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A New Theory of Inter-temporal Equilibrium

for Security Markets

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Inter-temporal equilibrium

A new theory of inter-temporal equilibrium for security markets in a continuous time setting with Brownian Filtrations for complete and incomplete markets is developed. A *simple* characterization of equilibrium when agents maximize a state dependent utility functional, as proposed in J.A. Londoño (2009) State Dependent Utility, J. Appl. Prob. 46(1): 55-70, is given. It is shown that any equilibrium market is consistent with non-arbitrage. Some simple examples that include economies when securities pay no dividends or when there are no income for agents are presented. The theoretical framework used is a generalization of markets when the processes are Brownian Flows on Manifolds.

Motivation

- Lack of agreement with empirical data for the standard model of Optimal Consumption and Investment “Equity Premium Puzzle”, “risk-free rate puzzle” and “risk-aversion puzzle”.

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- State independent utilities are not appropriate for modeling the behavior of human beings.
- Existence of Inter-temporal equilibrium is troublesome. Very restrictive conditions. Most assume *a priori* complete markets. It is impossible to determine from primitives whether markets are complete.

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- In case of complete markets very restrictive conditions are imposed towards finding characterization of market completeness for inter-temporal equilibrium models. (For instance evolution of relevant variables to be deterministic functions of the underlying Brownian motion)

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- In case of incomplete markets essentially nothing is known. For instance some authors assume constant endowment process, etc.

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- Numerical representations for the structure of the optimal portfolios and consumption processes based on PDEs are difficult to evaluate
- Conditional expectation of semi-martingale processes are in general non computable.

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- A characterization and general theory of arbitrage and valuation for consistent processes assuming continuous coefficients and Brownian filtrations.(J.A. Londoño. A more General Valuation and Arbitrage Theory for Itô Processes. Stoch. Anal. Appl. 26, 809–831, 2008)

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- A simple (algebraic) and most general theory of arbitrage and valuation assuming continuous coefficients, and Brownian filtrations. (State Tameness: A New Approach for Credit Constrains, Elect. Comm. Prob., 9, (2004), 1-13.)

Some Definitions

- We assume a d-dimensional Brownian Motion starting at 0 $\{W(t), \mathcal{F}_t; 0 \leq t \leq T\}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

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- Let $(\mathcal{F}_{s,t}) = \{\mathcal{F}_{s,t}, 0 \leq s \leq t \leq T\}$ be the two parameter filtration $\mathcal{F}_{s,t} \equiv \sigma(W(u) - W(s) \mid s \leq u \leq t) \vee \mathcal{N}$.

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- Let $\varphi(s,t,x)$ and $\psi(s,t,x)$ be $C^{m,\delta}(\mathbb{D} \colon \mathbb{R}^n)$ and $C^{m,\delta}(\mathbb{D} \colon \mathbb{D})$-semi-martingale with $\psi(s,s,x) = x$ for all $x \in \mathbb{D}$. We say that φ is a ψ-*consistent process* if almost everywhere $\varphi(s,t,x) = \varphi(s',t,\psi(s,s',x))$. We say that the process φ is a *consistent process* if φ is a φ-consistent process.

- A **Utility function** $U:\ (0,\infty) \mapsto \mathbb{R}$ is continuous, strictly increasing, strictly concave and continuous differentiable, with $U'(\infty) = \lim_{x\to\infty} U'(x) = 0$ and $U'(0+) \triangleq \lim_{x\downarrow 0} U'(x) = \infty$ (We extend U by $U(0) = U(0^+)$, and we keep the same notation to the extension to $[0,\infty)$ of U).

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- We say a couple (U_1, U_2) is a **state preference structure** if $U_1\colon [0,T]\times(0,\infty) \mapsto \mathbb{R}$ is a continuous function where $U_1(t,\cdot)$ for all t and $U_2\colon (0,\infty)\mapsto \mathbb{R}$ are utility functions We denote by $I_1(t,x) \triangleq (\partial U_1(t,x)/\partial x)^{-1}$, and by $I_2(t,x) \triangleq (\partial U_2(t,x)/\partial x)^{-1}$ and $\mathcal{X}$ by

$$\mathcal{X}(t,y) \triangleq I_2(y) + \int_t^T I_1(t',y)\,dt'.$$

Some Definitions

- We assume a **state preference structure** (U_1, U_2) where there exist constants $\alpha_{s,t}$ and α_s^I such that $\alpha(s,t)(x) = \alpha_{s,t}x$, and $\alpha^I(s,s)(x) = \alpha_s^I x$ where $\alpha(s,t) = \mathcal{X}(s, \mathcal{X}^{-1}(t,\cdot))$ and $\alpha^I(s,t) \triangleq I_1(s, \mathcal{X}^{-1}(t,\cdot))$.

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- *Example 1* Assume $h\colon [0,T] \to (0,\infty)$, $U_1(t,x) = x^\alpha h(t)$ and $U_2(x) = cx^\alpha$ with $\alpha \in (0,1)$ and $c \geq 0$.

$$\alpha_{s,t} = \frac{c^{1/(1-\alpha)} + \int_s^T h^{1/(1-\alpha)}(t')\,dt'}{c^{1/(1-\alpha)} + \int_t^T h^{1/(1-\alpha)}(t')\,dt'},$$

$$\alpha_t^I = \frac{h^{1/(1-\alpha)}(t)}{c^{1/(1-\alpha)} + \int_t^T h^{1/(1-\alpha)}(t')\,dt'}$$

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- *Example 2.* $U_1(t,x) = h(t)\log(x)$ and $U_2(x) = c\log(x)$, with $c \geq 0$.

$$\alpha_{s,t} = \frac{c + \int_s^T h(t')\,dt'}{c + \int_t^T h(t')\,dt'}, \qquad \alpha_t^I = \frac{h(t)}{c + \int_t^T h(t')\,dt'}$$

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- *Example 3.* $U_1(t,x) = h(t)u(x/h(t))$ and $U_2(x) = cu(x/c)$, $u(\cdot)$ is a utility function, $h(\cdot)$ is a positive continuous function and $c > 0$.

$$\alpha_{s,t} = \frac{c + \int_s^T h(t')\,dt'}{c + \int_t^T h(t')\,dt'}, \qquad \alpha_t^I = \frac{h(t)}{c + \int_t^T h(t')\,dt'}$$

A **financial market with terminal time** T **and initial time** 0 $\mathcal{M} = (P, \Theta, \mathbb{D}, b, \sigma, \delta, \theta, \rho, \varrho, r, p^0, \vartheta^0)$ with feasible set of values $\mathbb{D}$ consist of:

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State process A m-dimensional Itô process Θ, for $m > 1$ of two parameters with values in $C^{2,0+}(\mathbb{K} \colon \mathbb{K})$ ($\mathbb{K}(s) \subset \mathbb{R}_+ \times \mathbb{R}^{m-1}$) and $\mathbb{K} \subset \mathbb{R}_+ \times \mathbb{R}^{m-1} \times [0, T]$ is some measurable set with a differential structure. For each initial condition $\vartheta = (\vartheta_0, \cdots, \vartheta_{m-1})^T \in \mathbb{K}(s)$:

$$\frac{d\Theta^0_{s,t}}{\Theta^0_{s,t}} = \left(r(t, \Theta_{s,t})\, dt + \|\theta(t, \Theta_{s,t})\|^2\right) dt + \sum_{1 \le j \le d} \theta^j(t, \Theta_{s,t})\, dW^j(t) \qquad \Theta^0_{s,s}(\vartheta) = \vartheta_0$$

State Process

$$\Theta^i_{s,t} = \rho^i(t, \Theta_{s,t})\, dt + \sum_{1 \le j \le d} \varrho^{i,j}(t, \Theta_{s,t})\, dW^j_s(t)$$

$$\Theta^i_{s,s}(\vartheta) = \vartheta_i, i = 1 \cdots, m-1$$

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Price of shares outstanding We assume a $n+m$-dimensional Itô process (P,Θ) of two parameters with values in $C^{2,0+}(\mathbb{D}\colon \mathbb{D})$ where $\mathbb{D}$ is a measurable set with sections $\mathbb{D}(s) \subset \mathbb{R}_+^n \times \mathbb{K}(s)$ for each $(p_1, \cdots, p_n, \vartheta_0, \cdots, \vartheta_{m-1})^T = (p^T, \vartheta^T)^T \in \mathbb{D}(s)$

$$\frac{dP^i_{s,t}}{P^i_{s,t}} = b^i(t, P_{s,t}, \Theta_{s,t})dt + \sum_{1\leq j\leq d} \sigma^{i,j}(t, P_{s,t}, \Theta_{s,t})\, dW^j_s(t)$$

$$P^i_{s,s}(p,\vartheta) = p_i, i = 1, \ldots, n$$

Non-state tame arbitrage opportunities: A Market price of risk $\theta^T(t,\vartheta) = \left(\theta^1(t,\vartheta), \cdots, \theta^d(t,\vartheta)\right)$, $\theta(t,\vartheta) \in \ker^{\perp}(\sigma(t,p,\vartheta))$, is of class $C^{2,0+}$ that satisfies,

$$b(t,p,\vartheta) + \delta(t,p,\vartheta) - r(t,\vartheta)\mathbf{1}_n = \sigma(t,p,\vartheta)\theta(t,\vartheta)$$

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Bond price process

$$B_{s,t} = \exp\{\int_s^t r(u,\Theta_{s,u})\,du\}.$$

State price density process is the $C(\mathbb{K} \colon \mathbb{R}_+)$ semi-martingale

$$H_{s,t}(\vartheta) = B_{s,t}^{-1} \times \exp\left\{ -\int_s^t \theta'(u, \Theta_{s,u})\, dW(u) - \frac{1}{2} \int_s^t \left(\|\theta(u, \Theta_{s,u})\|^2 \right) du \right\}$$

A **portfolio evolution structure** (π_0, π) **with rate of consumption** c**, and rate of endowment** Q **with feasible set of values** $\mathbb{X} \subset [0, T] \times \mathbb{R}^{n+m}$ consist of:

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- A **wealth evolution structure** (X, P, Θ) with values in $\mathbb{X}$ and a (X, P, Θ)-consistent process of class C^{0+}, (π^0, π) defined as

$$\{(\pi^0_{s,t}(x, p, \vartheta), \pi^i_{s,t}(x, p, \vartheta)); (x, p^T, \vartheta^T)^T \in \mathbb{X}(s), 0 \le s \le t \le T\}$$

with $\pi_0 + (\pi)^T \mathbf{1}_n = X$.

- A couple of non-negative C^{0+} consistent (X, P, Θ) processes $c = \{c_{s,t}(x,p,\vartheta), (x,p,\vartheta) \in \mathbb{X}(s)\}$ and $Q = \{Q_{s,t}(p,\vartheta), (p,\vartheta) \in \mathbb{D}(s)\}$ with

$$\mathbf{E}\left[\int_s^T H_{s,t}(\vartheta) c_{s.t}(x,p,\vartheta)\, dt\right] + \mathbf{E}\left[\int_s^T H_{s,t}(\vartheta) Q_{s,t}(p,\vartheta)\, dt\right] < \infty$$

with

$$B^{-1}_{s,t}(\vartheta)X_{s,t}(x,p,\vartheta) = x + \int_s^t B^{-1}_{s,u}(\vartheta)\left(Q_{s,u}(p,\vartheta) - c_{s,u}(x,p,\vartheta)\right) du$$

$$+ \int_s^t B^{-1}_{s,u}(\vartheta)(\pi)'_{s,u}(x,p,\vartheta)\sigma_{s,u}(p,\vartheta)\, dW_s(u)$$

$$+ \int_s^t B^{-1}_{s,u}(p)(\pi)'_{s,u}(x,p,\vartheta)(b_{s,u}(p,\vartheta) + \delta_{s,u}(p,\vartheta) - r_{s,u}(\vartheta)\mathbf{1}_n)\, du$$

where $b_{s,t} \equiv b(t, P_{s,t}, \Theta_{s,t})$, $\sigma_{s,t} \equiv \sigma(t, P_{s,t}, \Theta_{s,t})$, etc.

Let L be the **subsistence random field**

$$L_{s,t}(p,\vartheta) = \frac{-1}{H_{s,t}(\vartheta)} \mathbf{E}\left[\int_t^T H_{s,u}(\vartheta) Q_{s,u}(p,\vartheta)\, du \mid \mathcal{F}_{s,t} \right],$$

where (X, Q, c) is a hedgeable (by a state tame portfolio) cumulative consumption and endowment evolution structure, with portfolio evolution structure (π_0, π) and feasible set of values $\mathbb{X}$.

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(π,c) **is admissible for** (L,Q)**, and** $(\pi,c)\in\mathcal{A}(L,Q)$**:** if for any $(x,p^T,\vartheta^T)^T \in \mathbb{X}(s)$ with $x \geq L_{s,s}(p,\vartheta)$

$$X_{s,t}(x,p,\vartheta) \geq L_{s,t}(p,\vartheta) \qquad \text{for all } t$$

Smooth market condition There exist a $C^{2,0+}$ matrix valued function κ defined on $\mathbb{D}$ with the property that

$$\sigma^T(t,p,\theta)\kappa(t,p,\vartheta) = \theta(t,\vartheta).$$

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Remember: Market price of risk $\theta(t,\vartheta) \in \ker^{\perp}(\sigma(t,p,\vartheta)) = Im(\sigma^T(t,p,\vartheta))$ there is always existence of a measurable function κ. The condition is a weak condition on the smoothness of this property.

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Theorem Assume a consumer with hedgeable income structure Q (and hedging portfolio $\pi^{Q}_{s,t}$). Define $\mathbb{X} = \{(s, x, p^T, \vartheta^T)^T | x > \Pi(s, p, \vartheta), (p, \vartheta) \in \mathbb{D}(s)\}$.
Let ξ be defined as

$$\xi_{s,t}(x, p, \vartheta) \triangleq \Pi(t, P_{s,t}(p, \vartheta), \Theta_{s,t}(p, \vartheta)) + \alpha_{s,t} H^{-1}_{s,t}(\vartheta)(x - \Pi(s, p, \vartheta))$$

and let c be defined as

$$c_{s,t}(x, p, \vartheta) \triangleq ((\alpha^I_t / \alpha_{s,t}) H^{-1}_{s,t}(\vartheta)(x - \Pi(s, p, \vartheta))$$

for any $(x, p, \vartheta) \in \mathbb{X}(s)$ where Π is

$$\Pi(t, p, \vartheta) \triangleq -\mathbf{E}\left[\int_t^T H_{t,u}(\vartheta) Q_{t,u}(p, \vartheta)\, du\right]$$

is a $C^{2,0+}$ and it is assumed smooth market condition.

Then,

(ξ, c, Q) is a hedgeable cumulative consumption and endowment structure with values in $\mathbb{X}$, with portfolio $(\pi, c) \in \mathcal{A}(L, Q)$ that is optimal for the problem of optimal consumption and investment. An optimal portfolio is

$$\frac{\alpha_{t,s}(x - \Pi(s,p,\vartheta))}{H_{s,t}} \kappa(t, P_{s,t}, \Theta_{s,t}) - \pi^{Q}_{s,t}(-\Pi(s,p,\vartheta), p, \vartheta).$$

Moreover, $\pi_{s,t} + \eta_{s,t}$, is an optimal portfolio for any (X, P, Θ) consistent process $\eta_{s,t}(x,p,\vartheta) \in \ker \sigma^{T}_{s,t}(x,p,\vartheta)$ for $(x,p,\vartheta) \in \mathbb{X}(s)$.

Assume an economy $\mathcal{E}$ consisting of a financial market $\mathcal{M} = (P, \Theta, \mathbb{D}, \mathbb{K}, b, \sigma, \delta, \theta, \rho, \varrho, r, p^0, \vartheta^0)$ and m agents characterized by hedgeable (by state tame portfolios) rates of consumption and endowment evolution structures (X^j, c^j, Q^j) with feasible sets $\mathbb{X}^j$ and portfolio structures (π_0^j, π^j), with current value of future endowments L^j for defined by

$$L^j_{s,t}(p, \vartheta) = \frac{-1}{H_{s,t}(\vartheta)} \mathbf{E} \left[\int_t^T H_{s,u}(\vartheta) Q^j_{s,u}(p, \vartheta) \, du \mid \mathcal{F}_{s,t} \right]$$

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Assume a set $\mathbb{X}$ (with an appropriate differential structure) that satisfies that for any

$$(x_1, x_2, \cdots, x_m, p^T, \vartheta^T)^T \in \mathbb{X}(s) \text{ then } (x_i, p, \vartheta) \in \mathbb{X}^i(s) \quad \forall i$$

We say that the economy is at equilibrium with feasible set of values $\mathbb{X}$, if for any $(x_1, x_2, \cdots, x_m, p^T, \vartheta^T)^T \in \mathbb{X}(s)$:

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Martingale condition:

$$H_{s,t}(\vartheta) \sum_{j=1}^{m} X_{s,t}^{j}(x_j, p, \vartheta) + \int_s^t H_{s,u}(\vartheta) \sum_{j=1}^{m} \Big(c_{s,u}^{j}(x_j, p, \vartheta) - Q_{s,u}^{j}(p, \vartheta)\Big) du$$

is a martingale.

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Clearing of the money market:

$$\sum_{j=1}^{m}(X_{s,t}^{j}(x_j, p, \vartheta) - (\pi_{s,t}^{j})^{T}(x_j, p, \vartheta)\mathbf{1}_n) = 0$$

for all $s \le t \le T$.

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Clearing of the money market:

$$\sum_{j=1}^{m}(X^j_{s,t}(x_j,p,\vartheta)-(\pi^j_{s,t})^T(x_j,p,\vartheta)\mathbf{1}_n)=0$$

for all $s \le t \le T$.

Clearing of the commodity Market: For all $s \le t \le T$

$$\sum_{j=1}^{m} c^j_{s,u}(x_j,p,\vartheta)$$

$$=\sum_{j=1}^{m} Q^j_{s,u}(p,\vartheta)+\sum_{i=1}^{n}\delta^i(t,P_{s,t}(p,\vartheta),\Theta_{s,t}(\vartheta))P^i_{s,t}(p,\vartheta)$$

Clearing of the stock market:

$$\sum_{j=1}^{m} \pi_{s,t}^{j}(x_j, p, \vartheta) = \sum_{i=1}^{n} P_{s,t}^{i}(p, \vartheta)$$

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- The **aggregate endowment process** $Q^{\Sigma}_{s,t} = \sum_i Q^i_{s,t}(p,\vartheta)$ (where $Q^i_{s,s}(p,\vartheta) = q_i(s,p,\vartheta)$) satisfy

$$dQ^{\Sigma}_{s,t} = Q^{\Sigma}_{s,t}\gamma^0(t,\Theta_{s,t})\,dt + Q^{\Sigma}_{s,t}\sum_{1\le j\le d}\gamma^j(t,\Theta_{s,t})\,dW^j_s(t)$$

$$Q^{\Sigma}_{s,s}(s,\vartheta) = q_{\Sigma}(s,\vartheta) = \sum_{i=1}^{m} q_i(s,p,\vartheta) \quad (q_{\Sigma}, \text{etc. in } C^{2,0+})$$

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- The **aggregate dividend process** $D^{\Sigma}_{s,t} = \sum_i P^i_{s,t}\delta^i(t,P_{s,t},\Theta_{s,t})$ satisfy

$$dD^{\Sigma}_{s,t}(\vartheta) = D^{\Sigma}_{s,t}\rho^0(t,\Theta_{s,t})\,dt + \sum_{1\le j\le d} D^{\Sigma}_{s,t}\varrho^j(t,\Theta_{s,t})\,dW^j_s,$$

$$D^{\Sigma}_{s,s} = d_{\Sigma}(s,\vartheta), \qquad (d_{\Sigma}, \text{etc. in } C^{2,0+})$$

Main Theorem

- The **aggregate market price process** $P^{\Sigma}_{s,t}(\vartheta) = \sum_{1\le i\le n} P^{i}_{s,t}(p,\vartheta)$ only depends on the state variables, namely,

$$dP^{\Sigma}_{s,t}(\vartheta) = P^{\Sigma}_{s,t} b_{\Sigma}(t, \Theta_{s,t})\, dt + \sum_{1\le j\le d} P^{\Sigma}_{s,t}\sigma^{j}_{\Sigma}(t,\Theta_{s,t})\, dW^{j}_{s}$$

$$P^{\Sigma}_{s,s} = p_{\Sigma}(s,\vartheta) = \sum_{i} p_i. \qquad (p_{\Sigma},\ \text{etc. in } C^{2,+})$$

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$$P^{\Sigma}_{s,s} = p_{\Sigma}(s,\vartheta) = \sum_{i} p_i. \qquad (p_{\Sigma}, \text{ etc. in } C^{2,+})$$

- Assume the function

$$\Pi_{\Sigma}(t,\vartheta) = -\mathbf{E}\left[\int_t^T H_{t,u}(\vartheta) Q^{\Sigma}_{t,u}(\vartheta)\, du\right]$$

as well as each Π_j is in $C^{2,0+}$,

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Theorem Let $\mathcal{E}$ be an economy consisting of a free of (state) arbitrage opportunities smooth market $\mathcal{M}$, $(P, \Theta, \mathbb{D}, b, \sigma, \delta, \theta, \rho, \varrho, r, p^0, \vartheta^0)$ with feasible set of values $\mathbb{D}$, and m agents with identical preference structure (U_1, U_2) and *hedgeable endowment processes* Q^j and current value of future endowments L^j. Assume rate of consumption and endowment evolution structure (X_j, c_j, Q_j) with portfolio $((\pi)^j, c^j) \in \mathcal{A}(L^j, Q^j)$ given by the previous theorem. Define

$$\mathbb{X} = \{(s, x_1, x_2, \cdots, x_m, p^T, \vartheta^T)^T \mid (x_j, p^T, \vartheta^T)^T \in \mathbb{X}^j(s)$$

$$\text{for} \quad j = 1, \cdots, m, \text{ and } \sum_j x_j = p_\Sigma(s, \vartheta)\}$$

Also, assume previous conditions on aggregate endowment process, the aggregate dividends process and the market price process

Then there exist a choice of portfolio for each agent with $(\pi^j, c^j) \in \mathcal{A}(L^j, Q^j)$, $j = 1, \cdots, m$ where the economy is at equilibrium with feasible set of values $\mathbb{X}$ if and only if

$$H_{s,t}^{-1} = \frac{\Theta_{s,t}^0}{\vartheta_0} = \frac{\alpha_{s,t}\alpha_s^I}{\alpha_t^I} \frac{Q_{s,t}^{\Sigma} + D_{s,t}^{\Sigma}}{q_{\Sigma}(s,\vartheta) + d_{\Sigma}(s,\vartheta)},$$

and,

$$P_{s,t}^{\Sigma}(\vartheta) = \Pi_{\Sigma}(t, \Theta_{s,t}(\vartheta)) + \frac{1}{\alpha_t^I}\left\{Q_{s,t}^{\Sigma}(\vartheta) + D_{s,t}^{\Sigma}(\vartheta)\right\}.$$

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Proposition Assume a financial market $\mathcal{M}$ that is not necessarily free of state arbitrage opportunities. Assume that $r_{s,t}(\vartheta)$, $\theta_{s,t}(\vartheta)$, $b^{\Sigma}_{s,t} = b_{\Sigma}(t, \Theta_{s,t})$ and $\sigma^{j,\Sigma}_{s,t} = \sigma^{j}_{\Sigma}(t, \Theta_{s,t})$, $\forall j$ are the coefficients implied by the relations of the main theorem, and let $\delta^{\Sigma}_{s,t}(\vartheta) P^{\Sigma}_{s,t}(\vartheta) = D^{\Sigma}_{s,t}(\vartheta)$.

Then, for all s

$$b^{\Sigma}_{s,t} + \delta^{\Sigma}_{s,t} - r_{s,t} = \sum_{1 \leq j \leq d} \sigma^{j,\Sigma}_{s,t} \theta^{j}_{s,t}$$

for all t almost everywhere.

It is possible the construction of equilibrium markets when there exist a continuous function $h(s,t,\vartheta)$ (differentiable in t) such that

$$h(s,t,\vartheta)\frac{Q^{\Sigma}_{s,t}(\vartheta)}{Q^{\Sigma}_{s,t}(\vartheta)+D^{\Sigma}_{s,t}(\vartheta)}$$

Example

It is possible the construction of equilibrium markets when there exist a continuous function $h(s,t,\vartheta)$ (differentiable in t) such that

$$h(s,t,\vartheta)\frac{Q^{\Sigma}_{s,t}(\vartheta)}{Q^{\Sigma}_{s,t}(\vartheta)+D^{\Sigma}_{s,t}(\vartheta)}$$

For instance when $h(s,t,\vartheta)=1$ the condition of equilibrium implies that

$$P^{\Sigma}_{s,t}=\frac{1}{\alpha^{I}_{t}}D^{\Sigma}_{s,t}+\frac{1}{\alpha^{I}_{t}\alpha_{tT}}Q^{\Sigma}_{s,t}.$$

Some consequences

- When a consumer has a hedgeable endowment the problem of optimal consumption and investment becomes equivalent to one where the entire endowment process is replaced by its present value, in the form of an augmented initial wealth and no income is assumed.

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- When a consumer has a hedgeable endowment the problem of optimal consumption and investment becomes equivalent to one where the entire endowment process is replaced by its present value, in the form of an augmented initial wealth and no income is assumed.
- We may assume for purposes of pricing that there is just one agent who is endowed with the sum of the original endowments and have as initial wealth the aggregate initial wealth for the economy.
- The total market value of everything $P^{\Sigma}_{s,t} - \Pi_{\Sigma}(t, \Theta_{s,t})$ (shares plus future endowments) is proportional to the current rate of consumption, with a constant (for time t) of proportionality that is given by the preference structure.

$$\mathbf{E}[\int_0^T U_1(t, H_{0,t}(\vartheta)c_{0,t}(x,p,\vartheta))\,dt + U_2(H_{0,T}(\vartheta)\xi_{0,T}(x,p,\vartheta))]$$

$$\geq \mathbf{E}[\int_0^T U_1(t, H_{0,t}(\vartheta)\tilde{c}_{0,t}(x,p,\vartheta))\,dt + U_2(H_{0,T}(\vartheta)\tilde{\xi}_{0,T}(x,p,\vartheta))]$$

for $(x, p^T, \vartheta^T)^T \in \mathbb{X}(s)$, $(\tilde{\xi}, \tilde{c}, Q)$ is any hedgeable cumulative consumption and endowment $(\tilde{\pi}, \tilde{c}) \in \mathcal{A}(L, Q)$, with

$$\mathbf{E}\left[\int_0^T U_1^-(t, H_{0,t}(\vartheta)\tilde{c}_{0,t}(x,p,\vartheta))\,dt\right] + E\left[U_2^-(H_{0,T}(\vartheta)\tilde{\xi}_{0,T}(x,p,\vartheta))\right] < \infty$$

$U_1^-(t,x) = -(U_1(t,x) \wedge 0)$ and $U_2^-(x) = -(U_2(x) \wedge 0)$.

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