Affine Springer fibers and compactified Jacobians

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As I said in my talk at the workshop, the p-adic orbital integrals which occur in the Langlands-Shelstad fundamental lemma for the unitary groups are of geometric nature. They are traces of Frobenius endomorphisms acting on ℓ-adic cohomology groups of the so-called affine Springer fibers for the full linear groups.

Today, I will expand the geometrical part of my talk at the workshop. I apologize in advance for the overlap between the two talks.

1. Affine Springer fibers

Let k be an arbitrary algebraically closed field and $F = k((\varpi_F))$. For every finite separable extension E of F, I denote as usual by $v_E : E^\times \to \mathbb{Z}$ the discrete valuation of E and by $\mathcal{O}_E$ its ring of integers.

We fix a non empty finite family $(E_i)_{i\in I}$ of finite separable

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extensions of F and

$$\gamma_I = \oplus_{i\in I}\gamma_i \in \bigoplus_{i\in I} E_i = E_I$$

satisfying the following properties :

- for every $i \in I$, $v_{E_i}(\gamma_i) > 0$,
- for every $i \in I$, γ_i generates E_i over F, so that $E_i = F[x]/(P_i(x))$ where $P_i(x) \in F[x]$ is the minimal polynomial of γ_i over F,
- the polynomials $P_i(x)$ are two-by-two distinct.

The affine Grassmannian Grass_I is the ind-scheme over k of $\mathcal{O}_F$-lattices $M \subset E_I$. Basic examples of lattices are $\mathcal{O}_{E_I} = \bigoplus_{i\in I} \mathcal{O}_{E_i} \subset E_I$ and $A_I = \mathcal{O}_F[\gamma_I] \subset \mathcal{O}_{E_I}$.

Any $\mathcal{O}_F$-lattice $M \subset E_I$ has an index $[M : A_I]$ with respect to the lattice A_I. For each integer d, we set $\mathrm{Grass}_I^d = \{M \in \mathrm{Grass}_I \mid [M : A_I] = d\}$. Then $(\mathrm{Grass}_I^d)_{d\in\mathbb{Z}}$ is the family of the connected components of Grass_I.

The *affine Springer fiber* at γ_I is the closed reduced ind-subscheme X_I of Grass_I whose closed points are the $\mathcal{O}_F$-lattices $M \subset E_I$ such that

$$\gamma_I M \subset M.$$

For each integer d, we set $X_I^d = X_I \cap \mathrm{Grass}_I^d$.

The lattice $A_I \subset E_I$ is in fact an $\mathcal{O}_F$-subalgebra of dimension 1 with total ring of fractions equal to E_I and with normalization $\widetilde{A}_I = \mathcal{O}_{E_I} \subset E_I$. The group $E_I^\times/A_I^\times$ is an extension

$$1 \to \widetilde{A}_I^\times/A_I^\times \to E_I^\times/A_I^\times \xrightarrow{v_I} \Lambda_I \to 0$$

where $\Lambda_I = \mathbb{Z}^I$ and where v_I is induced by the family $(v_{E_i})_{i\in I}$. Therefore, $E_I^\times/A_I^\times$ is in a natural way the group of k-points of a commutative group scheme $P_I^\natural$ over k which is smooth of dimension

$$\delta_I = \dim_k(\widetilde{A}_I/A_I),$$

and which has Λ_I as group of connected components. The connected component of the identity $P_I^{\natural,0}$ of P_I, the group of k-points of which is $\widetilde{A}_I^\times/A_I^\times$, is an extension of a torus

$$\mathbb{G}_{\mathrm{m},k}^I/\mathbb{G}_{\mathrm{m},k},$$

($\mathbb{G}_{\mathrm{m},k}$ is diagonally embedded into $\mathbb{G}_{\mathrm{m},k}^I$) by a commutative unipotent group scheme.

The action by homotheties of $E_I^\times/A_I^\times$ on the lattices in X_I obviously comes from an algebraic action of $P_I^\natural$ on X_I.

The lattice $A_I \subset E_I$ is a particular point $x_I \in X_I$. Its stabilizer in $P_I^\natural$ is trivial. The orbit $U_I = P_I^\natural \cdot x_I \subset X_I$ is thus an open subset of X_I which is isomorphic to $P_I^\natural$.

For every $a \in P_I^\natural$, we have

$$a \cdot X_I^d = X_I^{d+|v_I(a)|}$$

where $|\lambda| = \sum_{i\in I} \lambda_i$.

Theorem (Kazhdan-Lusztig). — *The affine Springer fiber X_I is k-scheme which is locally of finite type and of finite dimension. Its connected components are the X_I^d's, $d \in \mathbb{Z}$.*

For every section $\sigma : \Lambda_I^0 \to E_I^\times / A_I^\times = P_I^\natural(k)$ *of* v_I *over the subgroup* $\Lambda_I^0 = \{\lambda \in \Lambda_I \mid |\lambda| = 0\}$, *the quotient space* $Z_I = X_I/\sigma(\Lambda_I^0)$ *is the disjoint sum of projective* k*-schemes* $Z_I^d = X_I^d/\sigma(\Lambda_I^0)$ *for* $d \in \mathbb{Z}$. *Moreover the quotient map* $X_I \to Z_I$ *is an étale Galois covering with Galois group* Λ_I^0.

□

Examples : (i) Let us assume first that $|I| = 1$, $E = F[\varpi_E]/(\varpi_E^n - \varpi_F)$ and $\gamma = \varpi_E^m$ for two relatively prime positive integers n, m. Let V be a k-vector space of dimension $2\delta = (n-1)(m-1)$ and let N a regular nilpotent endomorphism of V. Then $\Lambda_I = \mathbb{Z}$ and $X_I^0 = Z_I^0$ is the closed subset of the Grassmannian variety of δ-planes in V whose points are the $W \subset V$ such that

$$N^n(W) \subset W \text{ et } N^m(W) \subset W.$$

Lusztig and Smelt have shown that this closed subset may be paved by standard affine spaces $\mathbb{A}_k^{d_\Gamma}$ parametrized by the subsets $\Gamma \subset \mathbb{Z}$ having the following two properties :

- $\Gamma - m\mathbb{N} - n\mathbb{N} \subset \Gamma$,
- there exists an integer $N \geq 0$ for which $\Gamma \subset]-\infty, N]$ and $|]-\infty, N] - \Gamma| = N$.

In particular, the dimension of the Springer fiber is δ and its Euler-Poincaré characteristic is $\frac{1}{m+n}\binom{m+n}{m}$.

For $n = 2$ and $m \geq 3$, we have $2\delta = m - 1$, $N^m = 0$ and the affine Springer fiber is an ordinary Springer fiber which admits the same paving

$$\mathbb{A}_k^0 \cup \mathbb{A}_k^1 \cup \cdots \cup \mathbb{A}_k^\delta$$

as the standard projective space $\mathbb{P}_k^\delta$ but which is not isomorphic to $\mathbb{P}_k^\delta$.

(ii) If $I=\{1,2\}$, $\mathrm{Car}(k)\neq 2$, $E_1=E_2=F$, $\gamma_1=\varpi_F$ and $\gamma_2=-\varpi_F$, X_I^0 is the chain of projective lines which is obtained by taking $\mathbb{Z}$ copies of $\mathbb{P}_k^1$ and by identifying the origin of the λ-th copy to the point at infinity of the $(\lambda+1)$-the copy. The group $\sigma(\Lambda_I^0)\cong\mathbb{Z}$ acts by translation on X_I^0 and Z_I^0 is the plane cubic curve with an ordinary double point which is obtained by identifying the origin and the point at infinity of $\mathbb{P}_k^1$. □

2. Compactified Jacobians

A trivial but nevertheless essential remark is that a lattice $M\in X_I$ is nothing else than a rank 1 torsion free A_I-module M which is equipped with a trivialization $E_I\otimes_{A_I}M\cong E_I$.

As $A_I=k[[\varpi_F,x]]/(P_I(x))$ where $P_I(x)=\prod_{i\in I}P_i(x)$, $\mathrm{Spf}(A_I)$ is a formal germ of singular plane curve. Thus it makes sense to try to find a projective curve C over k together with a closed point c of C for which A_I is the completion of the local ring $\mathcal{O}_{C,c}$. In fact there is a nice way to do that.

Lemma. — *There exists an integral projective curve C over k equipped with a close point c having the following properties :*

(1) *C is smooth over k outside c ;*

(2) $\widehat{\mathcal{O}}_{C,c}=A_I$ *;*

(3) *the normalization $\widetilde{C}$ of C is isomorphic to the standard*

projective line.

Proof : We consider the standard affine line $\widetilde{C}-\{\infty\} = \mathbb{A}^1_k \subset \mathbb{P}^1_k = \widetilde{C}$ and we arbitrarily choose a family $(\widetilde{c}_i)_{i\in I}$ of two-by-two distinct closed points in $\widetilde{C}-\{\infty\}$. Then, for each $i \in I$, we arbitrarily fix an isomorphism of the completion of the local ring $\mathcal{O}_{\widetilde{C},\widetilde{c}_i}$ of $\widetilde{C}$ at $\widetilde{c}_i$ onto $\mathcal{O}_{E_i}$. Finally, we construct C by simply replacing in $\widetilde{C}$ the semi-local ring $\prod_{i\in I} \mathcal{O}_{\widetilde{C},\widetilde{c}_i}$ by its intersection with A_I inside $\prod_{i\in I} \widehat{\mathcal{O}}_{\widetilde{C},\widetilde{c}_i} \cong \mathcal{O}_{E_I}$. □

Let $P = \mathrm{Pic}_{C/k}$ be the Picard scheme of C and $\overline{P}$ be the compactified Picard scheme of C :

- P is the moduli space of isomorphism classes of line bundles $\mathcal{L}$ over C; it is a commutative group scheme over k, whose group of connected components is $\mathbb{Z}$, the connected components being cut by the degree of $\mathcal{L}$; the connected component of the identity P^0 in P is affine and is thus an extension of a torus by a commutative unipotent group scheme;
- $\overline{P}$ is the moduli space of isomorphism classes of rank 1 torsion free coherent $\mathcal{O}_C$-Modules $\mathcal{M}$; it contains P as an open subset; its connected components $\overline{P}^d$ are again cut by the degree of $\mathcal{M}$ and are now projective over k; P acts on $\overline{P}$ by tensor product and

$$P^d \cdot \overline{P}^{d'} = \overline{P}^{d+d'},\ \forall d, d' \in \mathbb{Z}.$$

Theorem (Rego; Altman, Iarrobino et Kleiman). — *As the formal germ of C at its unique singular point c is the* *formal germ of a plane curve, every connected component* $\overline{P}^d$ *of* $\overline{P}$ *is integral and locally of complete intersection.*

Therefore, each connected component $\overline{P}^d$ *is a* P^0*-equivariant compactification of* P^d*.* □

What is the link between $\overline{P}$ and our affine Springer fiber X_I ?

We have a scheme morphism

$$X_I \to \overline{P}$$

which maps any lattice $M \subset E_I$ in X_I, viewed as a rank 1 torsion free module over A_I together with a trivialization $E_I \otimes_{A_I} M \cong E_I$, to the rank 1 torsion free coherent $\mathcal{O}_C$-Module which is obtained by gluing $\mathcal{O}_{C-\{c\}}$ and M along

$$\mathrm{Spec}(E_I) = (C - \{c\}) \times_C \mathrm{Spec}(A_I).$$

That morphism is surjective and equivariant with respect to the surjective homomorphism

$$P_I^\natural \twoheadrightarrow P$$

which maps $a \in E_I^\times / A_I^\times$ to the line bundle over C which is obtained by gluing as before $\mathcal{O}_{C-\{c\}}$ and $A_I = \widehat{\mathcal{O}}_{C,c}$ along $\mathrm{Spec}(E_I)$ with the transition function a.

We may identify $P_I^\natural$ with the moduli space of pairs $(\mathcal{L}, \iota)$ where $\mathcal{L}$ is a line bundle on C and ι is a trivialization of the restriction of $\mathcal{L}$ to $C - \{c\}$. In this identification, the surjective homomorphism $P_I^\natural \twoheadrightarrow P$ is simply the map which forgets ι and its kernel $K \subset P_I^\natural$ is nothing else that the abelian group

$$K = H^0(C - \{c\}, \mathbb{G}_{\rm m})/H^0(C, \mathbb{G}_{\rm m}).$$

Let us denote by $\{\widetilde{c}_i \mid i \in I\} \subset \widetilde{C}$ the fiber at c of the normalization morphism $\widetilde{C} \to C$. We then have

$$K = H^0(\widetilde{C} - \{\widetilde{c}_i \mid i \in I\}, \mathbb{G}_{\rm m})/H^0(\widetilde{C}, \mathbb{G}_{\rm m})$$

and K is canonically isomorphic to the group of degree 0 divisors on $\widetilde{C}$ the support of which is contained in $\{\widetilde{c}_i \mid i \in I\}$.

The morphism $v_I : P_I^\natural \twoheadrightarrow \Lambda_I$ maps any such divisor $\sum_{i\in I} \lambda_i[\widetilde{c}_i]$ to $\lambda \in \Lambda_I^0 \subset \Lambda_I$ and K is thus the image of a section σ of v_I over Λ_I^0.

PROPOSITION. — *The surjective morphism $X_I \to \overline{P}$ factors through a radicial morphism*

$$Z_I = X_I/K \to \overline{P}. \qquad \square$$

Remark : The above radicial morphism $Z_I \to \overline{P}$ is not in general an isomorphism. For example, if $|I| = 1$, $E = F[\varpi_E]/(\varpi_E^2 - \varpi_F)$ and $\gamma = \varpi_E^3$, then $Z_I^0 \cong \mathbb{P}_k^1$ but $\overline{P}^0$ is isomorphic to the plane cubic C with an ordinary cusp $x^3 = y^2$ as unique singularity.

More generally, if $|I| = 1$, so that C is unibranch, it may be true that the radicial morphism $Z_I^0 \to \overline{P}^0$ is simply the normalization map. $\square$

Corollary. — *Each connected component* Z_I^d *of* Z_I *is irreducible of dimension* δ*. The open subset* $U_I = P_I^\natural .x_I$ *is dense in* X_I*.* □

3. Galois coverings of the compactified Jacobians

Another consequence of our last proposition is that the étale Galois covering $X_I \to X_I/K = Z_I$ with Galois group $K \cong \Lambda_I^0$ descents to an étale Galois covering

$$\overline{P}^\natural \to \overline{P}$$

with the same Galois group, which extends, and which is equivariant with respect to, the surjective homomorphism $P_I^\natural \twoheadrightarrow P$ with kernel K.

In some sense, $\overline{P}^\natural$ is the moduli space of pairs $(\mathcal{M}, \iota)$ where $\mathcal{M}$ is a rank one torsion free coherent $\mathcal{O}_C$-Module and ι is a trivialization of $\mathcal{M}$ over the smooth locus $C - \{c\}$ of C.

It is useful to have the more direct definition of $\overline{P}^\natural \to \overline{P}$ that I will give now.

We may identify the Galois group

$$K = H^0(C - \{c\}, \mathbb{G}_{\mathrm{m}})/H^0(C, \mathbb{G}_{\mathrm{m}}) \cong \Lambda_I^0$$

to the character group $\mathrm{X}^*(T)$ of the maximal torus $T \cong \mathbb{G}_{\mathrm{m},k}^I/\mathbb{G}_{\mathrm{m},k}$ sitting inside P^0.

Estèves, Gagné and Kleiman have extended to singular curves the definition of the autoduality morphism for the Jacobians of the smooth projective curves. This morphism

$$\theta : P^0 \to \mathrm{Pic}_{\overline{P}/k}$$

maps $\mathcal{L} \in P^0$ to the line bundle $\theta(\mathcal{L})$ over $\overline{P}$ the fiber at $\mathcal{M} \in \overline{P}$ of which is

$$\theta(\mathcal{L})_{\mathcal{M}} = \mathcal{D}(\mathcal{M}) \otimes \mathcal{D}(\mathcal{L} \otimes \mathcal{M})^{\otimes -1}.$$

Here, for any coherent $\mathcal{O}_C$-Module $\mathcal{F}$, $\mathcal{D}(\mathcal{F}) = \det R\Gamma(C, \mathcal{F})$ is the determinant of the cohomology of $\mathcal{F}$.

PROPOSITION. — *The Grothendieck isomorphism*

$$H^1(\overline{P}, \mathrm{X}^*(T)) \xrightarrow{\sim} \mathrm{Hom}_{k-\mathrm{gr.sch}}(T, \mathrm{Pic}_{\overline{P}/k})$$

maps the $X^*(T)$*-torsor* $\overline{P}^\natural \to \overline{P}$ *to the restriction of* $\theta : P^0 \to \mathrm{Pic}_{\overline{P}/k}$ *to* $T \subset P^0$. □

4. Deformations

A major tool in the study of classical Springer fiber is the Grothendieck-Springer simultaneous resolution of the nilpotent cone.

For the affine Springer fibers, such a tool does not seem to be available. Indeed, let us consider the regular semisimple topologically nilpotent locus in the Lie algebra $\mathrm{gl}_n(F)$. Our data $\gamma_I = (\gamma_i \in E_i)_{i \in I}$ describe a $\mathrm{GL}_n(F)$-conjugacy class in this locus for $n = \dim_F(E_I)$ and any conjugacy class may be obtained in this way. Therefore, for any point γ in the regular semisimple topologically nilpotent locus, we have a corresponding Springer fiber X_γ ($X_\gamma \cong X_I$ if γ is conjugated to γ_I). But, if we let γ vary, it seems impossible to put in family the corresponding affine Springer fibers X_γ.

For example, let us consider the one parameter family

$$\gamma_z = \begin{pmatrix} z\varpi_F & \varpi_F^2 \\ \varpi_F & -z\varpi_F \end{pmatrix}$$

of regular semisimple topologically nilpotent elements in $\mathrm{gl}_2(F)$. The element γ_z is conjugated to γ_I where

$$(I = \{1,2\}, E_1 = E_E = F, \gamma_i = (-1)^i z\varpi_F(1 + z^{-1}\varpi_F)^{\frac{1}{2}})$$

if $z \neq 0$ and where

$$(|I| = 1, E = F[\varpi_E]/(\varpi_E^2 - \varpi_F), \gamma = \varpi_E^3)$$

if $z = 0$. Therefore, the connected component of the affine Springer fiber $X_{\gamma_z}^0$ is a chain of projective line if $z \neq 0$ and a single projective line if $z = 0$. It is hard to imagine an algebraic family with fibers $X_{\gamma_z}^0$.

The problem remains the same if we replace $X_{\gamma_z}^0$ by its quotient $Z_{\gamma_z}^0$. Indeed, $Z_{\gamma_z}^0$ is a plane cubic with an ordinary double point if $z \neq 0$ and a single projective line again if $z = 0$. But, if we replace the Z_{γ_z}'s by the corresponding compactified Jacobians, the problem disappears : there exists a one parameter family the generic fiber of which is a plane cubic with an ordinary double point and the special fiber of which is a plane cubic with an ordinary cusp!

The link between the affine Springer fibers and the compactified Jacobians is thus a powerful tool. It allows us to deform, up to homeomorphisms, the affine Springer fiber by deforming first the curve C, and then by taking the relative compactified Jacobian of the resulting relative curve over the base of the deformation.

5. Back to the geometric fundamental lemma

For simplicity, I will restrict myself to the key case where $I = \{1, 2\}$.

We have an action of the multiplicative group $\mathbb{G}_{\mathrm{m},k}$ on the affine Springer fiber $X_{1,2}^0$. It is induced by the action of $k^\times$ on $E_I = E_1 \oplus E_2$ which is given by

$$t \cdot (x_1 \oplus x_2) = tx_1 \oplus x_2 .$$

The fixed point set of this action is the set of lattices $M \subset E_{1,2}$ in $X_{1,2}^0$ which splits into a direct sum $M = M_1 \oplus M_2$ of lattices $M_\alpha \subset E_\alpha$. This fixed point set is thus nothing else than the disjoint sum of $X_1^d \times_k X_2^{-d}$ for $d \in \mathbb{Z}$, or equivalently $\mathbb{Z} \times (Z_1^0 \times_k Z_2^0)$ (recall that $X_\alpha^0 = Z_\alpha^0$).

Let us consider the map in $\mathbb{G}_{\mathrm{m},k}$-equivariant ℓ-adic homology which is induced by the inclusion of that fixed point set. It is a morphism

$$(*) \qquad H_\bullet(Z_1^0 \times_k Z_2^0)[\mathbb{Z}][t] \to H_\bullet^{\mathbb{G}_{\mathrm{m},k}}(X_{1,2}^0)$$

of graded $\mathbb{Q}_\ell[\mathbb{Z}][\partial]$-modules where t is of degree 2, where ∂ is the first Chern class of the universal line bundle on the classifying stack of $\mathbb{G}_{\mathrm{m},k}$ and where ∂ acts as the derivation which is defined by $\partial(t) = 1$.

The non trivial character $\kappa = \Lambda_I^0 = \mathbb{Z} \to \{\pm 1\}$ defines a maximal ideal $\mathfrak{m}$ of $\mathbb{Q}_\ell[\mathbb{Z}]$ and we may localize $(*)$ at $\mathfrak{m}$.

Let us assume from now on the purity conjecture :

Conjecture (Goresky, Kottwitz et MacPherson). — *For every integer m, the m-th groups of ordinary ℓ-adic cohomology of the Springer fibers $X_{1,2}^0$, Z_1^0 and Z_2^0 are all pure of weight m.*

Then the map $(*)$ and its localization $(*)_{\mathfrak{m}}$ are both surjective. Moreover, in order to complete the proof of the fundamental lemma for unitary groups, one only needs to show that the kernel N_s of $(*)_{\mathfrak{m}}$ is equal to the kernel of the action of ∂^r on $H_\bullet(Z_1^0 \times_k Z_2^0)[\mathbb{Z}]_{\mathfrak{m}}[t]$, that is

$$N_\eta = \bigoplus_{\rho=0}^{r-1} H_\bullet(Z_1^0 \times_k Z_2^0)[\mathbb{Z}]_{\mathfrak{m}} t^\rho \subset H_\bullet(Z_1^0 \times_k Z_2^0)[\mathbb{Z}]_{\mathfrak{m}}[t]$$

where $r = r_{1,2}$ is the power of q in the transfer factor.

In order to get this result, I proceed in two steps :

- first, I use a deformation, up to homeomorphisms, of $Z_{1,2}^0$ and its Galois covering $X_{1,2}^0 \to Z_{1,2}^0$ in order to obtain a specialization map which will gives the inclusion

$$N_\eta \subset N_s \subset H_\bullet(Z_1^0 \times_k Z_2^0)[\mathbb{Z}]_{\mathfrak{m}}[t];$$

- then, by considering the Leray spectral sequence for the non algebraic morphism $X_I \to X_1 \times_k X_2$ which maps $M \subset E_{1,2}$ to $(M_1 = M \cap M_1 \subset E_1, M_2 = \mathrm{pr}_{E_2}(M) \subset E_2)$, I show that the length of N_s as a $\mathbb{Q}_\ell[\mathbb{Z}]_{\mathfrak{m}}$-module is precisely r times the Euler-Poincaré characteristic of $Z_1^0 \times_k Z_2^0$.

Let me conclude this talk by describing the deformation that I use in the first step.

We have

$$A_{1,2} = k[[\varpi_F]][x]/(P_1(x)P_2(x))$$

where $P_\alpha(x) \in (\varpi_F, x) \subset k[[\varpi_F]][x]$, $\alpha = 1, 2$, are irreducible and relatively prime. The integer r is the intersection number of the two formal germs of irreducible plane curves $\mathrm{Spf}(A_1)$ and $\mathrm{Spf}(A_1)$ where $A_\alpha = k[[\varpi_F]][x]/(P_\alpha(x))$.

Let us denote $S = \mathrm{Spec}(k[[z]])$, s the closed point of S, $\eta = \mathrm{Spec}(K)$ its generic point and $\overline{\eta} = \mathrm{Spec}(\overline{K})$ a geometric point over η.

PROPOSITION. — *There exists a flat projective morphism $C \to S$ having the following properties :*

- *its geometric fibers are all integral and one dimensional, so that the singular locus D of C over S is finite over S ;*
- *the reduced special fiber of $D \to S$ is made of only one closed point c of C_s and the reduced generic fiber of $D \to S$ is made of two K-rational point c_1, c_2 and r closed points $(d_\rho)_{\rho=1,\ldots,r}$;*
- *the completion of the local ring of C_s at c is isomorphic to $A_{1,2}$;*
- *the completion of the local ring of $C_{\overline{\eta}}$ at c_α is isomorphic $\overline{K}\widehat{\otimes}_k A_\alpha$;*
- *each point d_ρ is an ordinary double point of the curve $C_{\overline{\eta}}$;*
- *the normalization $\widetilde{C}$ of C is S-isomorphic to $\mathbb{P}^1_S$ and the normalization map $\nu : \mathbb{P}^1_S \to C$ has the properties that*

its base changes $\nu_s : \mathbb{P}^1_s \to C_s$ *and* $\nu_{\overline{\eta}} : \mathbb{P}^1_{\overline{\eta}} \to C_{\overline{\eta}}$ *are the normalization of the special and the generic fiber of* C.

□

In fact, we construct C in such a way that the completion of the local ring of C at c is isomorphic to

$$k[[z, \varpi_F]][x]/(P_1(x)P_2(x-t)).$$

Having this deformation, we take its relative compactified Jacobians $\overline{P}^0$ over S and we checks that its geometric generic fiber $\overline{P}^0_{\overline{\eta}}$ is homeomorphic to

$$Z_1^0 \times_k Z_2^0 \times_k E^r$$

where E is the plane cubic over $\overline{K}$ with an ordinary double point which is the quotient of the chain $E^\natural$ of projective lines that we considered earlier. Moreover, we prove that the action of $\mathbb{G}_{\mathrm{m},k}$ on the special fiber $\overline{P}^0_s$ lifts to a natural action of $\mathbb{G}_{\mathrm{m},\overline{K}}$ on $\overline{P}^0_{\overline{\eta}}$ which is homeomorphic to the action on $Z_1^0 \times_k Z_2^0 \times_k E^r$ which is trivial on the first two factors and which is the diagonal action on the third one ($\mathbb{G}_{\mathrm{m},\overline{K}}$ acts in the usual way on E).

In equivariant homology as well as in ordinary homology there is a functorial specialization map from the homology of the generic fiber to the homology of the special fiber each time we have an algebraic family over S. Therefore, we get a commutative diagram of graded $\mathbb{Q}_\ell[\mathbb{Z}^r][\partial]$-modules

$$\begin{array}{ccc} H_\bullet(Z_1^0 \times_k Z_2^0)[\mathbb{Z}^r][t] & \longrightarrow & H_\bullet(Z_1^0 \times_k Z_2^0)[\mathbb{Z}][t] \\ \downarrow & & \downarrow \\ H_\bullet(Z_1^0 \times_k Z_2^0) \otimes H_\bullet^{\mathbb{G}_{\mathrm{m},k}}((E^\natural)^r) & \longrightarrow & H_\bullet^{\mathbb{G}_{\mathrm{m},k}}(X_{1,2}^0) \end{array}$$

where the action of $\mathbb{Q}_\ell[\mathbb{Z}^r]$ on the right column factors through the augmentation epimorphism $\mathbb{Q}_\ell[\mathbb{Z}^r] \twoheadrightarrow \mathbb{Q}_\ell[\mathbb{Z}]$.

More precisely,

- the two horizontal arrows are specialization maps in equivariant homology and the top one is nothing else than the tensor product of the augmentation epimorphism by $H_\bullet(Z_1^0 \times_k Z_2^0)[t]$;
- the vertical arrows are induced by the inclusions of the fixed point sets of the $\mathbb{G}_{\mathrm{m},k}$-action on the generic fiber for the left one, and on the special fiber for the right one, and they are surjective thanks to the purity conjecture.

Therefore our N_η, which is nothing else than the localization at $\mathfrak{m}$ of the image of the kernel of the left vertical arrow by the top horizontal arrow, is contained in the kernel of the localization at $\mathfrak{m}$ of the right vertical arrow map, which is by definition our N_s. Therefore, we are done.