The Numeraire Portfolio
Under Proportional Transaction Costs

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joint work with

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Outline

- Numeraire portfolio and growth optimality without transaction costs
- Growth optimality under transaction costs
- Discrete time model with transaction costs
- A “simple” example
- Price systems under transaction costs
- Numeraire portfolio under transaction costs
- Extensions and some background

The basic model (without transaction costs)

- Discrete time, arbitrage-free model with bond B and stock prices S

$$B_n \equiv 1, \quad S_n > 0, \quad n = 0, \dots N,$$

whose evolution is given w.r.t. the physical measure P.
As Filtration we use $\mathcal{F}_n = \sigma(S_0, S_1, \dots, S_n)$, $\mathcal{F}_0$ trivial.

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- **Fundamental theorems of asset pricing**
  - No arbitrage opportunities exist if and only if there is at least one EMM Q.
  - In this case: Market is complete if and only if Q is unique.

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  - No arbitrage opportunities exist if and only if there is at least one EMM Q.
  - In this case: Market is complete if and only if Q is unique.
- For any EMM Q an arbitrage free price for a claim C is

$$\mathrm{pr}_Q(C) := \mathrm{E}_Q[C] = \mathrm{E}[Z_N^Q\, C], \quad Z_N^Q := \frac{dQ}{dP}.$$

The numeraire portfolio

- For a self-financing trading strategy φ we can compute wealth X_n^φ, $n = 0, \dots, N$. We would like to find φ such that

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- This can be done, if we can find a trading strategy φ^* for $X_0^* = x_0 = 1$ such that $X_n^* > 0$ and we have martingales
$$1/X_n^*, \quad S_n/X_n^*, \quad n = 0, \ldots, N.$$
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- Ansatz: Choose φ^* growth optimal, $\mathrm{E}[\log X_N^*] = \max_\varphi \mathrm{E}[\log(X_N^\varphi)]$. This works in a **complete market**, since $X_N^* = x_0/Z_N^Q$ is maximizing
$$\mathrm{E}[\log(X_N)] - y(\mathrm{E}_Q[X_N] - x_0) = \mathrm{E}[\log(X_N) - yZ_N^Q X_N] + y\, x_0.$$

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  In an **incomplete market** this might not work, since for $V(z) = \sup\{\log(x) - x\, z \,:\, x > 0\} = \log(1/z) - 1$

$$X_N^* = \frac{x_0}{Z_N^*}, \quad \mathrm{E}[V(Z_N^*)] = \inf\{\mathrm{E}[V(Z)] \,:\, Z \text{ EMM}\}$$

Model specification

- We assume $S_{n+1} = S_n(1 + R_{n+1})$ with returns R_n i.i.d. like R,
  $\underline{R} \leq R \leq \overline{R}, \quad -1 < \underline{R} < 0 < \overline{R}, \quad \mathrm{E}[(R-\underline{R})^{-1}] = \mathrm{E}[(\overline{R}-R)^{-1}] = \infty.$

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- For traded amount Δ_n and proportional costs $\lambda, \mu \in [0, 1)$ we get

$$\begin{aligned} \text{stock account} \qquad & \overline{\pi}_n \overline{X}_n = \pi_n X_n + \Delta_n, \\ \text{bond account} \qquad & (1 - \overline{\pi}_n)\overline{X}_n = (1 - \pi_n)X_n - \beta(\Delta_n)\Delta_n, \\ \text{where} \qquad & \beta(\Delta) = \begin{cases} 1 + \lambda & \Delta > 0 \quad \text{buy}, \\ 1 - \mu & \Delta < 0 \quad \text{sell}. \end{cases} \end{aligned}$$

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- At N we liquidate our portfolio and may have liquidation costs L, i.e.

$$\overline{\pi}_N = 0, \quad \overline{X}_N = (1-\pi_N)X_N + L(\pi_N)\pi_N X_N.$$

  E.g. $L(\pi) = \beta(-\pi)$ or $L(\pi) = 1$, in general

$$L(\pi) = \begin{cases} 1-\mu_N & \pi > 0 \quad \text{sell}, \\ 1+\lambda_N & \pi < 0 \quad \text{buy}. \end{cases}$$

Growth optimality under transaction costs

Admissibility of a trading strategy $\varphi = (\Delta_n)_{n=0,\dots,N-1}$ is defined by

$$X_N^\varphi > 0, \quad 1 - \pi_N^\varphi + L(\pi_N^\varphi)\,\pi_N^\varphi > 0.$$

Theorem: An optimal admissible policy φ^* exists, i.e.

$$\mathrm{E}[\log(\overline{X}_N^*)] = \sup_{\varphi \text{adm.}} \mathrm{E}[\log(\overline{X}_N^\varphi)].$$

The optimal policy is characterized by risky fractions $a_n < b_n$ s.t.

$$\overline{\pi}_n^* = \begin{cases} a_n, & \text{if } \pi_n^* < a_n \quad \text{buy region,} \\ \pi_n^*, & \text{if } \pi_n^* \in [a_n, b_n] \quad \text{no-trading region,} \\ b_n, & \text{if } \pi_n^* > b_n \quad \text{sell region.} \end{cases}$$

References: Kamin 75, Constantinides 79, ...

Growth optimality under t.c. – about the proof

- Look at $Y_n = \pi_n X_n$, $Z_n = (1 - \pi_n) X_n$ and value function

$$V_n(y, z) = \sup_{\varphi} \mathrm{E}[\log(\overline{Y}_N + \overline{Z}_N) \mid Y_n = y, Z_n = z]$$

for those $\varphi = (\Delta_n)_{n=0,\ldots,N-1}$ for which (Y_n, Z_n) in solvency region.

Note that $V_0(0, x_0) = \mathrm{E}[\log(\overline{X}^*_N)]$.

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- Show by backward induction that
  - V_n is concave, increasing, and $V_n(\alpha y, \alpha z) = \log(\alpha) + V_n(y,z)$.
  - The maximum on the sell- and buy-lines is attained.
  - The optimality equation holds:

    $$V_n(y,z) = \max_{\Delta} \mathrm{E}[V_{n+1}((y+\Delta)R_{n+1}, z - \beta(\Delta))].$$

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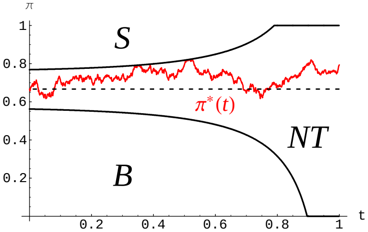
- Unique maximizers Δ^* exist. They define the optimal strategy φ^* and can be represented in terms of $\pi_n = Y_n / X_n$.
- Main problems: One-sided derivatives for first order conditions might not be continuous at 0. Since short selling and borrowing are allowed, existence of an optimizer can be delicate.

Properties of the boundaries of the trading regions

In continuous time for terminal trading time $T = 1$ and without short selling/borrowing we get (Kunisch/S. 07)



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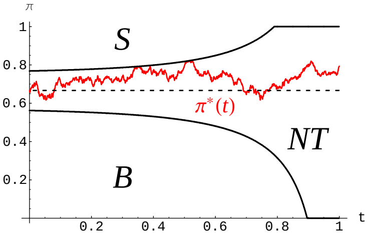

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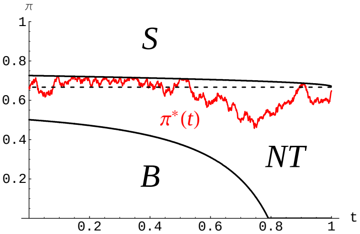

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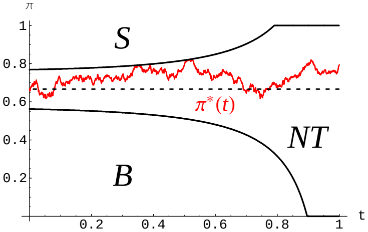

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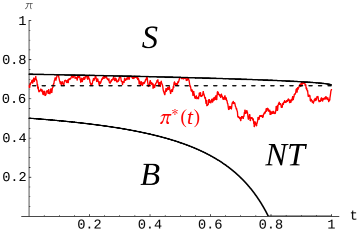

With liquidation costs

In our model we can prove

- Suppose $\lambda_N = \lambda$, $\mu_N = \mu$. If $(\mathrm{E}R)^N > \frac{1+\lambda}{1-\mu_N}$, then for $n_0 = \inf\{n \,:\, (\mathrm{E}R)^{N-n} \leq \frac{1+\lambda}{1-\mu_N}\}$ we have $a_n = 0$ for all $n \geq n_0$.
- Suppose $\lambda_N = 0$, $\mu_N = 0$. Then $0 \in (a_n, b_n)$ as long as $(ER)^{N-n} \in (1-\mu, 1+\lambda)$.

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- In a complete one-period binomial model with two states $u = up$, $d = down$ any claim can be hedged. E.g.

	$\nearrow$	$S_1(u) = 5$	$C(u) = 2$
$S_0 = 3$			Claim to be priced
	$\searrow$	$S_1(d) = 5/2$	$C(d) = 0$

Remember, $B_0 = B_1 = 1$.

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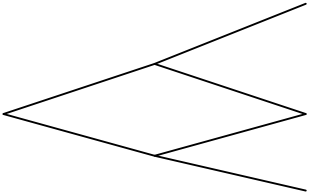
	↘	$S_1(d) = 5/2$	$C(d) = 0$

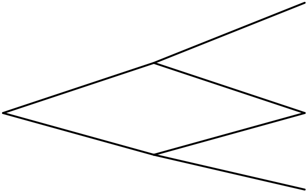
  Remember, $B_0 = B_1 = 1$.
- The replicating strategy (buy $4/5$ stocks, sell 2 bonds) leads to price $2/5$.

Transaction costs and bid/ask prices



Possible prices S in 2-period CRR

Transaction costs and bid/ask prices



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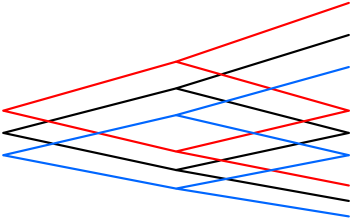

Bid, ask prices $(1+\lambda)S$, $(1-\mu)S$

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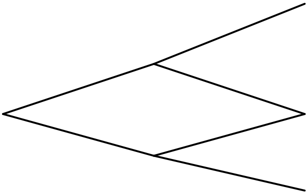

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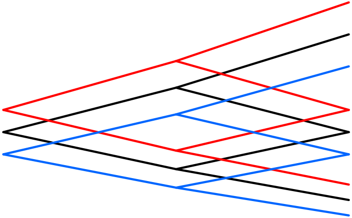

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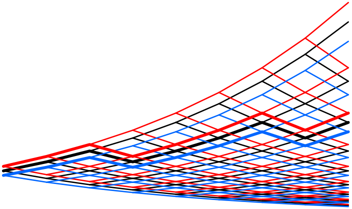
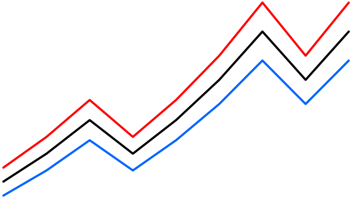


Possible pathes in 8-period CRR



One path in 8-period CRR

A "simple" example with transaction costs

- Roux and Zastawniak (2006) show that a price system based on replication may lead to arbitrage, using the following example:

$$
\begin{array}{lcll}
 & & S_1^{\text{ask}}(u) = 6 & (C^B(u), C^S(u)) = (2,0) \\
 & \nearrow & S_1^{\text{bid}}(u) = 4 & \\
S_0^{\text{ask}} = 5 & & & \text{Claim} \\
S_0^{\text{bid}} = 1 & & & \text{to be priced} \\
 & \searrow & S_1^{\text{ask}}(d) = 3 & \\
 & & S_1^{\text{bid}}(d) = 2 & (C^B(d), C^S(d)) = (0,0)
\end{array}
$$

This corresponds to prices and costs

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\begin{array}{ll}
S_0 = 3 & S_u = 5, S_d = 5/2 \\
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$$\nearrow$$

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Claim to be priced

$$\searrow$$

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- A replicating strategy (buy 1 stock, sell 2 bonds) leads to price 3.

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- A replicating strategy (buy 1 stock, sell 2 bonds) leads to price 3.
- A simple superhedging strategy costs 2 (buy 2 bonds).

Option pricing by hedging under t.c.

- Pricing by (super)replication
  - Leland 85
  - Merton 90
  - Bensaid/Lesne/Pagès 92
  - Boyle/Vorst 95
  - Stettner 97
  - Roux/Tokarz/Zastawniak 08

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  - Jouini/Kallal 95
  - Kabanov 99
  - Koehl/Pham/Touzi 01
  - Schachermayer 04
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EMM under transaction costs for one period

- For one period the gain following a self-financing trading strategy is

$$G^{\varphi} = \varphi(S_1 - S_0), \quad \varphi \quad \text{(number of stocks bought at 0)}.$$

For an EMM Q we have $\mathrm{E}_Q[G^{\varphi}] = 0$.
Thus $G^{\varphi} \geq 0$ implies $P(G^{\varphi} > 0) = 0$ for any φ (no arbitrage).

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EMM under transaction costs for one period

- For one period the gain following a self-financing trading strategy is

$$G^\varphi = \varphi(S_1 - S_0), \quad \varphi \quad \text{(number of stocks bought at 0)}.$$

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- With liquidation at 1, we have for $\varphi \geq 0$ gain

$$G^\varphi = \varphi\left((1-\mu)S_1 - (1+\lambda)S_0\right)$$

and thus

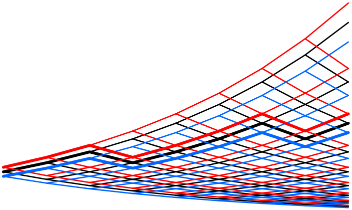
$$\mathrm{E}_Q[G^\varphi] \leq \varphi\, \mathrm{E}_Q[\rho_1 S_1 - \rho_0 S_0] = 0.$$

Similar for $\varphi < 0$

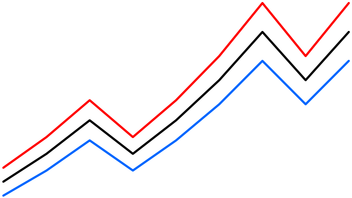
$$\mathrm{E}_Q[G^\varphi] = \mathrm{E}_Q[\varphi((1+\lambda)S_1 - (1-\mu)S_0)] = |\varphi|\, \mathrm{E}_Q[(1-\mu)S_0 - (1+\lambda)S_1] \leq 0.$$

Thus $\mathrm{E}_Q[G^\varphi] \leq 0$ for any φ and $G^\varphi \geq 0$ implies $P(G^\varphi > 0) = 0$.

Towards EMMs / consistent price systems under t.c.



Possible paths in 8-period CRR



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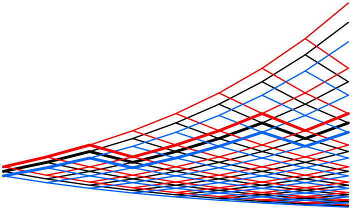

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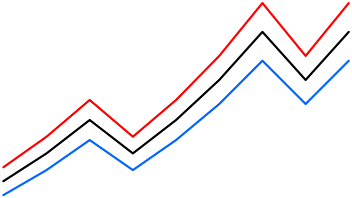

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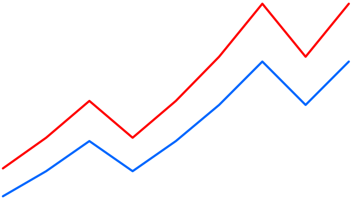

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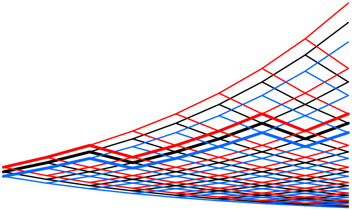

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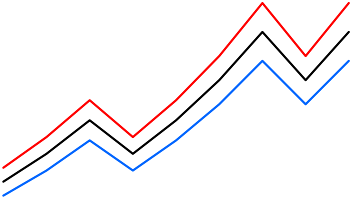

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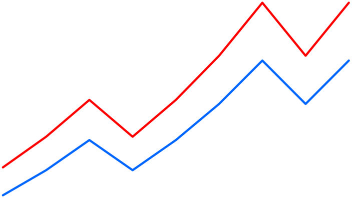

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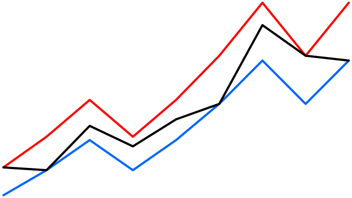

Path of ajusted price process

Consistent price systems under transaction costs

- Q is an EMM for factor $(\rho_n)_{n=0,\ldots,N}$, if
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- Ansatz: Choose $\rho_N = L(\pi_N^*)$ and $H_N = \overline{X}_N^*$ (growth optimal), i.e.

$$H_N = X_N^*(1 - \pi_N^* + \rho_N \pi_N^*).$$

Numeraire portfolio under transaction cots

Choosing $\rho_N = L(\pi_N^*)$ and $H_N = \overline{X}_N^* = X_N^*(1 - \pi_N^* + \rho_N \pi_N^*)$, we need to define ρ_n and H_n such that

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Corollary: E.g. for $x_0 = 1$, $\pi_0 = 0$ we have $H_0 = 1$. Thus (N4) holds.

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Interpretation: ρ_n is a liquidation factor adjusted to the fact that one behaves optimally in $n, \ldots, N$ and liquidates at N according to L.

Relation to shadow prices:Cvitanič/Karatzas 96,Kallsen/Muhle-Karbe 08

Some extensions

- Time dependent λ_n, μ_n.
- Previsible interest rates.
- Not identically distributed R_n.
  - Okay for $L \equiv 1$.
  - Otherwise more conditions needed to guarantee $\rho_n \in [1-\mu, 1+\lambda]$.
- Using power utility $U_\alpha(x) = \alpha^{-1}x^\alpha$ instead of $U(x) = \log(x)$ works similar, yielding price systems

$$\mathrm{pr}_\alpha(C) = \mathrm{E}_{\widetilde{Q}^\alpha}\left[\frac{C^B + \rho_N^\alpha C^S}{H_N^\alpha}\right],$$

where

$$\frac{d\widetilde{Q}^\alpha}{dP} = \frac{U'(H_N^\alpha)H_N^\alpha}{\mathrm{E}[U'((H_N^\alpha)H_N^\alpha]}.$$

$(\widetilde{Q}^\alpha, H_N^\alpha)$ is called numeraire pair.

Summing up: Numeraire portfolio under t.c.

To price contingent claims $C = (C^B, C^S)$ under prop. t.c. we proceed by

- choosing a liquidation factor $L = L(\pi)$,
- finding π_n^*, X_n^* for growth optimal φ^* by solving $\sup_{\varphi \text{adm.}} \mathrm{E}[\log(\overline{X}_N^{\varphi})]$,
- setting $\rho_N = L(\pi_N^*)$ and defining the adjusted value process

$$H_N := \overline{X}_N^* = X_N^*(1 - \pi_N^* + \rho_N \pi_N^*), \quad H_n^{-1} := \mathrm{E}[H_N^{-1} \mid \mathcal{F}_n],$$

  getting the adjustment factor from $H_n = X_n^*(1 - \pi_n^* + \rho_n \pi_n^*)$,
- starting with $H_0 = 1$ (e.g. $x_0 = 1$, $\pi_0 = 0$), define Q by $\frac{dQ}{dP} = H_N^{-1}$.

Then, Q is an EMM for factor ρ and thus

$$\mathrm{pr} : C = (C^B, C^S) \mapsto \mathrm{E}\left[\frac{C^B + \rho_N C^S}{H_N}\right]$$

is a consistent price system.