# Stochastic Perturbations and Smooth Condition Numbers

FIELDS INSTITUTE, November 26, 2009

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## Introduction

Let $X$ and $Y$ be Riemannian manifolds, $\dim_{\mathbb{R}} X = m$, $\dim_{\mathbb{R}} Y = n$ associated to some computational problem, where

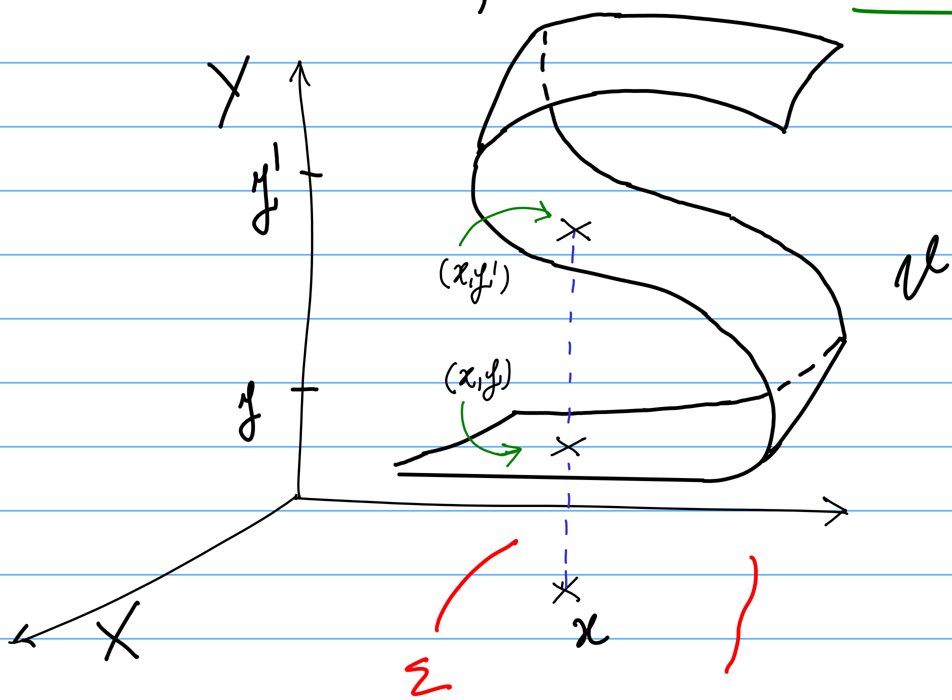
$X$ is the space of inputs and $Y$ is the space of outputs

Let $\mathcal{V} := \{ (x,y) \in X \times Y : y$ is an output corresponding to $x \}$ be the solution variety. Let $\pi_1$ and $\pi_2$ be the canonical projections:

$$\begin{tikzcd} & \mathcal{V} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

$\Sigma' \subset \mathcal{V}$ is the set of critical points of $\pi_1$, and $\Sigma := \pi_1(\Sigma')$.

Assume $\dim X = \dim \mathcal{V}$, then we have Jean-Pierre's Picture:



For each $(x,y) \in \mathcal{V} \setminus \Sigma$ there exist a smooth map $G$ locally defined, namely

$$G := \pi_2 \circ \pi_1^{-1}\big|_{U_x} : U_x \longrightarrow U_y$$

from some neighborhoods $U_x$ and $U_y$ of $x$ an $y$ respectively.

Then $DG(x) : (T_x X, \langle , \rangle_x) \longrightarrow (T_y Y, \langle , \rangle_y)$ is the condition operator at $(x,y)$

Def: The condition number at $(x,y) \in \mathcal{V} \setminus \Sigma'$ is defined as:

$$K(x,y) := \sup_{\substack{\dot{x} \in T_x X \\ \|\dot{x}\|_x^2 = 1}} \| DG(x)\dot{x} \|_y .$$

Remark 1: $K(x,y)$ is an uper-bound – to first-order approximation – of the **worst-case** sensitivity of the output error with respect to small perturbations of the input.

In many practical situations, however, there exist a discrepancy between worst case theoretical analysis and observed accuracy of algorithm.
There exist several approaches that attempt to rectify this discrepancy. Among them we find:

- average case-anaysis
- smooth analysis

In many problems, the space of inputs $X$ has a much larger dimension than the one of the space of outputs $Y$ ($m \gg n$).
Then, infinitesimal perturbations of the input will produce drastics changes in the output only when they are performed in a few directions.

Then, a possibly different approach to analyse accuracy of algorithm is to replace "worst-case" by a certain mean over all possible directions.
(alternative already suggest by Weis, Wasilkowski, Wozniakowski, Shub for linear system solving, and more generally by Stewart for matrix perturbations).

Def: We define the **pth-stochastic condition number** at $(x,y)$ as:

$$K_{st}^{[p]}(x,y) := \left[ \frac{1}{vol(S_x^{m-1})} \cdot \int_{\dot{x} \in S_x^{m-1}} \| DG(x)\dot{x} \|_y^p \, dS_x^{m-1}(\dot{x}) \right]^{1/p}$$

$$(p = 1, 2, \dots)$$

where $vol(S_x^{m-1}) = \frac{2\pi^{m/2}}{\Gamma(m/2)}$ is the measure of the unit sphere $S_x^{m-1} \subset T_x X$.

Notation: in the case $p=2$ we simply write $K_{st}$.

Def: We define the Frobenius condition number as

$$K_F(x,y)^2 = \| DG(x) \|_F^2 = \sigma_1^2 + \dots + \sigma_n^2$$

where $\| \cdot \|_F$ is the Frobenius norm and $\sigma_1, \dots, \sigma_n$ are the singular values of the condition operator.

**Theorem:** $$K_{st}^{[p]}(x,y) = \frac{1}{\sqrt{2}} \cdot \left[\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\right]^{1/p} \cdot \mathbb{E}\left(\|\eta_{\sigma_1,\dots,\sigma_n}\|^p\right)^{1/p},$$

where $\eta_{\sigma_1,\dots,\sigma_n}$ is a centered Gaussian vector in $\mathbb{R}^n$ with diagonal covariance matrix $\mathrm{Diag}(\sigma_1,\dots,\sigma_n)$, and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$.

In particular, for $p=2$

$$\boxed{K_{st}(x,y) = \frac{K_F(x,y)}{\sqrt{m}}} \qquad (m = \dim X)$$

**Remark:** This result is most interesting when $\dim X \gg \dim Y$ ($m \gg n$), for in that case

$$K_{st}(x,y) \leq \sqrt{\frac{n}{m}} \cdot K(x,y) \ll K(x,y), \quad \left(\text{since } K_F(x,y) \leq \sqrt{n} \cdot K(x,y)\right).$$

**Sketch of Proof:** (case $p=2$)

**Step 1:** $K_{st}(x,y)^2 = \frac{1}{m} \cdot \mathbb{E}\left(\|DG(x)\eta\|^2\right)$, where $\eta \sim \mathcal{N}(0, I_m)$. (Gaussian standard in $\mathbb{R}^m$)

Let $f: \mathbb{R}^m \longrightarrow \mathbb{R}$ be $f(v) = \|DG(x)v\|^2$. Since $f$ is homogeneous, integrating by polar coordinates we have:

$$\mathbb{E}(f(v)) \overset{\text{def.}}{=} \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} f(v) e^{\frac{-\|v\|^2}{2}} dv = C_m \cdot \int_{S^{m-1}} f(v)\, dS^{m-1}(v) = C_m' \cdot K_{st}(x,y)^2.$$

**Step 2:** $A \in M_{n\times m}$ then $\mathbb{E}\left(\|A\eta\|^2\right) = \|A\|_F^2$. $\quad (U \in \mathcal{O}_n, V \in \mathcal{O}_m)$

Let $A = UDV$ a s.v.d. $A$, where $D = \left(\begin{smallmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_n\end{smallmatrix} \middle| 0\right)$. Using the invariance of the Gaussian distribution and the Euclidean norm under the orthogonal group, we have

$$\mathbb{E}\left(\|UDV\eta\|^2\right) = \mathbb{E}\left(\|UD\eta\|^2\right) = \mathbb{E}\left(\|D\eta\|^2\right) = \mathbb{E}\left(\sigma_1^2\eta_1^2 + \dots + \sigma_n^2\eta_n^2\right) = \|A\|_F^2.$$

Remark: When $X$ and $Y$ are (finite dimensional) linear vector spaces, instead of considering the (absolute) condition number one can take the relative condition number defined as

$$K_{rel}(x;y) = \frac{\|x\|_X}{\|y\|_Y} \cdot K(x;y), \qquad x \neq 0,\ y \neq 0.$$

Theorem remains true if one exchange the condition number by the relative condition number.

## Some Examples

Systems of Linear Equations: solve for $y \in \mathbb{R}^n$: $Ay = b$.

Let $X = M_n(\mathbb{R})$ with the Frobenius inner product, and $Y = \mathbb{R}^n$ with the Euclidean inner product. In this case $\Sigma = \{A \in M_n(\mathbb{R}) : \det A = 0\}$.
The input-output map $G : M_n \setminus \Sigma \longrightarrow \mathbb{R}^n$ is given by

$$G(A) = A^{-1}b \ (= y)$$

Implicit differentiation yields $DG(A)\dot{A} = -A^{-1}\dot{A}A^{-1}b = -A^{-1}\dot{A}y$.
Then:

$$K(A) = \|A^{-1}\| \cdot \|y\| \quad \text{and} \quad K_F(A) = \|A^{-1}\|_F \cdot \|y\|.$$

From where we conclude:

$$K_{st}(A) = \frac{\|A^{-1}\|_F \cdot \|y\|}{n} \leq \frac{K(A)}{\sqrt{n}}.$$

Notice that $K_{rel}(A) = \|A\| \cdot \|A^{-1}\|$.
Edelman proved that
$$\boxed{\mathbb{E}(\log K_{rel}(A)) = \log n + c + o(1) \quad \text{as } n \to +\infty}$$
for $c \approx 1.537$,
where $A$ is a random matrix whose elements are i.i.d standard normal.
Then from Theorem we have:
$$\boxed{\mathbb{E}(\log K_{rel_{st}}(A)) \leq \frac{1}{2} \log n + c + o(1) \quad \text{as } n \to +\infty}$$

**Finding Roots Problem:** solve for $\zeta \in \mathbb{P}(\mathbb{C}^{n+1})$, $f(\zeta)=0$, where $f \in \mathbb{P}(\mathcal{H}_{(d)})$.

Let $(\mathcal{H}_{(d)}, \langle , \rangle_W)$ be the space of systems $f: \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n$, $f=(f_1, \dots, f_n)$ where $f_i$ is a homogeneous polynomial of degree $\underline{d_i}$, with the Weyl structure (see Blum-Cucker-Shub-Smale).
Let $X = \mathbb{P}(\mathcal{H}_{(d)})$ and $Y = \mathbb{P}(\mathbb{C}^{n+1})$ with the canonical Hermitian product
Then, the solution variety is given by:

$$V = \left\{ (f,\zeta) \in \mathbb{P}(\mathcal{H}_{(d)}) \times \mathbb{P}(\mathbb{C}^{n+1}) : f(\zeta) = 0 \right\}.$$

We denote by $N = \sum_{i=1}^{n} \binom{d_i+n}{n} - 1$ the complex dimension of $X$. We may think of $2N$ as the size of the input. For $(f,\zeta) \in V \setminus \Sigma'$ we have

$$DG(f)\dot{f} = -\left(Df(\zeta)|_{\zeta^\perp}\right)^{-1} \dot{f}(\zeta) \quad \text{then} \quad K_W(f,\zeta) = \left\| Df(\zeta)|_{\zeta^\perp}^{-1} \right\|$$

where some norm 1 affine representatives of $f$ and $\zeta$ have been chosen.

Associated with $K$ we consider $\quad K_W(f)^2 := \frac{1}{D} \sum_{\{\zeta : f(\zeta)=0\}} K_W(f,\zeta)^2, \quad (*)$

where $D = d_1 \cdots d_n$ is the number of roots of $f \in \mathbb{P}(\mathcal{H}_{(d)}) \setminus \Sigma$.
The expected value of $K_W^2$ with respect to the Weyl distribution is an essential ingredient in the complexity analysis of path-following methods (see Shub-Smale, Beltrán-Pardo, Bürgisser-Cucker).

Beltrán-Pardo proved that $\quad \boxed{\mathbb{E}(K_W(f)^2) \leq 8n \cdot N}$
The relation between $\underline{K_{st}}$ and $\underline{\text{complexity}}$ is not clear yet.
However is interesting to study the expected value of the $K_{st}$-analogous of $(*)$, namely $\quad K_{st_W}(f)^2 = \frac{1}{D} \sum_{\zeta : f(\zeta)=0} K_{st_W}(f,\zeta)^2.$
Then from Theorem: $\quad K_{st_W}(f,\zeta) \leq \dfrac{K_W(f,\zeta)}{\sqrt{N/n}}$ and $\quad \boxed{\mathbb{E}(K_{st_W}(f)^2) \leq 8n^2}$

Notice that the last bound depends on the number of unknowns $\underline{n}$, and NOT! on the size of the input $\underline{N \gg n}$.