Incorporating Managerial Cash-Flow Estimates and Risk Aversion to Value Real Options Projects

The Fields Institute for Mathematical Sciences

Sebastian Jaimungal
sebastian.jaimungal@utoronto.ca

Yuri Lawryshyn
yuri.lawryshyn@utoronto.ca

University of Toronto, Toronto, Canada

November 26, 2014
Agenda

- Introduction
  - Motivation
  - Real Options
- Matching cash-flows
  - General approach (numerical solution)
  - Normal distribution (analytical solution)
- Indifference pricing
  - General approach (numerical solution)
  - Normal distribution (analytical solution)
- Results
- Practical implementation
- Conclusions
Motivation

- To develop a theoretically consistent real options approach to value R&D type projects

**Theoretical Approaches**: Cash-flow determined by GBM

\[ df_t = \mu f_t dt + \sigma f_t dW_t \]

- **Practice**: Managerial supplied cash-flow estimates consist of low, medium and high values

<table>
<thead>
<tr>
<th>Economic Profit (Optimistic)</th>
<th>F'0</th>
<th>F'1</th>
<th>F'2</th>
<th>F'3</th>
<th>F'4</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>120</td>
<td>150</td>
<td>180</td>
<td>200</td>
<td></td>
</tr>
<tr>
<td>Economic Profit (Likely)</td>
<td>50</td>
<td>70</td>
<td>75</td>
<td>80</td>
<td>90</td>
</tr>
<tr>
<td>Economic Profit (Pessimistic)</td>
<td>20</td>
<td>25</td>
<td>25</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>
Valuation of R&D Projects: Managerial Sales and Cost Estimates

- Managers provide sales and cost estimates

**Table: Managerial Supplied Cash-Flow (Millions $).**

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sales</td>
<td>10.00</td>
<td>30.00</td>
<td>50.00</td>
<td>100.00</td>
<td>100.00</td>
<td>80.00</td>
<td>50.00</td>
<td>30.00</td>
</tr>
<tr>
<td>COGS</td>
<td>6.00</td>
<td>18.00</td>
<td>30.00</td>
<td>60.00</td>
<td>60.00</td>
<td>48.00</td>
<td>30.00</td>
<td>18.00</td>
</tr>
<tr>
<td>GM</td>
<td>4.00</td>
<td>12.00</td>
<td>20.00</td>
<td>40.00</td>
<td>40.00</td>
<td>32.00</td>
<td>20.00</td>
<td>12.00</td>
</tr>
<tr>
<td>SG&amp;A</td>
<td>0.50</td>
<td>1.50</td>
<td>2.50</td>
<td>5.00</td>
<td>5.00</td>
<td>4.00</td>
<td>2.50</td>
<td>1.50</td>
</tr>
<tr>
<td>EBITDA</td>
<td>3.50</td>
<td>10.50</td>
<td>17.50</td>
<td>35.00</td>
<td>35.00</td>
<td>28.00</td>
<td>17.50</td>
<td>10.50</td>
</tr>
<tr>
<td>CAPEX</td>
<td>1.00</td>
<td>3.00</td>
<td>5.00</td>
<td>10.00</td>
<td>10.00</td>
<td>8.00</td>
<td>5.00</td>
<td>3.00</td>
</tr>
<tr>
<td>Cash-Flow</td>
<td>2.50</td>
<td>7.50</td>
<td>12.50</td>
<td>25.00</td>
<td>25.00</td>
<td>20.00</td>
<td>12.50</td>
<td>7.50</td>
</tr>
</tbody>
</table>
Standard NPV Approach Using CAPM

- Ryan and Ryan (2002) report that 83% of businesses apply the WACC to value discounted cash-flows (DCF)
- CAPM: \( \mathbb{E}[r_E] = r_f + \beta_C(\mathbb{E}[r_M] - r_f) \)
- Use of CAPM implies beta: \( \beta_C = \frac{\rho_{M,C} \sigma_C}{\sigma_M} \)
Standard NPV Approach Using CAPM

- Ryan and Ryan (2002) report that 83% of businesses apply the WACC to value discounted cash-flows (DCF)
- CAPM: $\mathbb{E} [r_E] = r_f + \beta_C (\mathbb{E} [r_M] - r_f)$
- Use of CAPM implies beta: $\beta_C = \frac{\rho_{M,C} \sigma_C}{\sigma_M}$
- Some assumptions regarding $\beta$ when using WACC
  - Market volatility, $\sigma_M$, is known .................. ✓
  - Cash-flow volatility: $\sigma_{project} = \sigma_C$ .................. ?
  - Correlation of the cash-flows: $\rho_{project} = \rho_C$ .................. ?
Standard NPV Approach Using CAPM

- Ryan and Ryan (2002) report that 83% of businesses apply the WACC to value discounted cash-flows (DCF)
- CAPM: \( \mathbb{E}[r_E] = r_f + \beta_C (\mathbb{E}[r_M] - r_f) \)
- Use of CAPM implies beta: \( \beta_C = \frac{\rho_{M,C} \sigma_C}{\sigma_M} \)
- Some assumptions regarding \( \beta \) when using WACC
  - Market volatility, \( \sigma_M \), is known .................................................. ✓
  - Cash-flow volatility: \( \sigma_{\text{project}} = \sigma_C \) ..............................................?
  - Correlation of the cash-flows: \( \rho_{\text{project}} = \rho_C \) ...............................?
- Some further assumptions regarding DCF:
  - No managerial flexibility / optionality imbedded in the project
  - Financial risk profile of the value of the cash-flows matches that of the average project of the company
Standard NPV Approach Using CAPM

- Ryan and Ryan (2002) report that 83% of businesses apply the WACC to value discounted cash-flows (DCF)
- CAPM: \( \mathbb{E}[r_E] = r_f + \beta_C (\mathbb{E}[r_M] - r_f) \)
- Use of CAPM implies beta: \( \beta_C = \frac{\rho_{M,C}\sigma_C}{\sigma_M} \)
- Some assumptions regarding \( \beta \) when using WACC
  - Market volatility, \( \sigma_M \), is known \( \checkmark \)
  - Cash-flow volatility: \( \sigma_{\text{project}} = \sigma_C \)
  - Correlation of the cash-flows: \( \rho_{\text{project}} = \rho_C \)
- Some further assumptions regarding DCF:
  - No managerial flexibility / optionality imbedded in the project
  - Financial risk profile of the value of the cash-flows matches that of the average project of the company
- Proper beta: \( \beta_{\text{project}} = \frac{\rho_{M,\text{project}}\sigma_{\text{project}}}{\sigma_M} \)
Standard NPV Approach Using CAPM

- Ryan and Ryan (2002) report that 83% of businesses apply the WACC to value discounted cash-flows (DCF)
- CAPM: \( \mathbb{E}[r_E] = r_f + \beta_C (\mathbb{E}[r_M] - r_f) \)
- Use of CAPM implies beta: \( \beta_C = \frac{\rho_{M,C}\sigma_C}{\sigma_M} \)
- Some assumptions regarding \( \beta \) when using WACC
  - Market volatility, \( \sigma_M \), is known
  - Cash-flow volatility: \( \sigma_{\text{project}} = \sigma_C \)
  - Correlation of the cash-flows: \( \rho_{\text{project}} = \rho_C \)
- Some further assumptions regarding DCF:
  - No managerial flexibility / optionality imbedded in the project
  - Financial risk profile of the value of the cash-flows matches that of the average project of the company
- Proper beta: \( \beta_{\text{project}} = \frac{\rho_{M,\text{project}}\sigma_{\text{project}}}{\sigma_M} \)
- Matching method uses managerial supplied cash-flow estimates to determine \( \sigma_{\text{project}} \)
Real Options

Why real options?

- Superior to discounted cash flow (DCF) analysis for capital budgeting / project valuation
- Accounts for the inherent value of managerial flexibility
- Adoption rate ~12% in industry (Block (2007))

What is required?

- Consistency with financial theory
- Intuitively appealing
- Practical to implement
## Introduction: Real Options Approaches

<table>
<thead>
<tr>
<th></th>
<th>Intuitive</th>
<th>Practical / Easy to Implement</th>
<th>Financially Consistent</th>
<th>Minimal Subjectivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classic Approach</td>
<td>✓</td>
<td>✗</td>
<td>✓</td>
<td>-</td>
</tr>
<tr>
<td>Subjective Approach</td>
<td>✓</td>
<td>✓</td>
<td>-</td>
<td>✗</td>
</tr>
<tr>
<td>Market Asset Disclaimer</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
<td>✗</td>
</tr>
<tr>
<td>Revised Classic Approach</td>
<td>✓</td>
<td>✗</td>
<td>✓</td>
<td>-</td>
</tr>
<tr>
<td>Integrated Approach</td>
<td>✓</td>
<td>✗</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

* As classified by Borison (2005)
Berk et al.\textsuperscript{1} developed a real options framework for valuing early stage R&D projects

- Accounts for: technical uncertainty, cash-flow uncertainty, obsolescence, cost uncertainty
- Value of the project is a function of a GBM process representing the cash-flows
- \textbf{Main issue: how to fit real managerial cash-flow estimates to a GBM process}

Miao and Wang\textsuperscript{2}, and Henderson\textsuperscript{3}

- Present incomplete market real options models that show standard real options, which assume complete markets, can lead to contradictory results

\textsuperscript{1}See Berk, Green, and Naik (2004).
\textsuperscript{2}See Miao and Wang (2007).
\textsuperscript{3}See Henderson (2007).
Matching Method Advantages

- The approach utilizes managerial cash-flow estimates
- The approach is theoretically consistent
  - Provides a mechanism to account for systematic versus idiosyncratic risk
  - Provides a mechanism to properly correlate cash-flows from period to period
- The approach requires little subjectivity with respect to parameter estimation
- The approach provides a **missing link** between practical estimation and theoretical frameworks
**RO in R&D Applications: Managerial Cash-Flow Estimates**

- Managers provide cash flow estimates

<table>
<thead>
<tr>
<th>Scenario</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimistic</td>
<td>0</td>
<td>0</td>
<td>80</td>
<td>120</td>
<td>150</td>
<td>180</td>
<td>200</td>
<td>220</td>
<td>250</td>
</tr>
<tr>
<td>Most likely</td>
<td>0</td>
<td>0</td>
<td>50</td>
<td>70</td>
<td>75</td>
<td>80</td>
<td>90</td>
<td>100</td>
<td>110</td>
</tr>
<tr>
<td>Pessimistic</td>
<td>0</td>
<td>0</td>
<td>20</td>
<td>25</td>
<td>25</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Investment</td>
<td>450</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Diagram:**

- Cash flow estimates provided by managers
- Invest K to receive future cash flows
- **V₀**
- **K**

**Expected Cash Flows per Year**
RO in R&D Type Applications: Two Approaches

- Managers supply low, medium and high sales and cost estimates (numerical solution)
- Managers supply $\pm$ sales and cost estimates from which a standard deviation can be determined for a normal distribution (analytical solution)
### RO in R&D Type Applications: Low, Medium and High Sales and Cost Estimates

- Managers supply revenue and GM% estimates

<table>
<thead>
<tr>
<th>Scenario</th>
<th>End of Year Sales (Margin%)</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimistic</td>
<td></td>
<td>80</td>
<td>116</td>
<td>153</td>
<td>177</td>
<td>223</td>
<td>268</td>
<td>314</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(50%)</td>
<td>(60%)</td>
<td>(65%)</td>
<td>(60%)</td>
<td>(60%)</td>
<td>(55%)</td>
<td>(55%)</td>
</tr>
<tr>
<td>Most Likely</td>
<td></td>
<td>52</td>
<td>62</td>
<td>74</td>
<td>77</td>
<td>89</td>
<td>104</td>
<td>122</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(30%)</td>
<td>(40%)</td>
<td>(40%)</td>
<td>(40%)</td>
<td>(35%)</td>
<td>(35%)</td>
<td>(35%)</td>
</tr>
<tr>
<td>Pessimistic</td>
<td></td>
<td>20</td>
<td>23</td>
<td>24</td>
<td>18</td>
<td>20</td>
<td>20</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(20%)</td>
<td>(20%)</td>
<td>(20%)</td>
<td>(20%)</td>
<td>(15%)</td>
<td>(10%)</td>
<td>(10%)</td>
</tr>
<tr>
<td>SG&amp;A*</td>
<td></td>
<td>10%</td>
<td>5%</td>
<td>5%</td>
<td>5%</td>
<td>5%</td>
<td>5%</td>
<td>5%</td>
</tr>
<tr>
<td>Fixed Costs</td>
<td></td>
<td>30</td>
<td>25</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

* Sales / General and Administrative Costs
## RO in R&D Type Applications: ± Sales and Cost Estimates

<table>
<thead>
<tr>
<th>End of Year Sales (Margin)</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sales</td>
<td>52 ± 10</td>
<td>62 ± 12</td>
<td>74 ± 15</td>
<td>77 ± 15</td>
</tr>
<tr>
<td>COGS (31 ± 6)</td>
<td>(37 ± 7)</td>
<td>(44 ± 9)</td>
<td>(46 ± 10)</td>
<td></td>
</tr>
<tr>
<td>SG&amp; A 10%</td>
<td>5%</td>
<td>5%</td>
<td>5%</td>
<td>5%</td>
</tr>
<tr>
<td>CAPEX (30 ± 6)</td>
<td>(25 ± 5)</td>
<td>(20 ± 4)</td>
<td>(20 ± 14)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_S$ (Sales)</td>
<td>5.20</td>
<td>6.20</td>
<td>7.40</td>
<td>7.70</td>
</tr>
<tr>
<td>$\sigma_C$ (COGS)</td>
<td>3.12</td>
<td>3.72</td>
<td>4.44</td>
<td>4.62</td>
</tr>
<tr>
<td>$\sigma_{EX}$ (CAPEX)</td>
<td>3.00</td>
<td>2.50</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>$\sigma_{CF}$ (Cash-Flow)</td>
<td>4.61</td>
<td>4.94</td>
<td>5.55</td>
<td>5.75</td>
</tr>
</tbody>
</table>

\[
\sigma_{CF} = \sqrt{\sigma_S^2 + \sigma_C^2 + \sigma_{EX}^2 - 2\rho_{S,C}\sigma_S\sigma_C - 2\rho_{S,EX}\sigma_S\sigma_{EX} + 2\rho_{S,C}\rho_{S,EX}\sigma_S\sigma_C\sigma_{EX}}
\]
Real Options in R&D Type Applications

- **Problem:**
  - How should we value the cash flows?
  - How should we account for managerial risk aversion?
- **Approach:**
  - Apply “matching method” with MMM to value cash flows
  - Apply indifference pricing to determine the value with manager’s risk aversion
- **Why Account for Risk Aversion:**
  - MMM assumes investors are fully diversified
  - Impact of managerial risk aversion on the valuation of a real options project can enhance decision making
Market Stochastic Driver

- Traded index / asset

\[ dl_t = \mu l_t dt + \sigma l_t dW_t \]

- Assume there exists a Market Stochastic Driver / Indicator correlated to the traded index

\[ dS_t = \nu S_t dt + \eta S_t (\rho dW_t + \sqrt{1 - \rho^2} dW_t^\perp) \]

- Market stochastic driver
  - does not need to be traded
  - could represent market size / revenues
  - is not constrained to a GBM process

- Risk-neutral MMM

\[ dl_t = rl_t dt + \sigma l_t d\widetilde{W}_t \]

\[ dS_t = \hat{\nu} S_t dt + \rho \eta S_t \left( d\widetilde{W}_t + \sqrt{1 - \rho^2} dW_t^\perp \right) \]

\[ \hat{\nu} = \nu - \frac{\rho \eta}{\sigma} (\mu - r) \]
Match Cash Flow Payoff
Each cash flow is effectively an option on the market stochastic driver, \( V_T = \varphi(S_T) \), and so, we match probabilities

\[
\mathbb{P}(\varphi(S_T) < \nu) = F^*(\nu) \\
\mathbb{P}(S_T < \varphi^{-1}(\nu)) = F^*(\varphi(S)) \\
\mathbb{P}(S_0 e^{(\nu - \frac{\eta^2}{2})T + \eta \sqrt{T} Z} < S) = F^*(\varphi(S)), \quad Z \sim N(0, 1) \\
\mathbb{P}\left(Z < \frac{\ln \frac{S}{S_0} - (\nu - \frac{\eta^2}{2})T}{\eta \sqrt{T}}\right) = F^*(\varphi(S)) \\
\Phi\left(\frac{\ln \frac{S}{S_0} - (\nu - \frac{\eta^2}{2})T}{\eta \sqrt{T}}\right) = F^*(\varphi(S))
\]
Match Cash Flow Payoff

\[
\phi(S) = F_{\text{ind}}^{-1}\left\{ \Phi\left( \frac{\ln \frac{S}{S_0} - (\nu - \frac{\eta^2}{2}) T}{\eta \sqrt{T}} \right) \right\}
\]
Information Distortion

(a) Market Indicator Flow

(b) Distorted Distribution Flow
Theorem

The GBM Risk-Neutral Distribution. The conditional distribution function \( \hat{F}_{v_k|S_t}(v) \) of \( v_k \) conditional on \( S_t \) at \( t \), for \( 0 < t < T_k \), under the measure \( \mathbb{Q} \) is given by

\[
\hat{F}_{v_k|S_t}(v) = \Phi \left( \sqrt{\frac{T_k}{T_k-t}} \Phi^{-1} \left( F_k^*(v) \right) - \hat{\lambda}_k(t, S_t) \right)
\]

where the pseudo-market-price-of-risk

\[
\hat{\lambda}_k(t, S) = \frac{1}{\eta \sqrt{T_k - t}} \ln \frac{S}{S_0} + \frac{\nu - \frac{1}{2} \eta^2}{\eta} \sqrt{T_k - t} - \frac{\nu - \frac{1}{2} \eta^2}{\eta} \frac{T_k}{\sqrt{T_k - t}}.
\]

Note that as \( t \downarrow 0 \) and \( S \downarrow S_0 \) then \( \hat{\lambda}_k(t, S) \downarrow -\rho \frac{\mu - r}{\sigma} \sqrt{T_k} \), i.e. the valuation is independent of \( \nu \) and \( \eta \).
Option Pricing

- Value of the cash flows

\[ V_t = \sum_{i=1}^{n} e^{-r(t_i-t)} \mathbb{E}^{Q} \left[ V_{t_i} | \mathcal{F}_t \right] \]

\[ = \sum_{i=1}^{n} e^{-r(t_i-t)} \mathbb{E}^{Q} \left[ \varphi_i(S_{t_i}) | \mathcal{F}_t \right] \]

- Value of the project with option

\[ V = e^{-rt} \mathbb{E}^{Q} \left[ \max (V_t - K, 0) \right] \]

\[ = e^{-rt_K} \int_{-\infty}^{\infty} \left( \sum_{i=1}^{n} \left( e^{-r(t_i-t_K)} \int_{-\infty}^{\infty} \varphi_i(S_{t_i}) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \right) - K \right) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \]

\[ S_{t_i} = S_0 e^{(\bar{\nu} - \frac{1}{2} \eta^2) t_i + \eta (\sqrt{t_K x} + \sqrt{t_i - t_K} y)} \]
Matching Cash-Flows for Normally Distributed Estimates

- Assume that the managers have provided cash-flow estimates of the form $N(\mu_k, \sigma_k^2)$
- Assume the Market Stochastic Driver to be a Brownian motion
- Assume that there exists a cash-flow process: $F_t$
- Introduce a collection of functions $\varphi_k(S_t)$ such that at each $T_k$, $F_{T_k} = \varphi_k(S_{T_k})$

**Theorem**

The Replicating Cash-Flow Payoff. The cash-flow payoff function $\varphi_k(s)$ which produces the managerial specified distribution $\Phi\left(\frac{s-\mu_k}{\sigma_k}\right)$ for the cash-flows at time $T_k$, when the underlying driving uncertainty $S_t$ is a BM, and $S_0 = 0$, is given by

$$\varphi_k(s) = \frac{\sigma_k}{\sqrt{T_k}} s + \mu_k^F.$$
Value of the Cash-Flows for Normally Distributed Estimates

**Theorem**

**Value of the Cash-Flows.** For a given set of cash-flow estimates, normally distributed with mean $\mu_k$ and standard deviation $\sigma_k$, given at times $T_k$, where $k = 1, 2, \ldots, n$, the value of these cash-flows at time $t < T_1$ is given by

$$V_t(S_t) = \sum_{k=1}^{n} e^{-r(T_k-t)} \left( \frac{\sigma_k}{\sqrt{T_k}} (S_t + \hat{\nu}(T_k - t)) + \mu_k \right),$$

and for the case where $t = 0$,

$$V_0 = \sum_{k=1}^{n} e^{-rT_k} \left( \hat{\nu}\sigma_k \sqrt{T_k} + \mu_k \right).$$
Option Pricing for Normally Distributed Estimates

Theorem

Real Option Value of Risky Cash-Flows Estimates. For a given set of cash-flow estimates, normally distributed with mean \( \mu_k \) and standard deviation \( \sigma_k \), given at times \( T_k \), where \( k = 1, 2, ..., n \), the value of the option at time \( t < T_0 \) to invest the amount \( K \) at time \( T_0 < T_k \) to receive these cash flows is given by

\[
RO_t(S_t) = e^{-r(T_0-t)} \left[ (\xi_1(S_t) - K) \Phi \left( \frac{\xi_1(S_t) - K}{\xi_2} \right) + \xi_2 \phi \left( \frac{\xi_1(S_t) - K}{\xi_2} \right) \right]
\]

where \( \Phi(\bullet) \) and \( \phi(\bullet) \) are the standard normal distribution and density functions, respectively, and

\[
\xi_1(S_t) = \sum_{k=1}^{n} e^{-r(T_k-T_0)} \left( \frac{\sigma_k}{\sqrt{T_k}} (S_t + \hat{\nu}(T_k - t)) + \mu_k \right),
\]

\[
\xi_2 = \sqrt{T_0-t} \sum_{k=1}^{n} e^{-r(T_k-T_0)} \frac{\sigma_k}{\sqrt{T_k}}.
\]
Utility Maximization

- Assume exponential utility

\[ u(x) = -\frac{e^{-\gamma x}}{\gamma} \]

- \( \gamma \geq 0 \) represents managerial risk aversion
- Manager has two options: 1) invest in the market, or 2) invest in the real option
- Goal is to maximize the terminal utility in each of the two options and determine the indifference price
Optimal Investment in the Traded Index (Merton Model)

- Invest in market only, with $\pi_t$ invested in the risky asset
  
  $$dX_t = (rX_t + \pi_t(\mu - r))dt + \pi_t\sigma dW_t$$

- And maximize expected terminal utility
  
  $$V(t, x) = \sup_{\pi_t} \mathbb{E}[u(X_T) | X_t = x]$$

- Applying standard arguments leads to the PDE
  
  $$\partial_t V - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} (\partial_x V)^2 + rx \partial_x V = 0$$

- with $V(T, x) = u(x)$, and the solution is given by
  
  $$V(t, x) = -\frac{1}{\gamma} e^{-\frac{1}{2} \left( \frac{\mu-r}{\sigma} \right)^2 (T-t) - \gamma e^{r(T-t)} x}$$
Optimal Investment in the Real Option Project

- Wealth dynamics are given as
  \[ dX_t = (rX_t + \pi_t(\mu - r)) \, dt + \pi_t \sigma dW_t, \quad t \notin [T_0, T_1, \ldots, T_n] \]
  \[ X_{T_0} = X_{T_0^-} - K1_A \]
  \[ X_{T_j} = X_{T_j^-} + \varphi(S_j)1_A, \quad j \in [1, 2, \ldots, n] \]

  where \( 1_A \) represents the indicator function equal to 1 if the real option is exercised.

- The manager seeks to maximize his expected terminal utility as
  \[ U(t, x, s) = \sup_{\pi_t} \mathbb{E} [u(X_T)| X_t = x, S_t = s] \]

- Applying standard arguments, it can be shown that the solution to \( U(t, x, s) \) can be achieved by solving the following PDE
  \[ \partial_t U + rx\partial_x U + \nu s\partial_s U + \frac{1}{2} \partial_{ss} U \eta^2 s^2 - \frac{1}{2} \frac{((\mu - r)\partial_x U + \rho \sigma \eta s \partial_{sx} U)^2}{\sigma^2 \partial_{xx} U} = 0 \]
Boundary conditions

\[ U(T_j, x, s) = U(T_j^+, x, s)e^{-\gamma \varphi(s)}, \text{ for } j = 1, \ldots, n - 1 \]
\[ U(T_n, x, s) = u(x + \varphi_n(s)) \]

Using the substitution \( U(t, x, s) = V(t, x)(H(t, s))^{\frac{1}{1-\rho^2}} \) results in the simplified PDE

\[ \partial_t H + \hat{\nu} s \partial_s H + \frac{1}{2} \eta^2 s^2 \partial_{ss} H = 0 \]

with \( H(T_n, s) = e^{-\gamma(1-\rho^2)\varphi_n(S_{T_n})}, \text{ and } t \in (T_{n-1}, T_n] \)

Apply dynamic programming, where at each \( t = T_j, j = \{1, 2, \ldots, n - 1\} \), set \( H(T_j, s) = H(T_j^+, s)e^{-\gamma(1-\rho^2)\varphi_j(S_{T_j})} \)
The Indifference Price

- At $t = T_0$, we should invest in the real option if

$$
(H(T_0^+, s))^\frac{1}{1-\rho^2} e^{\gamma K e^{r(T_n-T_0^+)} \leq 1}
$$

- Defining $f$ as the *indifference price*, i.e. the value of the real option, and setting $U(t, x - f, s) = V(t, x)$ leads to

$$
f(t, s) = -\frac{1}{\gamma(1-\rho^2)} \ln H(t, s) e^{-r(T_n-t)}
$$
The Indifference Price for Normally Distributed Estimates

Theorem

Real Option Value of Risky Cash-Flows Accounting for Risk Aversion. For a given set of cash-flow estimates, normally distributed with mean $\mu_k$ and standard deviation $\sigma_k$, given at times $T_k$, where $k = 1, 2, ..., n$, the value of the option at time $t < T_0$ to invest the amount $K$ at time $T_0 < T_k$ to receive these cash flows accounting for risk aversion, where the utility of the investor is given by $u(x) = -\frac{e^{\gamma x}}{\gamma}$, is given by

$$f(t, s) = -\frac{1}{\gamma(1-\rho^2)} \ln H(t, s)e^{-r(T_n-t)}$$

where

$$H(t, s) = \Phi(\hat{B}(t, s)) + e^{\frac{\xi^2}{2}} \frac{\xi}{e^r} \hat{C}(t, s)\Phi(\xi_t - \hat{B}(t, s)).$$
The Indifference Price for Normally Distributed Estimates

\[ \xi_t = -\gamma(1 - \rho^2)\hat{a}_1 \sqrt{T_0 - t} \]

\[ \hat{a}_j = \sum_{k=j}^{n} \frac{\sigma_k}{\sqrt{T_k}} e^{r(T_n-T_k)}, \quad \hat{b}_j = \sum_{k=j}^{n} \mu_k e^{r(T_n-T_k)} \]

\[ A_j = \hat{a}_j \left( \frac{\gamma\hat{a}_j}{2} (1 - \rho^2) - \nu \right), \quad A_0 = \sum_{j=1}^{n} A_j (T_j - T_{j-1}) \]

\[ \hat{B}(t, s) = \frac{A_0 - \hat{b}_1 + Ke^{r(T_n - T_0)}}{\hat{a}_1} - s - \nu(T_0 - t) \]

\[ \hat{C}(t, s) = e^{\gamma(1-\rho^2)(A_0 - \hat{a}_1(s + \nu(T_0 - t))) - \hat{b}_1 + Ke^{r(T_n - T_0)}} \]
Real Option Value (MMM)

(a) Market Stochastic Driver as GBM

(b) Market Stochastic Driver as GMR

Project value and real option value of the UAV project for varying correlation (note that they are independent of $S_0$, $\nu$ and $\eta$)

<table>
<thead>
<tr>
<th>Correlation ($\rho$)</th>
<th>0.0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Project Value ($V_0$)</td>
<td>493.69</td>
<td>467.31</td>
<td>441.49</td>
<td>416.35</td>
<td>392.00</td>
<td>368.54</td>
</tr>
<tr>
<td>Option Value ($RO_0$)</td>
<td>199.82</td>
<td>173.83</td>
<td>148.71</td>
<td>124.82</td>
<td>102.45</td>
<td>82.02</td>
</tr>
</tbody>
</table>
Sensitivity to Risk - Standard Approach

![Graph showing real option value (MAD Method) versus risk (ΔV) for different strike prices (K = 40, 45, 50, 55).]
Sensitivity to Risk - MMM

Assumptions:

- Single cash-flow at $T_1 = 3$
- Expected value of the cash-flow: $\mu_1 = 50$
- Correlation to traded index: $\rho = 0.5$
- Investment time: $T_0 = 2$
Sensitivity to Risk - MMM ($\rho = 0.5$)
Sensitivity to Risk - MMM

For a single cash-flow, the real option value is given as

$$RO_0 = e^{-rT_0} \mathbb{E}^Q \left[ \left( e^{-r(T_1-T_0)} \left( \mu_1 + \hat{\nu} \sigma_1 \sqrt{T_1} \right) + e^{-r(T_1-T_0)} \sqrt{\frac{T_0}{T_1}} \sigma_1 Z - K \right) \right]$$

Recall $\hat{\nu} = -\rho \frac{\mu - r}{\sigma}$
Sensitivity to Risk - MMM

The graph illustrates the frequency distribution of distorted cash flows with a volatility (σ) of 5. The x-axis represents the distorted cash flows, while the y-axis shows the frequency. The peak at 50 indicates the most likely distortion in cash flows.
Sensitivity to Risk - MMM

The diagram illustrates the distribution of distorted cash flows (CF) for different values of the standard deviation ($\sigma$). The blue line represents $\sigma = 5$, and the red line represents $\sigma = 25$. The x-axis represents the distorted CF, and the y-axis represents the frequency of occurrence.
Sensitivity to Risk - MMM

![Graph showing the frequency distribution of distorted cash flows for different values of \(\sigma\).]
(c) Real option indifference price as a function of $S_t$ and $t$ at $\gamma = 0.01$.

(d) Real option indifference price at $S_t = S_0$ for varying levels of risk aversion.
Sensitivity to Risk - Indifference Price

<table>
<thead>
<tr>
<th>Risk ($\Delta V$), $\rho$</th>
<th>Indiff. Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.25</td>
<td>0.00</td>
</tr>
<tr>
<td>0.5</td>
<td>0.25</td>
</tr>
<tr>
<td>0.75</td>
<td>0.50</td>
</tr>
<tr>
<td>1.00</td>
<td>0.75</td>
</tr>
</tbody>
</table>
Sensitivity to Risk - Indifference Price

- Risk ($\Delta V$), $\rho = 0.00$
- Risk ($\Delta V$), $\rho = 0.25$
- Risk ($\Delta V$), $\rho = 0.50$
- Risk ($\Delta V$), $\rho = 0.75$
- Risk ($\Delta V$), $\rho = 1.00$
Practical Implementation of the Matching Method

- Assume managers supply revenue and GM% estimates

<table>
<thead>
<tr>
<th>Scenario</th>
<th>End of Year Sales / Margin</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimistic</td>
<td></td>
<td>80</td>
<td>116</td>
<td>153</td>
<td>177</td>
<td>223</td>
<td>268</td>
<td>314</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(50%)</td>
<td>(60%)</td>
<td>(65%)</td>
<td>(60%)</td>
<td>(60%)</td>
<td>(55%)</td>
<td>(55%)</td>
</tr>
<tr>
<td>Most Likely</td>
<td></td>
<td>52</td>
<td>62</td>
<td>74</td>
<td>77</td>
<td>89</td>
<td>104</td>
<td>122</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(30%)</td>
<td>(40%)</td>
<td>(40%)</td>
<td>(40%)</td>
<td>(35%)</td>
<td>(35%)</td>
<td>(35%)</td>
</tr>
<tr>
<td>Pessimistic</td>
<td></td>
<td>20</td>
<td>23</td>
<td>24</td>
<td>18</td>
<td>20</td>
<td>20</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(20%)</td>
<td>(20%)</td>
<td>(20%)</td>
<td>(20%)</td>
<td>(15%)</td>
<td>(10%)</td>
<td>(10%)</td>
</tr>
<tr>
<td>SG&amp;A*</td>
<td></td>
<td>10%</td>
<td>5%</td>
<td>5%</td>
<td>5%</td>
<td>5%</td>
<td>5%</td>
<td>5%</td>
</tr>
<tr>
<td>Fixed Costs</td>
<td></td>
<td>30</td>
<td>25</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

* Sales / General and Administrative Costs
Sales and GM% Stochastic Drivers

- **Traded index**
  \[ dl_t = \mu I_t dt + \sigma I_t dW_t \]

- **Sales stochastic driver to drive revenues**
  \[ dX_t = \rho_{SI} dW_t + \sqrt{1 - \rho_{SI}^2} dW_t^S \]

- **GM% stochastic driver to drive GM%**
  \[ dY_t = \rho_{SM} dX_t + \sqrt{1 - \rho_{SM}^2} dW_t^M \]

- **Cash flow**
  \[ V_k(t) = (1 - \kappa_k) \phi_k^S(X_t) \phi_k^M(Y_t) - \alpha_k \]
Theorem

The Bivariate Density of Sales and GM%. The bivariate probability density function between sales and GM% is given by

\[ u(s, m) = \phi_{\Omega_{PSM}} \left( \Phi^{-1} \left( F^*(s) \right), \Phi^{-1} \left( G^*(m) \right) \right) \bullet \frac{f^*(s)}{\phi \left( \Phi^{-1} \left( F^*(s) \right) \right)} \frac{g^*(m)}{\phi \left( \Phi^{-1} \left( G^*(m) \right) \right)} \]

where \( \phi_{\Omega_{\rho}} \) represents the standard bivariate normal PDF with correlation \( \rho \), and \( \phi \) is the standard normal PDF.
Project and Real Option Value

- **Project value**

\[
V_{T_0}(X_{T_0}, Y_{T_0}) = \sum_{k=1}^{n} e^{-r(T_k - T_0)} \mathbb{E}_Q^\mathbb{Q} \left[ v_k(X_{T_k}, Y_{T_k}) \mid X_{T_0}, Y_{T_0} \right]
\]

- **Real option value**

\[
RO_t(X_t, Y_t) = e^{-r(T_0 - t)} \mathbb{E}_Q^\mathbb{Q} \left[ (V_{T_0}(X_{T_0}, Y_{T_0}) - K)_+ \mid X_t, Y_t \right]
\]

- **Risk-neutral measure** \((\hat{\nu} = -\rho_{SI} \frac{\mu - r}{\sigma} \text{ and } \hat{\gamma} = -\rho_{SI} \rho_{SM} \frac{\mu - r}{\sigma})\)

\[
\frac{dl_t}{l_t} = r \, dt + \sigma \, d\widehat{W}_t, \\
\]

\[
dX_t = \hat{\nu} \, dt + \rho_{SI} \, d\widehat{W}_t + \sqrt{1 - \rho_{SI}^2} \, d\widehat{W}_t^S, \\
\]

\[
dY_t = \hat{\gamma} \, dt + \rho_{SI} \rho_{SM} \, d\widehat{W}_t + \rho_{SM} \sqrt{1 - \rho_{SI}^2} \, d\widehat{W}_t^S + \sqrt{1 - \rho_{SM}^2} \, d\widehat{W}_t^M
\]
Computing the Real Option

- Resulting PDE

\[ rH = \frac{\partial H}{\partial t} + \hat{\nu} \frac{\partial H}{\partial x} + \hat{\gamma} \frac{\partial H}{\partial y} + \frac{1}{2} \frac{\partial^2 H}{\partial x^2} + \frac{1}{2} \frac{\partial^2 H}{\partial y^2} + \rho_{SM} \frac{\partial^2 H}{\partial x \partial y} \]

Value of the real option for varying \( \rho_{SI} \) and \( \rho_{SM} \)
Matching Method Conclusions

- The approach utilizes managerial cash-flow estimates
- The approach is theoretically consistent
  - Provides a mechanism to account for systematic versus idiosyncratic risk
  - Provides a mechanism to properly correlate cash-flows from period to period
- The approach requires little subjectivity with respect to parameter estimation
- The approach provides a **missing link** between practical estimation and theoretical frame-works
References

Valuation and return dynamics of new ventures.

Are real options actually used in the real world?

Real options analysis: Where are the emperor’s clothes?
*Journal of Applied Corporate Finance* 17(2), 17–31.

Valuing the option to invest in an incomplete market.

Investment, consumption, and hedging under incomplete markets.