Mathematics-in-Industry Case Studies Journal, Volume 6, pp. 1-21 (2014)

A Mathematical Model of the Rainwater Flows in a Green Roof

Catherine C Adley * Mark J Cooker † Gemma L Fay ‡ Ian Hewitt §
Andrew A Lacey ¶ Niklas Mellgren ‖ Marguerite Robinson ** Michael Vynnycky ††

Abstract. A model is presented for the gravity-driven flow of rainwater descending through the soil layer of a green roof, treated as a porous medium on a flat permeable surface representing an efficient drainage layer. A fully saturated zone is shown to occur. It is typically a thin layer, relative to the total soil thickness, and lies at the bottom of the soil layer. This provides a bottom boundary condition for the partially saturated upper zone. It is shown that after the onset of rainfall, well-defined fronts of water can descend through the soil layer. Also the rainwater flow is relatively quick compared with the moisture uptake by the roots of the plants in the roof. In separate models the exchanges of water are described between the (smaller-scale) porous granules of soil, the roots and the rainwater in the inter-granule pores.

Keywords. Porous media, ground-water flow, mathematical modelling, method of lines, mesh refinement

1 Introduction

Green roofs are becoming increasingly popular around the world. The many benefits of a green roof include assistance in the management of storm water, pollution control, building insulation and recycling of carbon dioxide, in addition to being aesthetically pleasing. A green roof is subject to various stresses from the weather, in particular wind-loading, which we ignore in this report, and rainfall: it is the flow, drainage and uptake of rainwater that we model. An understanding of

*Department of Chemical and Environmental Sciences, University of Limerick, Limerick, Ireland
†School of Mathematics, University of East Anglia, Norwich, NR4 7TJ, U.K.
‡OCIAM, Mathematical Institute, University of Oxford, Andrew Wiles Building, Woodstock Road, Oxford, OX2 6GG, UK
§OCIAM, Mathematical Institute, University of Oxford, Andrew Wiles Building, Woodstock Road, Oxford, OX2 6GG, UK
¶Maxwell Institute for Mathematical Sciences, and School of Mathematical and Computer Sciences, Heriot-Watt University, Riccarton, Edinburgh, EH14 4AS, U.K.
‖KTH, Department of Mechanics, SE-100 44 Stockholm, Sweden
**Institut Català de Ciències del Clima (IC3), C/Doctor Trueta 203, 08005 Barcelona, Catalunya, Spain
††Division of Casting of Metals, Department of Materials Science and Engineering, Royal Institute of Technology, Brinellvägen 23, 100 44 Stockholm, Sweden

where the water goes is essential to design a roof able to achieve sustained healthy plants and loads that lie within the safe capacity of the supporting structure.

The main focus of this paper is on the transport of water through the structure of the green roof. Inadequate drainage can lead to the undesirable occurrence of a fully saturated soil which will cut off the air supply to the plants. Conversely, if the saturation levels are too low plants will die from lack of water. Ideally a degree of saturation that is less than eighty per cent should be maintained at all times. Our goal is to model the distribution of the degree of saturation through the depth of the soil layer, and to see how it changes due to spells of rain, and under the influence of moisture-uptake by plant roots.

This study was motivated by a problem brought to the 70th European Study Group with Industry, held in Limerick in 2009. The moisture input into the roof used later is, therefore, based on Irish weather data.

The basic structure of a common green roof is shown in Fig. 1. A waterproof root barrier protects the underlying roof structure. A drainage layer sits atop this barrier. The typical thickness of this layer is 8/15/20 mm depending on the type of roof. The drainage layer has not been modelled in this study, and any possible build-up of water there has been disregarded. Instead, any water entering this layer is assumed, perhaps unrealistically, to leave the system. The soil and drainage layers are separated by a thin sheet of perforated hard plastic containing holes approximately 2 mm in diameter and spaced roughly 2 cm apart. There are two layers of soil at the top of the structure separated by a layer of felt. A thin layer (< 2 cm) of refined rooting soil contains the plant life, mainly sedum for thinner roofs and, for thicker ones, low growing grasses such as common bent grass and/or other plants, such as cowslip and ladies bedstraw. Beneath the rooting soil are pellets of lightweight expanded clay. This layer is 5-10 cm thick. Grain sizes are typically < 2 mm for rooting soil and 4-8 mm for expanded clay pellets.

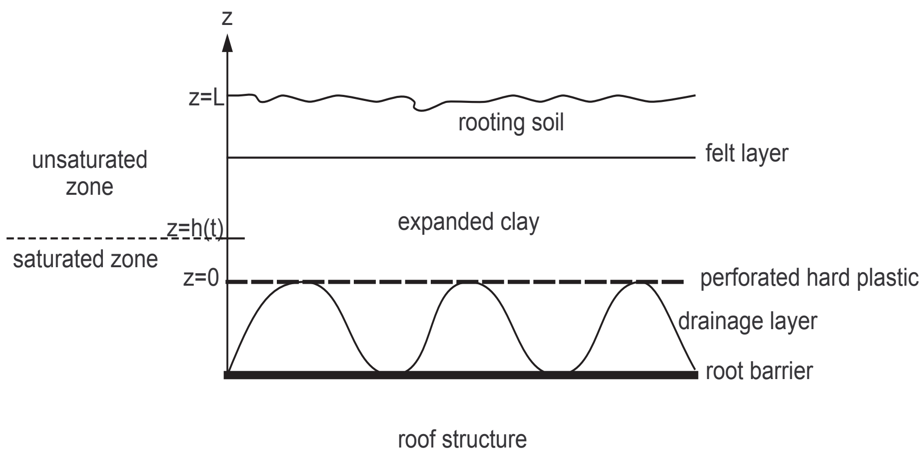

Figure 1: *Green roof structure. Gravity is in the negative z direction.*

2 The Model

We model the dynamics of water flow through the soil layer. We consider a single soil layer with thickness $L \approx 10^{-1}$ m and we ignore the presence of the felt layer. We assume that the soil-drainage-layer interface is located at $z = 0$ and the soil surface at $z = L$. We consider two possible scenarios: (i) the entire region $0 \leq z \leq L$ is unsaturated, so that the soil saturation S is everywhere less than 1 and (ii) a saturated region $0 \leq z \leq h$ lies at the bottom of the soil layer. Note that the model as presented here is one-dimensional, and represents a horizontal roof, but can be easily extended to two (or three) dimensions, and to account for sloping roofs.

2.1 The Unsaturated Region

We first assume the entire region $0 \leq z \leq L$ is unsaturated ($S < 1$). The basic model for this region follows that outlined in [6] and [4]. The equation for water flow in the unsaturated soil comes from having local water flux measured upwards (in the positive z direction)

$$q = -\left(D_0 D(S) \frac{\partial S}{\partial z} + K_0 K(S) \right) \tag{1}$$

and employing the balance law

$$\phi \frac{\partial S}{\partial t} + \frac{\partial q}{\partial z} = -R, \tag{2}$$

to give the one-dimensional Richards' equation (see [1] and [3])

$$\phi \frac{\partial S}{\partial t} = \frac{\partial}{\partial z} \left(D_0 D(S) \frac{\partial S}{\partial z} + K_0 K(S) \right) - R, \tag{3}$$

where $S = S(z, t)$, ϕ is the constant porosity of the soil, taken here to be 0.25, $D_0 D(S)$ and $K_0 K(S)$ are the water diffusivity and hydraulic conductivity respectively, with the functions $D(S)$ and $K(S)$

given by

$$K(S) = S^{1/2}[1-(1-S^{1/m})^m]^2, \tag{4}$$

$$D(S) = \frac{[1-(1-S^{1/m})^m]^2}{S^{1/m-1/2}(1-S^{1/m})^m}, \tag{5}$$

where $0 < m < 1$ (see [6] and [2]). The value of m for the expanded-clay soil should be found experimentally but, for later use in simulations and analysis of the model, is taken to be $m = \frac{1}{2}$. Likewise the values of the constants K_0, the conductivity for saturated soil, and D_0, a representative value of diffusivity, should be obtained empirically for particular roofing materials. However, in the absence of good experiments, values as found in [4] and [5] are assumed here. Water uptake by the plant roots is incorporated into the model through the last term in (3) and is given ([4] and [5]) by

$$R = 2\pi a k_r l_d \left(p_a - p_c f(S) - p_r\right), \tag{6}$$

where k_r is the root's radial conductivity of water, a is the root radius, l_d is the average number of roots per unit (horizontal) area, p_a is atmospheric pressure, p_r is an effective pressure in the roots (although it can be negative), $p_c f(S)$ is the capillary pressure in the soil, with p_c another constant characterising the partly saturated pellets, and

$$f(S) = (S^{-\frac{1}{m}} - 1)^{1-m}. \tag{7}$$

We take parameter values from Roose and Fowler [4] and let $2\pi a k_r = 7.85 \times 10^{-16}$ m^2 s^{-1} Pa^{-1}, $l_d = 5 \times 10^3$ m^{-2} and $p_c = 10^4$ N m^{-2}. The root pressure p_r will be determined from conservation of water within the root. Finally we must prescribe boundary conditions at the top and bottom of the soil layer. At the soil surface we take

$$D_0 D(S)\frac{\partial S}{\partial z} + K_0 K(S) = Q_{in}(t) \quad \text{at} \quad z = L, \tag{8}$$

where Q_{in} is the rainfall rate averaged over the surface area of the ground. We assume, in this unsaturated case, no outflow at the base of the soil layer and set

$$D_0 D(S)\frac{\partial S}{\partial z} + K_0 K(S) = 0 \quad \text{at} \quad z = 0. \tag{9}$$

We nondimensionalise the equations by scaling

$$z = L\hat{z}, \quad p_r = |P|\hat{p}_r, \quad t = \frac{L}{K_0}\hat{t}, \quad p = p_a + p_c\hat{p}, \quad R = 2\pi a k_r l_d |P|\hat{R}, \quad Q_{in} = Q_{typ}\hat{Q}, \tag{10}$$

where P is the (negative) root pressure at the soil surface and we set $|P| = 10^6$ N m^{-2}. The time scale used here is that for flow though the soil layer under the action of gravity, with saturation neither small nor close to one. The dimensionless Richards' equation (3) then has the form

$$\phi\frac{\partial S}{\partial \hat{t}} = \frac{\partial}{\partial \hat{z}}\left(\delta D(S)\frac{\partial S}{\partial \hat{z}} + K(S)\right) - \eta(\theta - \varepsilon f(S) - \hat{p}_r), \tag{11}$$

where

$$\delta = \frac{D_0}{LK_0} \approx 10^{-4}, \quad \eta = \frac{2\pi a k_r l_d |P| L}{K_0} \approx 4 \times 10^{-6}, \quad \theta = \frac{p_a}{|P|} \approx 10^{-1}, \quad \varepsilon = \frac{p_c}{|P|} \approx 10^{-2}. \tag{12}$$

Roose and Fowler [4] give values of D_0 for different soil types and we can reasonably take $D_0 = 10^{-6}$ m^2 s^{-1}. However, the value of K_0 is more difficult to determine as it varies significantly with different soil types. The parameter values in (12) are given for $K_0 = 10^{-1}$ m s^{-1}. We note that $\eta \ll 1$ suggesting that water uptake by the roots is negligible over the chosen time scale (of order 1 s). The dimensionless forms of the boundary conditions are given by

$$\delta D(S)\frac{\partial S}{\partial \hat{z}} + K(S) = \nu \hat{Q} \quad \text{at} \quad \hat{z} = 1, \tag{13}$$

$$\delta D(S)\frac{\partial S}{\partial \hat{z}} + K(S) = 0 \quad \text{at} \quad \hat{z} = 0, \tag{14}$$

where

$$\nu = \frac{Q_{typ}}{K_0} \approx 3 \times 10^{-6}, \tag{15}$$

with Q_{typ} taken to be some typical rainfall. We set $Q_{typ} = 3 \times 10^{-7}$ m s^{-1} for a "wet day" in Ireland. (This figure equates to about 2.6 cm in a day. Averaging a monthly precipitation would give a substantially lower figure.)[1] With this size of rainfall, on order one times, the saturation will generally be small, of order $\nu^{2/9}$ for $m = 1/2$. This might suggest a rescaling of the saturation S, but we delay such an approach until later. First, we address the possibility of the soil becoming fully saturated, the no-flux condition at the base in (14) indicating that the whole soil layer would fill up on a dimensionless time scale of $O(\nu^{-1})$. When the soil becomes saturated, however, the model must change, and this allows for drainage through the base as described below. (Note that the time scale for S to become locally order one near the base should depend on δ as well as on ν.)

2.2 The Saturated Region

When the soil starts to become fully saturated ($S = 1$) at $z = 0$, we assume that a moving boundary forms between the fully saturated soil below and the partially saturated soil above. This boundary lies at $z = h(t)$ and the soil saturation is identically one for $z \in [0, h]$. For saturated soil, water flux is given by Darcy's law,

$$q = -K_0 \left(\frac{\partial p}{\partial z} + \rho g \right), \tag{16}$$

instead of (1), and our governing equation in this lower region can now be written in the (dimensionless) form

$$\frac{\partial}{\partial \hat{z}} \left(1 + \frac{1}{\gamma} \frac{\partial \hat{p}}{\partial \hat{z}} \right) - \eta \left(\theta + \varepsilon \hat{p} - \hat{p}_r \right) = 0, \tag{17}$$

[1] Rainfall of 8 cm in 30 minutes was recorded at Eskdalemuir in southwest Scotland in 1953. Such a figure would make Q_{typ} over 100 times larger but still keep ν small.

where

$$\gamma = \frac{\rho g L}{p_c} \approx 10^{-1}. \tag{18}$$

The flux through the membrane at $z = 0$ is prescribed to occur at a rate proportional to the pressure difference across it: $Q_{mem} = \kappa(p - p_a)$ dimensionally, where p_a is the atmospheric pressure in the drainage layer beneath, p is the pressure at $z = 0$, and $\kappa \approx 10^{-5}$ m s^{-1} Pa^{-1} (determined experimentally in the next sub-section). This gives the dimensionless condition

$$1 + \frac{1}{\gamma}\frac{\partial \hat{p}}{\partial \hat{z}} = \alpha \hat{p} \quad \text{at} \quad \hat{z} = 0, \tag{19}$$

where $\alpha = \frac{\kappa p_c}{K_0} \approx 1$. At the saturation front $\hat{z} = \hat{h}(\hat{t}) \equiv \frac{h(t)}{L}$, $S = 1$, $\hat{p} = 0$ (atmospheric), and continuity of fluid flux requires

$$\lim_{\hat{z} \to \hat{h}+} \left[K(S) + \delta D(S) \frac{\partial S}{\partial \hat{z}} \right] = \lim_{\hat{z} \to \hat{h}-} \left[1 + \frac{1}{\gamma}\frac{\partial \hat{p}}{\partial \hat{z}} \right]. \tag{20}$$

Neglecting the η term in (17) for this saturated region, and using $\hat{p} = 0$ at $\hat{z} = \hat{h}$ along with (19), gives

$$\hat{p}(\hat{z}, \hat{t}) = \frac{\gamma(\hat{h}(\hat{t}) - \hat{z})}{1 + \alpha\gamma\hat{h}(\hat{t})}, \tag{21}$$

so that (20) becomes

$$\lim_{\hat{z} \to \hat{h}+} \left(K(S) + \delta D(S) \frac{\partial S}{\partial \hat{z}} \right) = \frac{\alpha\gamma\hat{h}}{1 + \alpha\gamma\hat{h}} \tag{22}$$

and then, on using $K(1) = 1$,

$$-\lim_{\hat{z} \to \hat{h}+} \left(\delta D(S) \frac{\partial S}{\partial \hat{z}} \right) = \frac{1}{1 + \alpha\gamma\hat{h}}. \tag{23}$$

In principle equation (22) and boundary condition $S = 1$ at $\hat{z} = \hat{h}(\hat{t})$ determine $\hat{h}$ in terms of the flux from the unsaturated region. However, we can simplify things if we notice from (13) that the dimensionless flux will in general be small, of order ν (due to the rainfall). If this is the case, then the value of $\hat{h}$ required to satisfy (22) will be small. Physically, this is because, for the typical size of fluid flux considered, the pressure required to force it through the membrane according to (19) is provided by the hydrostatic head of a very thin layer of water (dimensionally, h is calculated to be much less than 1 mm).

Thus if a saturated region is created at the bottom of the soil layer, it will quickly grow to a depth which is sufficient to drain exactly the same amount of water through the membrane as is arriving from the unsaturated region above. Provided this depth is substantially less than the depth of the soil, the saturated region can be 'collapsed' (mathematically) onto the line $\hat{z} = 0$, and the boundary condition applied to the problem in the unsaturated zone for some of the numerical solutions of sub-section 2.5 is then simply

$$S = 1 \quad \text{at} \quad \hat{z} = 0. \tag{24}$$

After computing the solution $S(\hat{z},\hat{t})$ of the problem with the simplified boundary condition (24), we can evaluate the limit in (23), and hence estimate the small non-zero depth $h(\hat{z},\hat{t})$.

Note that with this model, even with the η term restored in the saturated layer, once the layer forms, there is no mechanism by which it will entirely disappear:

Starting with a completely unsaturated roof, so that (14) is initially imposed at the base, if the roof attains saturation at some dimensionless time $\hat{t}_s$, the base condition (24) holds for all later times $\hat{t} > \hat{t}_s$. [2]

2.3 Experimental measurement of κ

The value of κ was deduced from a simple experiment, which involved puncturing a 2 mm diameter hole in a plastic bottle, made with material similar to that of the drainage membrane (this is normally made from high-density polyethylene). The rate of drainage through the hole driven by the hydraulic head in the bottle was measured, and used to determine the coefficient of proportionality between pressure difference across the membrane Δp and the water flux through it q. Writing

$$q = k\Delta p, \tag{25}$$

where $\Delta p = \rho g h$, the water depth in the bottle, h, satisfies the equation

$$A_{bottle}\frac{dh}{dt} = -k\rho g h, \tag{26}$$

where A_{bottle} is the cross-sectional area of the bottle. Thus

$$\log h = -\frac{k\rho g}{A_{bottle}} t. \tag{27}$$

Measurements of h against t made during the experiment are in Fig. 2, and the best fit value of the time constant $t_c = A_{bottle}/k\rho g$ was 74 seconds. The flux through an individual hole can be converted into an average velocity through a membrane, using the area of the membrane $A_{membrane}$ that is drained by each hole. Thus

$$\bar{u} = \kappa\Delta p, \qquad \kappa = \frac{A_{bottle}}{A_{membrane}\rho g t_c}. \tag{28}$$

Taking $A_{membrane} = \pi$ cm^2, and using the cross-sectional area of the bottle $A_{bottle} = 25$ cm^2, $\rho = 10^3$ kg m^{-3}, and $g = 10$ m s^{-2}, gives $\kappa \approx 10^{-5}$ m s^{-1} Pa^{-1}.

2.4 The Root Pressure

To determine the root pressure p_r in equation (11), we assume that the root extends through the full thickness of the soil layer of depth L. Conservation of water inside the root yields

$$k_z\frac{d^2 p_r}{dz^2} + 2\pi a k_r\left(p_a - p_c f(S) - p_r\right) = 0, \tag{29}$$

[2] Alternatively, the bottom condition might be specified in linear complementary form $(1-S)q = 0$ with $1-S \geq 0$ (for no super-saturation) and $q \leq 0$ (for downward flux).

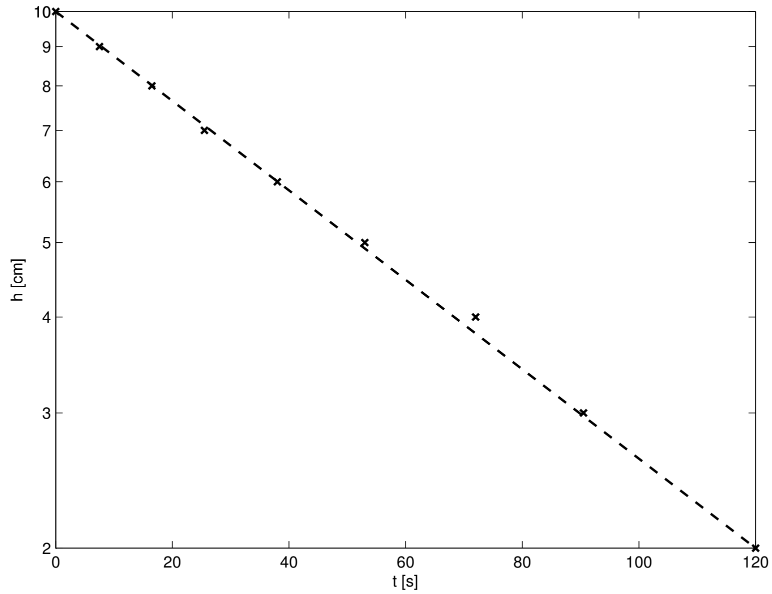

Figure 2: *Experimental measurements of h against t.*

where $k_z = 10^{-14}$ m^6 s^{-1} N^{-1} is the root axial conductivity and $f(S)$ is defined in equation (7). Zero axial flux at the root tip implies

$$\frac{dp_r}{dz} + \rho g = 0 \quad \text{at} \quad z = 0. \tag{30}$$

In addition we prescribe a driving pressure, P, at the root base yielding

$$p_r = p_a + P \quad \text{at} \quad z = L. \tag{31}$$

In dimensionless form the root pressure will satisfy

$$\frac{d^2\hat{p}_r}{d\hat{z}^2} + \tau\left(\theta - \varepsilon f(S) - \hat{p}_r\right) = 0 \quad \text{in } 0 < \hat{z} < 1\,, \tag{32}$$

subject to

$$\frac{d\hat{p}_r}{d\hat{z}} = -\varepsilon\gamma \quad \text{at} \quad \hat{z} = 0, \tag{33}$$

$$\hat{p}_r = \theta - 1 \quad \text{at} \quad \hat{z} = 1, \tag{34}$$

where

$$\tau = \frac{2\pi a k_r L^2}{k_z} \approx 10^{-3}. \tag{35}$$

The parameters $\tau \ll 1$ and $\varepsilon\gamma \ll 1$ which implies $\dfrac{d^2\hat{p}_r}{d\hat{z}^2} \approx 0$ subject to $\dfrac{d\hat{p}_r}{d\hat{z}} = 0$ on $\hat{z} = 0$. Note that having $\tau \ll 1$ means that varying saturation in the soil has negligible effect on the root pressure. The dimensionless root pressure is thus given by

$$\hat{p}_r = \theta - 1 \quad \text{for } 0 \leq \hat{z} \leq 1\,. \tag{36}$$

The complete model is now given by (11), with the definitions (4), (5), (7) and (36), with boundary condition (13) at $\hat{z} = 1$ and (14) if $S < 1$, or (24) otherwise. An initial condition is also needed.

The diffusion term which has δ as a factor in (11) is small, so the equation is essentially a first-order non-linear wave equation; the boundary condition (rainfall) is transmitted downwards as a wave. If rain starts suddenly, there is a sharp jump in saturation that propagates quickly down to the bottom of the soil; if the rain stops suddenly then, in the $z - t$ plane, the solution is described by a classical expansion fan.

2.5 Numerical Solutions

The governing equation for the unsaturated region (11) was solved numerically subject to boundary conditions (13), with $\hat{Q} = 1$, and (14). In these first simulations, a finite element method was used with 385 elements and significant refinement near $\hat{z} = 0$ and near $\hat{z} = 1$. As a first approach the η term in (11) is neglected so that we are just considering drainage of the soil layer under gravity. The initial saturation was taken to be uniform throughout the soil layer. Three different initial values of the saturation $S_{init} = 0.05, 0.1, 0.15$ were considered. The profiles obtained for S in all the cases, when the computations were stopped, are shown in Fig. 3; a corresponding semi-log plot is shown in Fig. 4, in order to demonstrate the boundary layer of thickness δ in S at $\hat{z} = 0$ that is predicted by comparing the two transport terms in (11), and which is captured by the numerical solution, but which is not visible in Fig. 3. For $S_{init} = 0.1$ and 0.15, computations were stopped when the value of S at $\hat{z} = 0$, S_{bottom}, reached 1; for $S_{init} = 0.05$, S_{bottom} is still far from 1, even for the value of dimensionless time (100) shown here. The time evolution of S_{bottom} is shown in Fig. 5, while that for S at $\hat{z} = 1$, S_{top}, is shown in Fig. 6.

Thus, the results suggest an appreciable difference in the time at which complete saturation is achieved at the bottom of the soil when S_{init} is increased from 0.05 to 0.1. The effect of the rainfall boundary condition (13) has (by the end of the simulations) only affected the tiny region at the right of Fig. 3, where there is the beginning of a shock front propagating downwards from $\hat{z} = 1$; since δ has been taken to be very small, the shock looks very sharp, and the values on either side of it are the initial condition (below, or left, of the shock), and the value given by $K(S) = \nu\hat{Q}$ (above, or right, of the shock - this value is expectedly independent of the initial condition, as shown in Fig. 6).

The complete problem, with a small saturated region allowed for by using boundary condition (24), and with $\eta \neq 0$, was also solved by discretising in space and solving with the method of lines using ode15s in Matlab. Rather than have a mesh refinement as employed earlier to cope with the stiffness produced by the small value of δ, the value of this parameter is now taken to be artificially large, $\delta = 10^{-2}$. We use a larger value of δ partly so as to avoid having to use a variable grid and partly so as to make the diffusive transition layers more clear visible in the solutions. Since the value is still small, using the larger value does not affect the overall dynamics – it simply

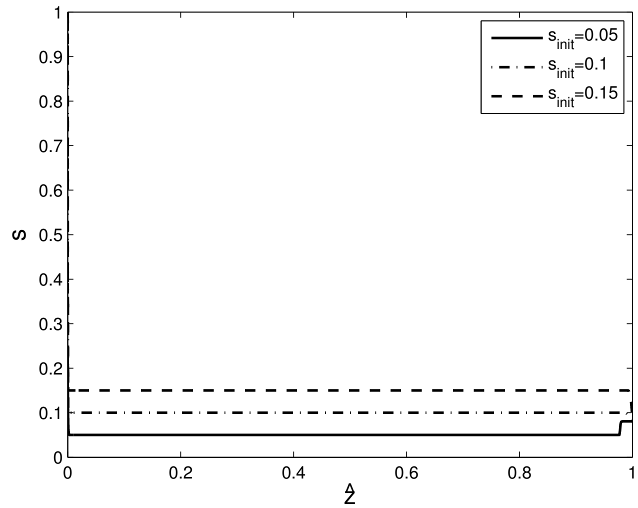

Figure 3: S *vs.* $\hat{z}$ *for three different initial conditions* ($S_{init} = 0.05, 0.1, 0.15$) *at either dimensionless time 100* ($S_{init} = 0.05$) *or when S reaches 1 at* $\hat{z} = 0$ ($S_{init} = 0.1, 0.15$). *Parameter values are* $m = 1/2$, $\delta = 10^{-4}$, $\nu = 3 \times 10^{-6}$. *The top condition has* $\hat{Q} = 1$. *Note a sudden increase to* $S = 1$ *for small* $\hat{z}$.

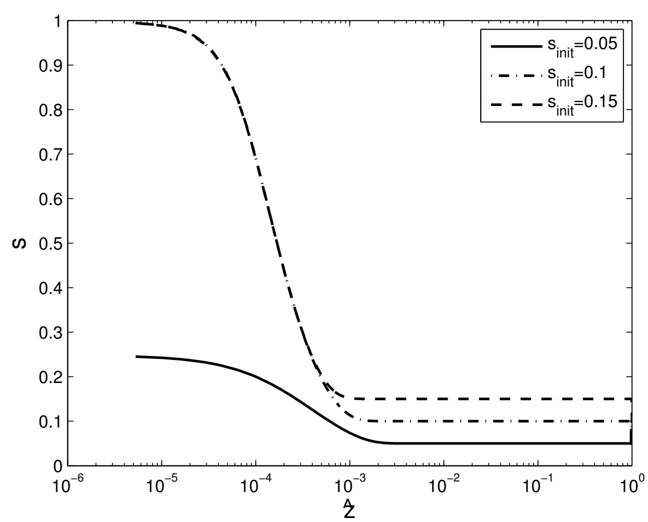

Figure 4: *A semi-log plot of* S *vs.* $\hat{z}$ *for three different initial conditions* ($S_{init} = 0.05, 0.1, 0.15$) *at either dimensionless time 100* ($S_{init} = 0.05$) *or when S reaches 1 at* $\hat{z} = 0$ ($S_{init} = 0.1, 0.15$). *Parameter values are* $m = 1/2$, $\delta = 10^{-4}$, $\nu = 3 \times 10^{-6}$.

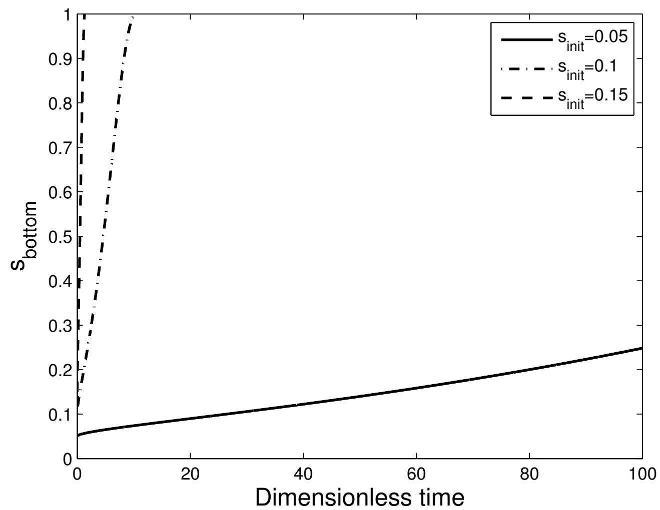

Figure 5: S_{bottom} *vs. dimensionless time for three different initial conditions ($S_{init} = 0.05, 0.1, 0.15$). Parameter values are $m = 1/2$, $\delta = 10^{-4}$, $\nu = 3 \times 10^{-6}$.*

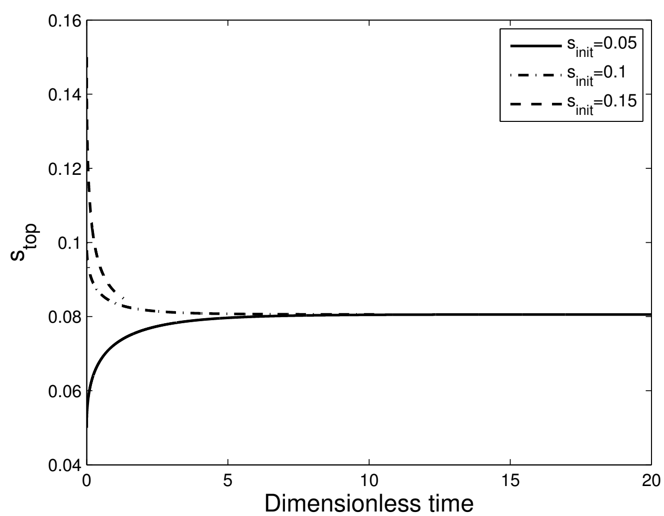

Figure 6: S_{top} *vs. dimensionless time for three different initial conditions ($S_{init} = 0.05, 0.1, 0.15$). Parameter values are $m = 1/2$, $\delta = 10^{-4}$, $\nu = 3 \times 10^{-6}$. The top condition has $\hat{Q} = 1$.*

exaggerates the width of the diffusive layers. To apply the switch in boundary conditions smoothly, the condition

$$q_0 = q_1 e^{-1000(1-S)}, \tag{37}$$

was applied for the flux at the bottom node q_0 in terms of the flux at the node above q_1; thus when S is close to 1 this becomes $\partial\hat{q}/\partial\hat{z} = 0$, and when S is less than 1 it becomes $q_0 = 0$. The diffusion coefficient is infinite when $S = 1$, but this does not cause any issues in the numerics, possibly because the above boundary condition ensures S never quite reaches 1.

This seems to allow for steady states when rainfall is constant; if there is more rainfall than is taken up by the roots, the saturation at the bottom is 1 and there is a boundary layer of width δ in which it adjusts to the value as determined by $K(S) \approx \nu\hat{Q} - \eta\int_0^1 \hat{R}\,\mathrm{d}\hat{z}$ (Fig. 7). If there is less rainfall than is taken up by the roots, the saturation at the bottom decreases almost to 0.

Fig. 7 shows the result of a sudden increase in rainfall from $\hat{Q} = 0.1$ to $\hat{Q} = 10$, which shows the initial shock front travelling down into the soil and the eventual steady state. The saturation at the bottom does not increase towards 1 until the shock front arrives there. Fig. 8 shows the result of a sudden decrease back to $\hat{Q} = 0.1$. Note that the time intervals shown are longer. Most of the apparent changes occur over a time scale suggested by following characteristics (neglecting the diffusion term) from $\hat{z} = 1$ where the saturation is given by $K(S) = \nu\hat{Q}$, say $S = S_1$. Along such a characteristic, S is given by

$$\phi\frac{\mathrm{d}S}{\mathrm{d}\hat{t}} = -\eta(\theta\epsilon f(S) - \hat{p}_r) \sim \eta\left(\frac{\epsilon}{S} - 1\right)$$

for S small, and the (dimensionless) time scale is of order $S_1/(\phi\eta) \approx 100$ for this particular problem. (This time scale may be associated with an expansion fan localised near $\hat{z} = 1$.) For this case, there appears to be a more substantial boundary layer, possibly of width $\delta^{1/2}$, near $\hat{z} = 0$.

The simple model presented in this section suggests that we can generally expect the soil to be partially saturated throughout most of its depth, with a small saturated layer at the base facilitating drainage through the underlying membrane. Even with quite large rainfall, the drainage is apparently sufficient to evacuate the water without the soil becoming fully flooded. This is of course dependent on the permeability of the membrane, which may vary considerably and may also decrease with time due to clogging; but given the values assumed here we may conclude that full saturation of the soil layer is unlikely. On the other hand the model suggests the opposite problem of having long periods of drought when there is no rainfall. We therefore turn to some alternative two-porosity models that could give longer-term water storage.

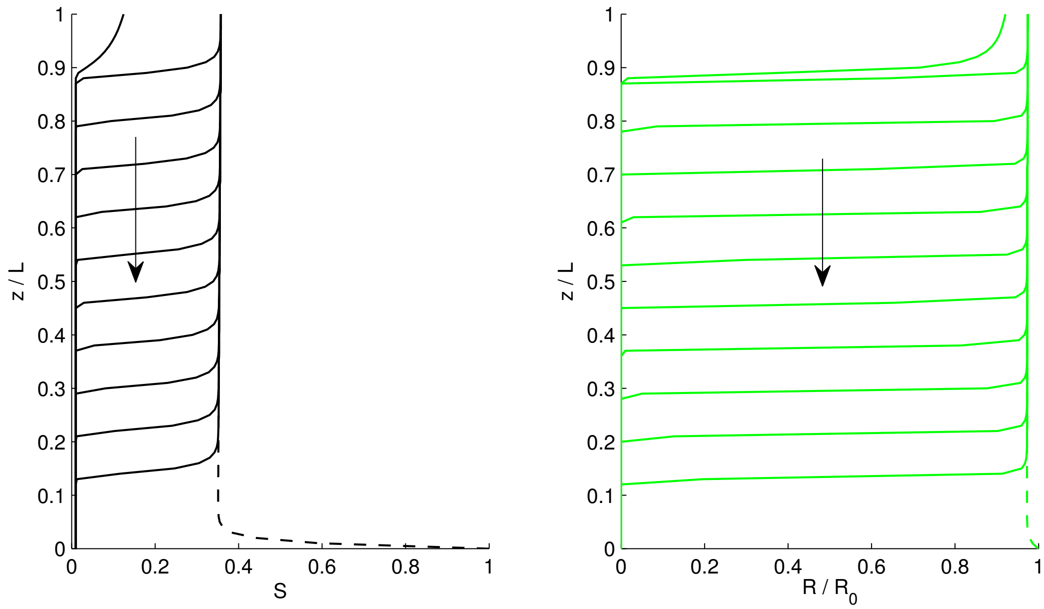

Figure 7: *Profiles of saturation and root uptake at time intervals of* 1 *(in the dimensionless units); the arrow shows the direction of increasing time. This is the result of a sudden increase in rainfall to* $\hat{Q} = 10$*, from the steady state when* $\hat{Q} = 0.1$*, and the dashed line shows the steady state that results. Parameter values are* $m = 1/2$, $\delta = 10^{-2}$, $\eta = 4 \times 10^{-4}$, $\nu = 3 \times 10^{-4}$, $\varepsilon = 10^{-2}$, $\gamma = 10^{-1}$.

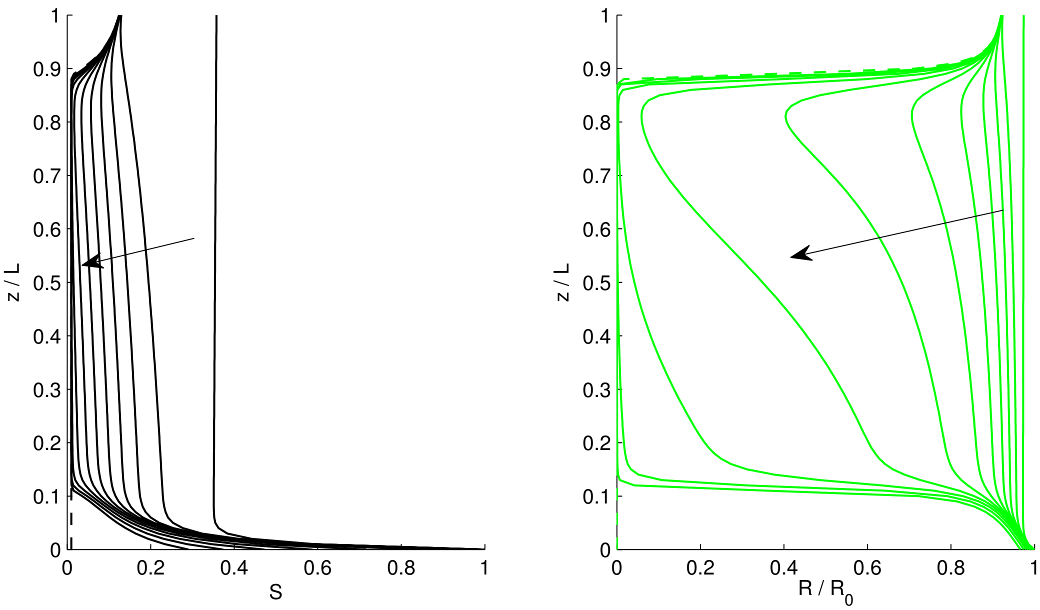

Figure 8: *Profiles of saturation and root uptake at time intervals of* 10 *(in the dimensionless units); the arrow shows the direction of increasing time. This is the result of a sudden decrease in rainfall from* $\hat{Q} = 10$, *to* $\hat{Q} = 0.1$. *Parameter values are* $m = 1/2$, $\delta = 10^{-2}$, $\eta = 4 \times 10^{-4}$, $\nu = 3 \times 10^{-4}$, $\varepsilon = 10^{-2}$, $\gamma = 10^{-1}$.

3 Two-Porosity Models

The expanded clay pellets used in green roof construction are quite large but contain lots of pore space. The difference in pore sizes between these and the inter-pellet space means water can be drawn into the pellets and retained there for longer than it would otherwise remain in the soil. Thus a two-porosity model would seem appropriate.

3.1 A Model with Slow Saturation

This is an outline of a "box" or "lumped" model for water storage in the macro-pores between soil particles, which have saturation S, and in the micro-pores within the particles, which have saturation S_P. Transport of water into or out of the particles is given by a rate constant λ times the saturation difference $S - S_P$ (the penultimate term in (38) and the right-hand side of (39), below).[3] The roots do not penetrate into individual particles so provide a sink term R only from the macro-pores. This root uptake $R(S)$ in (3) is primarily due to the large negative pressure in the root system, but as saturation decreases a large capillary pressure acts to counteract this; thus $R(S)$ is roughly constant for S close to 1 but decreases at small S (as in the model above).

The following equations are dimensionless, and the time scale has been chosen to be that due to uptake by the roots (the time scale differs from that used previously by a factor η, so that now $t = t_0\hat{t}$ with $t_0 = 1/(2\pi a k_r l_d|P|) \approx 2.5 \times 10^5 s \approx 3$ days). Drainage from the volume of soil is supposed to occur due to gravity at a rate $K(S)$, and occurs on a time scale η compared to the uptake by the roots (see above). Rainfall provides a source which is scaled to be the same size as the gravity drainage (note this is different to above – the scale for the rainfall here is large and is intended to represent the size of heavy showers; the dimensionless $r(\hat{t})$ will be 0 most of the time, when it is not raining, and $O(1)$ when it is raining heavily).

$$\phi\frac{\mathrm{d}S}{\mathrm{d}\hat{t}} = \frac{1}{\eta}r(\hat{t}) - \frac{1}{\eta}K(S) - \lambda(S - S_P) - R(S), \tag{38}$$

$$(1-\phi)\phi_P\frac{\mathrm{d}S_P}{\mathrm{d}\hat{t}} = \lambda(S - S_P), \tag{39}$$

where $r = \nu\hat{Q}$, ϕ_P is the porosity of the pellets, $\lambda > 0$ is a transport constant and

$$K(S) = S^{1/2}[1 - (1 - S^2)^{1/2}]^2, \tag{40}$$

which comes from equation (4) with $m = \frac{1}{2}$ and

$$R(S) = 1 - \varepsilon\frac{(1 - S^2)^{1/2}}{S}. \tag{41}$$

[3] A variant of this model might assume that water transfer into the particles occurs at a rate proportional to the *pressure* difference $p_{cP}f(S_P) - p_c f(S)$; since the capillary pressure in the micropores would be larger than in the macropores ($p_{cP} > p_c$), this would cause more water to be transferred into the micropores, and a larger supply would be maintained there for the roots to take up.

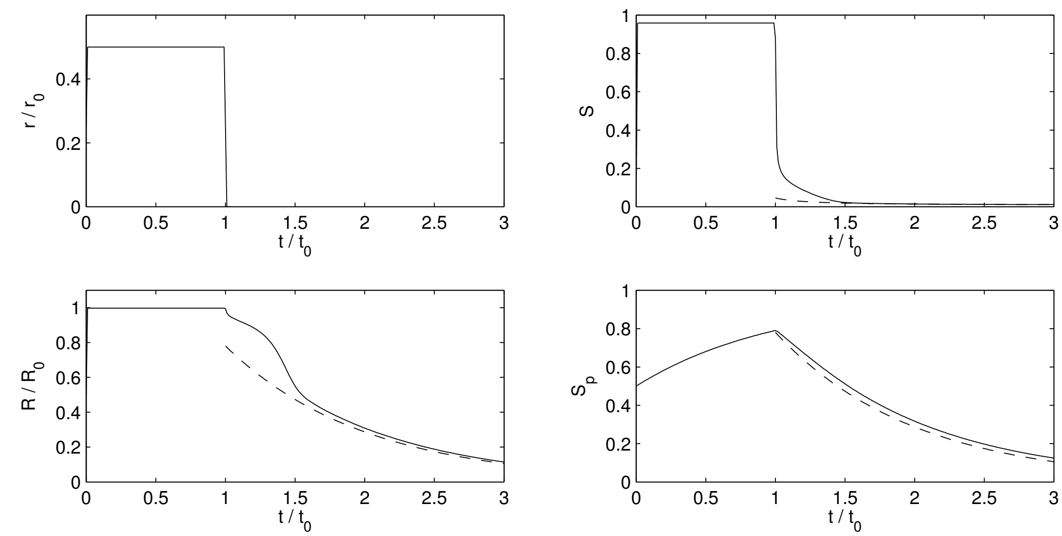

Figure 9: *Solutions for macro-scale and particle-scale saturations S and S_P, and root uptake R, as a result of rainfall $r(t)$ which represents a large rain shower. Dashed lines show the limiting behaviour. Parameter values are $\eta = 10^{-4}$, $\lambda = 1$, $\varepsilon = 10^{-2}$.*

The use of $K(S)$ for the gravity drainage in equation (38) is motivated by the fact that the water flow in Section 2 is essentially determined by this hydraulic conductivity (since δ is small). The time scale for water to diffuse into individual particles is estimated using their dimensions $L_P \sim 1$ cm and a diffusion coefficient $D_P \sim 10^{-9}$ m^2 s^{-1}. L_P^2/D_P is comparable to the time scale for uptake by the roots ($\sim 10^5$ s), so the parameter λ is order 1. In equation (38) η is very small and in equation (41) ε is also small, and we consider especially the distinguished case ε of order $\eta^{2/9}$, see (44) below.

The behaviour of solutions to this model is quite straightforward, and an example solution for a large rain storm followed by dry weather is in Fig. 9. When it is raining, r is order 1, and on a fast time scale, $\hat{t} \sim O(\eta)$, the saturation S relaxes towards the equilibrium given by $K(S) = r(\hat{t})$. This causes water to then transfer into the particles on an $O(1)$ time scale according to (39). When it stops raining $r = 0$, and the saturation S decreases quickly due to gravity drainage on an $O(\eta)$ time scale. In this fast regime, (38) is approximately

$$\phi \frac{\mathrm{d}S}{\mathrm{d}\hat{t}} \sim -\frac{1}{\eta} K(S), \tag{42}$$

where $K(S) \sim \frac{1}{4} S^{\frac{9}{2}}$ for S small. This suggests S tends towards 0 as

$$S \sim \left(\frac{8\phi\eta}{7\hat{t}} \right)^{2/7}. \tag{43}$$

Looking to balance the $\mathrm{d}S/\mathrm{d}\hat{t}$, $K(S)/\eta \sim S^{\frac{9}{2}}/(4\eta)$ (for S small), and $\lambda(S - S_P) \sim -\lambda S_P$ (for S small) terms in (38), we then take $\hat{t} = \eta^{2/9}\tilde{t}$ and $S = \eta^{2/9}\tilde{S}$. A complete balance from the final term,

$$R(S) \sim 1 - \varepsilon/(\eta^{2/9}\tilde{S}) \tag{44}$$

is achieved on taking

$$\varepsilon = \eta^{2/9}\tilde{\varepsilon}. \tag{45}$$

In this intermediate regime, (38) then reduces to

$$\phi\frac{\mathrm{d}\tilde{S}}{\mathrm{d}\tilde{t}} \sim -\frac{1}{4}\tilde{S}^{9/2} + \lambda S_P - 1 + \frac{\tilde{\varepsilon}}{\tilde{S}} \tag{46}$$

while (39) becomes simply, to leading order,

$$\frac{\mathrm{d}S_P}{\mathrm{d}\tilde{t}} = 0. \tag{47}$$

On the $O(1)$ time scale, S continues to be order $\eta^{2/9}$ and can be regarded as quasi-stationary, with (38) (or (46)) being replaced by

$$\frac{K(S)}{\eta} + R(S) \sim \frac{1}{4}\tilde{S}^{9/2} + 1 - \frac{\tilde{\varepsilon}}{\tilde{S}} \sim \lambda S_P \tag{48}$$

while S_P now reduces, with (39) (or (47)) being replaced by

$$(1-\phi)\phi_P\frac{\mathrm{d}S_P}{\mathrm{d}\hat{t}} \sim -\lambda S_P. \tag{49}$$

Thus, ignoring the small terms, S_p decays exponentially and the water coming out into the macropores is immediately either taken up by the roots or lost by drainage:

$$R + \frac{K}{\eta} = \lambda S_P = -(1-\phi)\phi_P dS_P/d\hat{t}, \tag{50}$$

In the example shown in Fig. 9, $\tilde{\varepsilon}$ is rather small (approximately 0.1) while λS_P is significantly less than one in this time regime. Equation (48) then indicates that $\tilde{S}$ is small and the water being lost by drainage is negligible; in this case the water coming out of the micropores is immediately taken up by the roots.

A longer-time regime will apply for S_P sufficiently small but this is not considered here.

In conclusion, root uptake is maintained for a much longer period (as it decreases slowly with time according to (50)) after it ceases to rain. This contrasts with the case with no micropores, when S decreases rapidly towards 0 (the time scale being a factor η shorter).

3.2 A Model for Fast Saturation

Assuming instead fast saturation of the pellets, so $S_P = \mathrm{H}(S)$, the intra-pellet water content is given by

$$\text{water density in individual pellets} = \phi_P S_P = \phi_P \mathrm{H}(S)\,, \tag{51}$$

where H denotes the Heaviside function, S_P denotes the saturation of the individual pellets, ϕ_P is the porosity of an individual pellet, and S is the saturation of the inter-pellet pores. The required short time scale can arise from high capillary pressures associated with the very small pores within the pellets.

Taking now $\phi = \frac{1}{4}$ to be the total proportion of space occupied by air and water within the soil, and $\phi_P = \frac{1}{5}$, then the inter-pellet porosity is $\varphi = \frac{1}{16}$ (given by $\varphi + \frac{1}{5}(1-\varphi) = \frac{1}{4}$). The total water content is now inter-pellet water content (porosity × inter-pellet saturation), φS, plus that of the pellets (the volume fraction occupied by the pellets × their porosity × their saturation), $(1-\varphi)\phi_P S_P$,

$$= \frac{1}{16}S + \frac{15}{16} \times \frac{1}{5}\mathrm{H}(S) = \frac{1}{16}(S + 3\mathrm{H}(S)).$$

The water flux, q, and rate of uptake of water by the roots, R, are assumed to depend on the inter-pellet saturation S in the same way as earlier. Equation (11) can then be replaced by

$$\frac{1}{16}\frac{\partial}{\partial \hat{t}}(S + 3\mathrm{H}(S)) = \frac{\partial}{\partial \hat{z}}\left(K(S) + \delta D(S)\frac{\partial S}{\partial \hat{z}}\right) - \eta \hat{R}\,, \tag{52}$$

with $\hat{R} \sim 1$, from (11) and (36). (Equation (52) might be better written in terms of the total water content, $S_T = \frac{1}{16}(S + 3\mathrm{H}(S))$, so that S on the right-hand side is replaced by $S(S_T) = 0$ for $0 \le S_T \le \frac{3}{16}$, $S(S_T) = 16(S_T - \frac{3}{16})$ for $\frac{3}{16} \le S_T \le \frac{1}{4}$.)

Where the pellets are saturated, $S > 0$ and $\mathrm{H}(S) = 1$, the equations are as in Section 2. Here, for simplicity, an initially dry soil is considered, so that at $\hat{t} = 0$, $S \equiv S_P \equiv 0$. For $\hat{t} > 0$, a region $\hat{W}(\hat{t}) < \hat{z} < 1$ has become wet:

$$S = \mathrm{H}(S) = 0 \ \text{ in } \ 0 < \hat{z} < \hat{W}\,, \quad S > 0 \ \text{ and } \ \mathrm{H}(S) = 1 \ \text{ in } \ \hat{W} < \hat{z} < 1\,. \tag{53}$$

To obtain an order-one sized wet region, the relevant time scale must be that for the rainfall (days). Hence time has to be rescaled by

$$\hat{t} = \tilde{t}/\nu\,. \tag{54}$$

Note that this time scale is similar to that for the up-take of water by the plants' roots. It is also appropriate, from the top boundary condition, to rescale the saturation:

$$S = \nu^{2/9}\tilde{S}\,, \tag{55}$$

where, since we have assumed that $m = \frac{1}{2}$, $K(S) \sim \frac{1}{4}S^{9/2}$ and $D(S) \sim \frac{1}{4}S^{5/2}$ for small S.

Neglecting the time-derivative term (now effectively of order $\nu^{2/9}$), the partial differential equation (52) becomes

$$\frac{1}{4}\frac{\partial}{\partial \hat{z}}\left(\tilde{S}^{9/2}+\tilde{\delta}\tilde{S}^{5/2}\frac{\partial \tilde{S}}{\partial \hat{z}}\right)=\tilde{\eta}\hat{R}\,. \tag{56}$$

Here $\tilde{\eta}=\eta/\nu\approx\frac{1}{3}$ and $\tilde{\delta}=\delta/\nu^{2/9}\approx\frac{1}{600}$, using the values of Section 2. Although the value of $\tilde{\delta}$ is small here, because of the uncertainty in the values of the physical parameters describing water transport through the soil, it could conceivably be of order one and it is therefore retained in (56), for the present. The $\tilde{\delta}$ term should also be kept as it contains the highest derivative in the equation, just as the diffusion term was retained in Section 2.

The differential equation is subject to the top boundary condition

$$\frac{1}{4}\left(\tilde{S}^{9/2}+\tilde{\delta}\tilde{S}^{5/2}\frac{\partial \tilde{S}}{\partial \hat{z}}\right)=\hat{Q}_{in}\ \text{ at }\ \hat{z}=1 \tag{57}$$

and, assuming that the diffusive, $\tilde{\delta}$, term is retained, to a lower boundary condition

$$\tilde{S}=0\ \text{ at }\ \hat{z}=\hat{W}(\tilde{t})\,. \tag{58}$$

Finally, to fix the position of the free boundary $\hat{z}=\hat{W}(\tilde{t})$ between dry and wet soil, conservation of mass of water at this point, where S_P jumps from 0 to ϕ_P, leads to

$$\frac{\mathrm{d}\hat{W}}{\mathrm{d}\tilde{t}}=-\frac{4}{3}\left(\tilde{S}^{9/2}+\tilde{\delta}\tilde{S}^{5/2}\frac{\partial \tilde{S}}{\partial \hat{z}}\right)\ \text{ at }\ \hat{z}=\hat{W}(\tilde{t})\,. \tag{59}$$

(Since, for $\tilde{\delta}>0$, $\tilde{S}=0$ at this point, the second term on the right-hand side should then be interpreted as $\tilde{\delta}\lim_{\hat{z}\to\hat{W}}\left\{\tilde{S}^{5/2}\frac{\partial \tilde{S}}{\partial \hat{z}}\right\}$.)

Of course, if the pellets were already partially saturated, (59) would be suitably modified, leading to a faster-moving free boundary.

Note also that if the diffusion can be neglected, (56) and (57) lead to $\frac{1}{4}\tilde{S}^{9/2}=\hat{Q}_{in}+\hat{z}-1$ so (59) gives

$$-\frac{\mathrm{d}\hat{W}}{\mathrm{d}\tilde{t}}=\frac{16}{3}\left(\hat{Q}_{in}+\hat{W}-1\right)\,. \tag{60}$$

The free-boundary condition (59) only applies for an advancing wet region, $\mathrm{d}\hat{W}/\mathrm{d}\tilde{t}\le 0$. An alternative form is needed for when this region contracts, which will happen when the rainfall decreases sufficiently. In any part of the soil between the lowest location of the free boundary and its current position, the roots can continue to remove water from the pellets, thereby reducing S_P.

As described in this paper we could now have at least four types of region within the soil layer:

1. Dry zone, where $S=S_P=0$;

2. Damp or moist (unsaturated) zone I, where $S = 0$, $0 < S_P < 1$;

3. Damp or moist (unsaturated) zone II, where $0 < S < 1$, $S_P = 1$;

4. Wet (saturated) zone, where $S = 1$, $S_P = \phi_P$.

4 Conclusions

In this paper a one-dimensional time-dependent mathematical model has been described for the development of the saturation in the soil layer of a flat green roof. Our model suggests that a fully saturated ($S = 1$) region forms at the base of the soil layer and this region can be thin relative to the total soil thickness.

From an initial dry state and from the onset of persistent rain, fronts of saturation were computed to descend through the layer. The decrease of saturation from unity following a decrease in rainfall was also described. The end result is that most of the rainwater falls through the soil layer and exits through the network of holes in the bottom supporting sheet.

On a smaller scale, the pellets and soil particles are themselves porous and made up of micropores. The water flow in and out of a typical particle is modelled using the flux between (a) the macropores (whose saturation is as modelled above) and (b) the root system. This two-porosity model suggests that during the time between spells of rain the micropores can retain (for long periods of time) water that is available to be taken up by the roots. For green roof design it is important to ensure that the membrane supporting the soil is sufficiently permeable to prevent any risk of full saturation. It is also important to use soil which has sufficient micro-pores to soak up large quantities of water during rainfall and allow slow release during dry periods.

Further work might include adapting the soil thickness L to rainfall at the site of the building with the aim of making L as small as possible, while avoiding problems with saturation and aridity. A first step towards this goal would be to carry out experiments to more accurately determine the values of the constants. Further simulations using more extensive rainfall data could then be carried out to determine the optimum soil thickness. In addition small modifications could be made to include the influence of a sloped roof.

Acknowledgements

We acknowledge the support of the Mathematics Applications Consortium for Science and Industry (www.macsi.ul.ie) funded by the Science Foundation Ireland mathematics initiative grant 06/MI/005.

References

[1] G.R. Fulford and P. Broadbridge, *Industrial mathematics: Case studies in the diffusion of heat and matter*, CUP, Cambridge, 2002. 3

[2] Y. Mualem, *A new method for predicting the hydraulic conductivity of unsaturated porous media*, Water Resour. Res **12** (1976), 513–522. 4

[3] L.A. Richards, *Capillary conduction of liquids through porous mediums*, Physics **1** (1931), 318–333. 3

[4] T. Roose and A.C. Fowler, *A model for water uptake by plant roots*, J. Theoret. Biol. **228** (2004), 155–171. 3, 4, 5

[5] ———, *A mathematical model for water and nutrient uptake by plant root systems*, J. Theoret. Biol. **228** (2004), 173–184. 4

[6] M.Th. Van Genuchten, *A closed-form equation for predicting the hydraulic conductivity of unsaturated soil*, Soil Sci. Soc. Am. J. **48** (1980), 892–898. 3, 4