for} \quad A \in \mathcal{F}_T \]

Reward function  
\[ E_{t,x,p}[U(X^h_T)] = \frac{x^{\theta}}{\theta} E_t^h \left[ \exp \left\{ -\int_t^T b^{(\theta)}(p_s, h_s) ds \right\} \right] \]

\[ =: v(t, p, h) \quad \text{independent of} \ x \]

where  
\[ b^{(\theta)}(p, h) := -\theta \left( h^\top M p - \frac{1 - \theta}{2} h^\top \sigma \sigma^\top h \right) \]

Admissible strategies  
\[ \mathcal{A} = \mathcal{H} \cap \{ (h_t) \mid E[Z^h] = 1 \} \]

Value function  
\[ V(t, p) = \sup_{h \in \mathcal{A}} v(t, p, h) \]

Find  \( h^* \in \mathcal{A} \) such that  
\[ V(0, \pi) = v(0, \pi, h^*) \]
HJB-Equation

\[ V_t(t, p) + \sup_{h \in \mathbb{R}^n} \left\{ \mathcal{L}^h V(t, p) - b^{(\theta)}(p, h) V(t, p) \right\} = 0 \]

terminal condition \( V(T, p) = 1 \)

where \( \mathcal{L}^h \) generator of the filter process \( p_t \) under measure \( P^h \)
HJB-Equation

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Optimal Strategy

\[ h^* = h^*(t, p) = \frac{1}{(1 - \theta)(\sigma \sigma^\top)^{-1}} \left\{ Mp + \frac{1}{V(t, p)} \sigma \sum_{k=1}^d a_k(p) V_{p^k}(t, p) \right\} \]
HJB-Equation

\[ V_t(t, p) + \sup_{h \in \mathbb{R}^n} \left\{ \mathcal{L}^h V(t, p) - b^{(\theta)}(p, h) V(t, p) \right\} = 0 \]

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**Optimal Strategy**

\[ h^* = h^*(t, p) = \frac{1}{(1 - \theta)(\sigma \sigma^\top)^{-1}} \left\{ Mp + \frac{1}{V(t, p)} \sigma \sum_{k=1}^{d} a_k(p) V_{p^k}(t, p) \right\} \]

- myopic strategy + correction

Certainty equivalence principle does not hold
Plugging in $h^*$ into the HJB equation and substituting $V = G^{1-\theta}$ we derive a

**Transformed HJB-Equation for $G = G(t, p)$**

$$G_t + \frac{1}{2} tr[A^\top(p)A(p)D^2G] + B^\top(p) \nabla G + C(p)G$$

$$+ \frac{\lambda}{1 - \theta} \int_{Z} \frac{G^{1-\theta}(t, p + \Delta(p, z)) - G^{1-\theta}(t, p)}{G^{-\theta}(t, p)} \bar{f}(p, z) dz = 0,$$

$$G(T, p) = 1,$$

The functions $A, B, C, \Delta$ are defined in the paper.

Note that the equation has a **linear diffusion part** but **nonlinear integral term**.
Starting approximation is the myopic strategy

\[ h_t^{(0)} = \frac{1}{1-\theta} (\sigma \sigma^T)^{-1} M \rho_t \]

The corresponding reward function is

\[ V^{(0)}(t, p) := v(t, p, h^{(0)}) = E_{t,p} \left[ \exp \left( - \int_t^T b^{(\theta)}(p_{s}^{h^{(0)}}, h_{s}^{(0)}) ds \right) \right] \]
Policy Improvement

Starting approximation is the myopic strategy

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The corresponding reward function is

\[ V^{(0)}(t, p) := v(t, p, h^{(0)}) = E_{t,p} \left[ \exp \left( -\int_t^T b^{(\theta)}(p_{s}^{h^{(0)}}, h_{s}^{(0)}) ds \right) \right] \]

Consider the following optimization problem

\[
\max \left\{ \mathcal{L}^h V^{(0)}(t, p) - b^{(\theta)}(p, h) V^{(0)}(t, p) \right\}
\]

with the maximizer

\[
h^{(1)}(t, p) = h^{(0)}(t, p) + \frac{1}{(1 - \theta) V^{(0)}(t, p)} (\sigma^\top)^{-1} \sum_{k=1}^{d} a_k(p) V_{p_k}^{(0)}(t, p)
\]
Policy Improvement

Starting approximation is the myopic strategy

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The corresponding reward function is

\[ V^{(0)}(t, p) := v(t, p, h^{(0)}) = E_{t,p} \left[ \exp \left( - \int_t^T b^{(\theta)}(p_s^{h^{(0)}}, h_s^{(0)}) ds \right) \right] \]

Consider the following optimization problem

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\max_h \{ \mathcal{L}^h V^{(0)}(t, p) - b^{(\theta)}(p, h) V^{(0)}(t, p) \}
\]

with the maximizer

\[
h^{(1)}(t, p) = h^{(0)}(t, p) + \frac{1}{(1-\theta) V^{(0)}(t, p)} (\sigma^\top)^{-1} \sum_{k=1}^d a_k(p) V_{p_k}^{(0)}(t, p)
\]

For the corresponding reward function \( V^{(1)}(t, p) := v(t, p, h^{(1)}) \) it holds

**Lemma** ( \( h^{(1)} \) is an improvement of \( h^{(0)} \))

\[ V^{(1)}(t, p) \geq V^{(0)}(t, p) \]
Numerical Results

**Drift**

- Drift
- $Z_n$
- Filter

**Strategy $h_t$ ($\theta = 2/3$)**

- Myopic
- Policy Impr.
For $t = T_n$: nearly full information $\implies$ correction $\approx 0$