Simulation of Diversified Portfolios in a Continuous Financial Market

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Platen, E. & Heath, D.: A Benchmark Approach to Quantitative Finance


Benchmark Approach

Pl. & Heath (2006)

- Diversification Theorem
  well diversified portfolio approximates numeraire portfolio

- growth optimal portfolio equals numeraire portfolio, Long (1990)

- benchmark in portfolio optimization

- numeraire in derivative pricing
Figure 1: EWI114 and MCI.
Supermartingale Property

Assume numeraire portfolio $S_{tn}^{\delta^*}$ as benchmark s.t.

for all nonnegative portfolios $S_{tn}^\delta$

$$\frac{S_{tn}^\delta}{S_{tn}^{\delta^*}} = \hat{S}_{tn}^\delta \geq E_{tn} (\hat{S}_{tn+1}^\delta)$$

$S^{\delta^*}$ - best performing portfolio, numeraire portfolio, growth optimal

Kelly (1956), Long (1990), Becherer (2001), Pl. (2002),
• Benchmarked numeraire portfolio

\[ \hat{S}_{t_n}^{\delta_*} = 1 \implies \frac{\hat{S}_{t_{n+1}}^{\delta_*} - \hat{S}_{t_n}^{\delta_*}}{\hat{S}_{t_n}^{\delta_*}} = 0 \text{ a.s.} \]

• approximate \( \hat{S}^{\delta_*} \)

strictly positive
sequence of strictly positive, benchmarked portfolios

\[ (\hat{S}^{\delta_{\ell}})_{\ell \in \{1, 2, \ldots\}} \]

with \( \hat{S}_{0}^{\delta_{\ell}} = 1 \)

is a sequence of approximate numeraire portfolios

if for \( \varepsilon > 0 \)

\[
\lim_{\ell \to \infty} P_{t_{n}} \left( \frac{1}{\sqrt{t_{n+1} - t_{n}}} \left\lvert \frac{\hat{S}_{t_{n+1}}^{\delta_{\ell}} - \hat{S}_{t_{n}}^{\delta_{\ell}}}{\hat{S}_{t_{n}}^{\delta_{\ell}}} \right\rvert \geq \varepsilon \right) = 0
\]

for all \( n \in \{0, 1, \ldots\} \)
• financial market is **semi-regular** if for all $\varepsilon > 0$

$$\lim_{\ell \to \infty} P \left( \left| \frac{1}{\sqrt{\ell}} \sum_{j=1}^{\ell} \sigma_{t_n}^{j,k} \right| \geq \varepsilon \right) = 0$$

for all $k \in \{1, 2, \ldots\}$ and $n \in \{0, 1, \ldots\}$
Diversification Theorem:

In a semi-regular market each sequence of EWIs is a sequence of approximate numeraire portfolios.

Pl. (2005), Pl. & Rendek (2009)

Numeraire portfolio has only non-diversifiable risk!
Equally Weighted Index

EWI

\[ \pi_{j}^{\delta_{\text{EWI}},t} = \frac{1}{d} \]

\[ j \in \{1, 2, \ldots, d\} \]
Inverse Transform Method

- random variable $Y$ - distribution function $F_Y$
- uniformly distributed random variable $0 < U < 1$
  $F_Y$ distributed random variable $y(U)$

\[
U = F_Y(y(U))
\]

\[
y(U) = F_Y^{-1}(U)
\]

More generally

\[
y(U) = \inf\{y : U \leq F_Y(y)\}
\]
Transition Density of a Matrix SR-process

\(d \times m\) matrix

\[
p(s, X; t, Y) = \frac{p \delta \left( \varphi(s), \frac{X}{s_s}; \varphi(t), \frac{Y}{s_t} \right)}{s_t}
\]

- \(\varphi\)-time

\[
\varphi(t) = \varphi(0) + \frac{\bar{b}^2}{4\bar{c} s_0} \left(1 - \exp\{-\bar{c} t\}\right)
\]

\[
s_t = s_0 \exp\{\bar{c} t\} \text{ for } t \in [0, \infty), s_0 > 0, \bar{c} < 0 \text{ and } \bar{b} \neq 0
\]
Figure 2: Matrix valued square root process
Figure 3: Simulated benchmarked primary security accounts under the Black-Scholes model
Figure 4: Simulated GOP, EWI and MCI under the Black-Scholes model
Figure 5: Simulated benchmarked GOP, EWI and MCI under the Black-Scholes model
Multi-asset Heston Model

Heston (1993)

- matrix SDEs

\[ d\hat{S}_t = \text{diag} \left( \sqrt{V_t} \right) \text{diag} \left( \hat{S}_t \right) \left( A d\tilde{W}^1_t + B d\tilde{W}^2_t \right) \]

\[ dV_t = (a - EV_t) dt + F \text{diag} \left( \sqrt{V_t} \right) d\tilde{W}^1_t \]

\[ \hat{S}_t = (\hat{S}^0_t, \hat{S}^1_t, \ldots, \hat{S}^d_t)^\top \]

- independent vectors of correlated Wiener processes

\[ \tilde{W}^k_t = C^k W^k_t \]
• exact simulation in Broadie & Kaya (2006) (complicated, slow)

• simplified almost exact simulation Pl. & Rendek (2010)

• squared volatility $V_{t_{i+1}}^j$

  sampling directly from the noncentral chi-square distribution

  exact

• almost exact simulation

  $X_t = \ln(\hat{S}_t)$
• log-asset price

\[ X_{t_{i+1}}^j = X_{t_i}^j + \frac{\rho_j}{\gamma_j} (V_{t_{i+1}}^j - V_{t_i}^j - \alpha_j \Delta) + \left( \frac{\rho_j \kappa_j}{\gamma_j} - \frac{1}{2} \right) \int_{t_i}^{t_{i+1}} V_u^j \, du \]

\[ + \sqrt{1 - \rho_j^2} \int_{t_i}^{t_{i+1}} \sqrt{V_u^j} \, dW_u^{2,j} \]

with

\[ \int_{t_i}^{t_{i+1}} \sqrt{V_u^j} \, dW_u^{2,j} \]

conditionally Gaussian:

mean zero, variance \( \int_{t_i}^{t_{i+1}} V_u^j \, du \)
• approximate

\[ \int_{t_i}^{t_{i+1}} V^j_u \, du \]

• trapezoidal rule

\[ \int_{t_i}^{t_{i+1}} V^j_u \, du \approx \frac{\Delta}{2} \left( V^j_{t_i} + V^j_{t_{i+1}} \right) \]

\[ \Rightarrow \]

\[ \int_{t_i}^{t_{i+1}} \sqrt{V^j_u} \, dW^{2,j}_u \approx \mathcal{N} \left( 0, \frac{\Delta}{2} \left( V^j_{t_i} + V^j_{t_{i+1}} \right) \right) \]

achieved with high accuracy
efficient almost exact simulation technique
Figure 6: Simulated squared volatility under the Heston model
Figure 7: Simulated benchmarked GOP, EWI and MCI under the Heston model
Multi-asset ARCH-diffusion Model

Nelson (1990), Frey (1997)

\[
d\hat{S}_t = \text{diag} \left( \sqrt{V_t} \right) \text{diag} \left( \hat{S}_t \right) \left( A d\tilde{W}^1_t + B d\tilde{W}^2_t \right)
\]

\[
dV_t = \left( a - EV_t \right) dt + F \text{diag} \left( V_t \right) d\tilde{W}^1_t
\]

- squared volatility

\[
V_{t_{i+1}}^j = \exp \left\{ \left( -\kappa_j - \frac{1}{2} \gamma_j^2 \right) t_{i+1} + \gamma_j W_{t_{i+1}}^{1,j} \right\}
\]

\[
= \times \left( V_{t_0}^j + a_j \sum_{k=0}^{t_{k+1}} \left[ \exp \left\{ \left( \kappa_j + \frac{1}{2} \gamma_j^2 \right) s - \gamma_j W_s^{1,j} \right\} ds \right] \right)
\]
Figure 8: Simulated squared volatility under the ARCH-diffusion model
Figure 9: Simulated benchmarked GOP, EWI and MCI under the ARCH-diffusion model
Geometric Ornstein-Uhlenbeck Volatility Model

\[ d\hat{S}_t = \text{diag}(\exp\{V_t\}) \text{diag}(\hat{S}_t) \left( A d\tilde{W}_t^1 + B d\tilde{W}_t^2 \right) \]

\[ dV_t = (a - EV_t) \, dt + F d\tilde{W}_t^1 \]

simulation for \( \exp\{V^j\} \) exact

log-asset price as before
Figure 10: Simulated squared volatility under the geometric Ornstein-Uhlenbeck volatility model
Figure 11: Simulated benchmarked GOP, EWI and MCI under the geometric Ornstein-Uhlenbeck volatility model
Minimal Market Model


• $\varphi$-time

$$\varphi^j(t) = \frac{\alpha^j_0}{4\eta^j_0} \exp\{\eta^j_0 t\}$$

• benchmarked primary security account

$$d\hat{S}^j(\varphi^j(t)) = -2 \left( \hat{S}^j(\varphi^j(t)) \right)^{\frac{3}{2}} d\bar{W}^j(\varphi^j(t))$$

strict supermartingale

$$\hat{S}^j(\varphi^j(t_{i+1})) = \frac{1}{\sum_{k=1}^{4} \left( w^k + \bar{W}_{t_{i+1}}^{k,j} \right)^2}$$

exact simulation
Figure 12: Simulated benchmarked primary security accounts under the MMM
Figure 13: Simulated squared volatility under the MMM
Figure 14: Simulated benchmarked GOP, EWI and MCI under the MMM model
References


