

Painting the Office

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Abstract. A particular instance of a wireless channel assignment problem for office blocks takes this form: If a box is partitioned into cuboids, and each cuboid must be assigned a channel different from those of the cuboids it is in face-to-face contact with, how many channels do you need? We show that the number of channels may be arbitrarily large.

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1 Introduction

In most modern office complexes, many different companies can be working in the same building, each with its own “territory” but with shared public spaces such as hallways. The use of wireless local area networks (WLAN) introduces a potential loss of security for data that is company-confidential. There are various ways of reducing the risk of this, one of which is to arrange that walls and ceilings between *different* territories have a special treatment, “stealthy wallpaper”, that provides reasonable attenuation at the relevant frequencies. However, it is still necessary that companies whose territories are in face-to-face contact on opposite sides of just one wall or floor-ceiling should be on different channels. Territories that are separated by two walls or floors, or that touch only along an edge or at a corner can share a frequency. For this reason, BAE Systems (the company who developed “stealthy wallpaper” from their radar work) asked the 53rd European Study Group with Industry to address the problem of Graph Colouring for Office Blocks. If the territories in an office block are regarded as the vertices of a graph, and two vertices are joined when their territories are in contact (meaning face-to-face), then a valid channel assignment for WLAN in the building corresponds to a vertex-colouring of the graph: an assignment of channels to the territories is an assignment of colours to the vertices. So the questions that BAE Systems asked were:

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1. Can the class of adjacency graphs produced by arrangements of office territories be described mathematically?
2. What is the maximum chromatic number of this class of adjacency graphs? (Recall that the chromatic number of a graph is the number of colours needed to colour its vertices in such a way that joined vertices have different colours.)
3. Can reasonably natural restrictions be placed on the allowable configurations of office territories in order to yield a class of adjacency graphs with known maximum chromatic number (or a reasonably tight upper bound U for the maximum chromatic number)? The restrictions should be as weak as possible, *i.e.* ideally they should only rule out extreme or unlikely configurations. The chromatic number (or upper bound) should be as small as possible. There will be a trade-off between the strength of the restrictions and the maximum chromatic number, which may be worth investigating.
4. Is there an algorithm to produce a colouring using U colours or fewer for any given adjacency graph in the class? The efficiency of this algorithm is probably not of great concern, but if the number of nodes is 100 or fewer, the colouring should be found in a day or less on a modern PC.

We here summarize in Section 2 the work of the Study Group [3]. This led to an attractive problem whose subsequent solution is in Section 3.

2 Some basic observations

The problem has a history going back to Möbius, though his interest was motivated by curiosity rather than WLAN, of course. Tietze [2] discusses the problem and shows that the chromatic number is unbounded, by the example of Figure 1. In our terminology, this is an office block of 2 floors, each consisting of n parallel long rooms, which run east-west on the lower floor and north-south on the upper.

Suppose each of the n territories consists of one room on the lower floor and one on the upper. This clearly results in each pair of territories being in face-to-face contact and therefore requires n colours—the adjacency graph is complete on n vertices. Indeed, by adjusting this building slightly we see that *any* graph can arise as an adjacency graph, thus answering questions (1) and (2). Question (4) turns out not to be difficult and several graph-colouring algorithms were tried on the adjacency graph of some real offices (OCIAM in fact) treating each room, corridor, cupboard *etc.* as a territory. This example had 82 vertices and 280 edges, and the algorithms quickly produced a 5-colouring. Studying the floor plans showed there were “cliques” of 5 rooms all in contact, so 5 colours are also necessary. Recall that the clique number of a graph is the largest number of vertices that are all joined to each other, so in any graph the chromatic number is at least as large as the clique number, and in our example both were 5.

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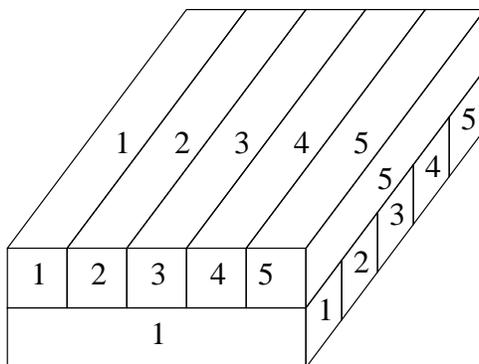


Figure 1: Territories of $n = 5$ companies each in contact with all of the others.

So, the interesting question (3) remained: Are there reasonable restrictions that can be placed on the shapes of territories in an office block that result in the chromatic number being bounded? Restricting the territories to be convex does not achieve this, for Tietze [4] and Besicovitch [5] have shown that there can be n convex polyhedra all in contact with each other. On the other hand, if the building is divided into floors in the usual way and each territory lies on only one floor, then the chromatic number is bounded by 8. For the territories on each floor form a planar graph which is therefore 4-colourable, so we can use colours 1–4 on the odd floors and 5–8 on the even. In fact the clique number can be 8 in such a building even if the territories are restricted to be convex.

An interesting question was identified though: If all the territories are cuboids (aligned with a fixed set of Cartesian axes), is the chromatic number bounded or not? The clique number in the adjacency graph is then bounded by 4, and it is not hard to make an example that needs 6 colours, but the Study Group could neither prove any bound nor make an example needing more than 6. However, this problem caught the attention of some graph theorists.

3 The adjacency graph of cuboids

It will be shown now that the adjacency graph of cuboids, in the sense we have been discussing, can have arbitrarily high chromatic number. In fact, given any integer k we shall construct an arrangement of blocks such that if they can be coloured with k colours then *all* of the colours are needed.

We begin with a *staircase* of a_k blocks as in Figure 2.

These all have width 1, length l_1 and heights $1, 2, \dots, a_k$ where we shall define $a_r = 2^r r!$ (for each $r = 1, \dots, k$). We call these the blocks of Type 1, or for short the 1-blocks. The construction now proceeds in k stages: we encode certain subsets of the 1-blocks into an arrangement of some blocks above them, called the 2-blocks. Then we encode certain subsets of the 2-blocks into arrangements of some blocks called the 3-blocks that touch both them and the 1-blocks, and so on until we have

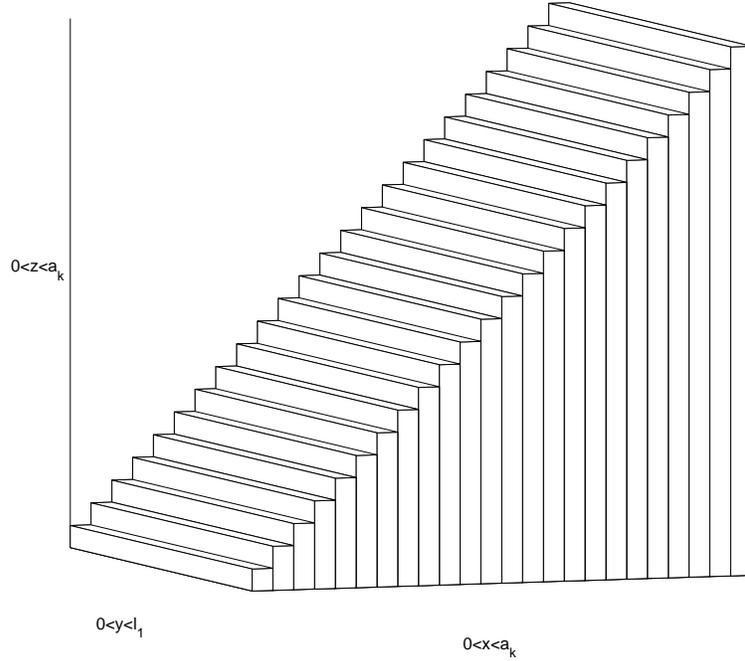


Figure 2: *The staircase of a_k blocks of Type 1.*

added the k -blocks. The blocks get *very* short in the y -direction, and of course the result is totally unlike any real office building. In detail, we first let $b_k = a_k/k$. A set of b_k of the 1-blocks will be called a 1-set, so there are $n_k = \binom{a_k}{b_k}$ 1-sets. Our Type 2 blocks are going to be of length $l_2 = l_1/n_k$. For each 1-set, we allocate a section of length l_2 in the y -direction, and place $b_k/2 = a_{k-1}$ blocks of Type 2 that are arranged in the (x, z) -plane as described in Figure 3.

The total number of 2-blocks added is therefore $n_k a_{k-1}$, since there are a_{k-1} in each of the n_k sections. The arrangements of the 3-blocks, 4-blocks *etc.* are going to be obtained by a recursive iteration of this construction. So, consider any particular set of a_{k-1} 2-blocks that have been constructed in this way from a particular 1-set. A set of $b_{k-1} = a_{k-1}/(k-1)$ of these 2-blocks will be called a 2-set, so there are $n_{k-1} = \binom{a_{k-1}}{b_{k-1}}$ such 2-sets. We let $l_3 = l_2/n_{k-1}$ and for each 2-set we allocate a section of length l_3 and place $b_{k-1}/2 = a_{k-2}$ blocks of Type 3 that are arranged in the (x, z) -plane as shown in Figure 4.

The total number of 3-blocks is therefore $n_k n_{k-1} a_{k-2}$ since for each of the n_k 1-sets there are n_{k-1} 2-sets, and each of those has a_{k-2} 3-blocks placed in its section. When we go on like this, a diagram showing some blocks up to one of Type 4 is shown in Figure 5.

Now we have to describe the consequences of this arrangement for the colouring problem. First, since there are a_k 1-blocks, if they are coloured in k colours, there must be some set of $b_k = a_k/k$ that are all the same colour, which we may call colour 1. Somewhere along the y -direction is the section of length l_2 where those blocks were the 1-set, and therefore the 2-blocks are arranged as shown in say Figure 3. So the 2-blocks in that section must all avoid colour 1, and are therefore

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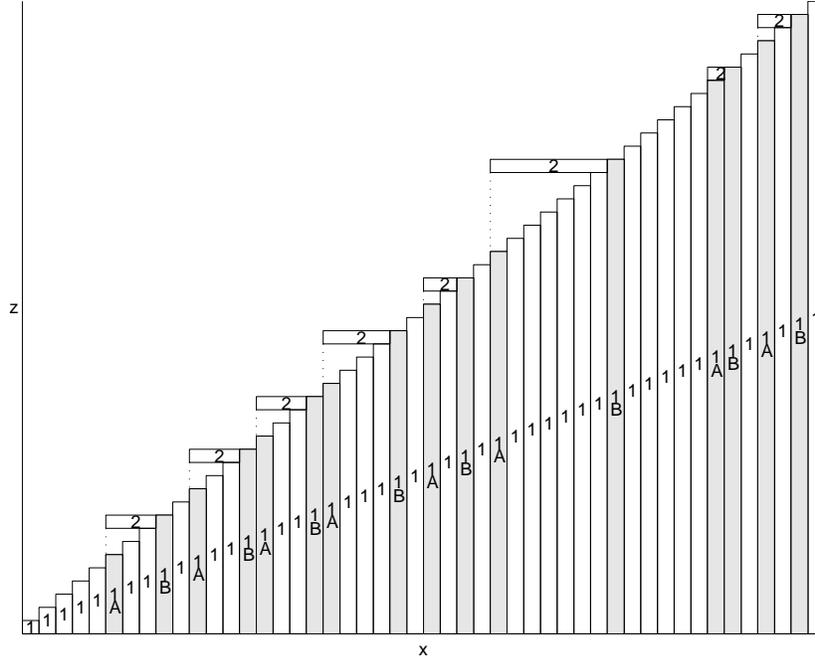


Figure 3: Arrangement of Type 2 blocks for the 1-set that consists of the shaded 1-blocks. The shaded 1-blocks are taken in order and temporarily labelled A, B, A, B, \dots as shown. For each consecutive pair A, B , a 2-block is placed with its right face in contact with B and its left face vertically above the left face of A .

coloured using the $k - 1$ colours $2, \dots, k$. However, since there are a_{k-1} of these 2-blocks, there must be some subset of $b_{k-1} = a_{k-1}/(k - 1)$ of them that are all the same colour, which we may call colour 2. Somewhere along our section of length l_2 in the y -direction is the arrangements of 3-blocks that corresponds to this particular monochromatic 2-set. So within that section of length l_3 the 3-blocks are arranged as shown in Figure 4. So the 3-blocks in that section must all avoid colours 1 and 2, and are therefore coloured using the $k - 2$ colours $3, \dots, k$. Repeating this argument through each stage of the construction we see that in fact all k colours must be used. (Note that we have not shown the chromatic number actually is k , just that it is at least k .)

The number of blocks in this construction is

$$X_k = a_k + n_k a_{k-1} + n_k n_{k-1} a_{k-2} + \dots = a_k + n_k X_{k-1}, \quad (1)$$

with $X_1 = a_1 = 2$. To get a reasonable upper bound on X_k one can show by induction that $X_k \leq a_k^{b_k} - 1$, and hence

$$X_k \leq a_k^{b_k} = \left(2^k k!\right)^{2^{k(k-1)!}} \leq k^{k^k} \quad \text{for } k \geq 6. \quad (2)$$

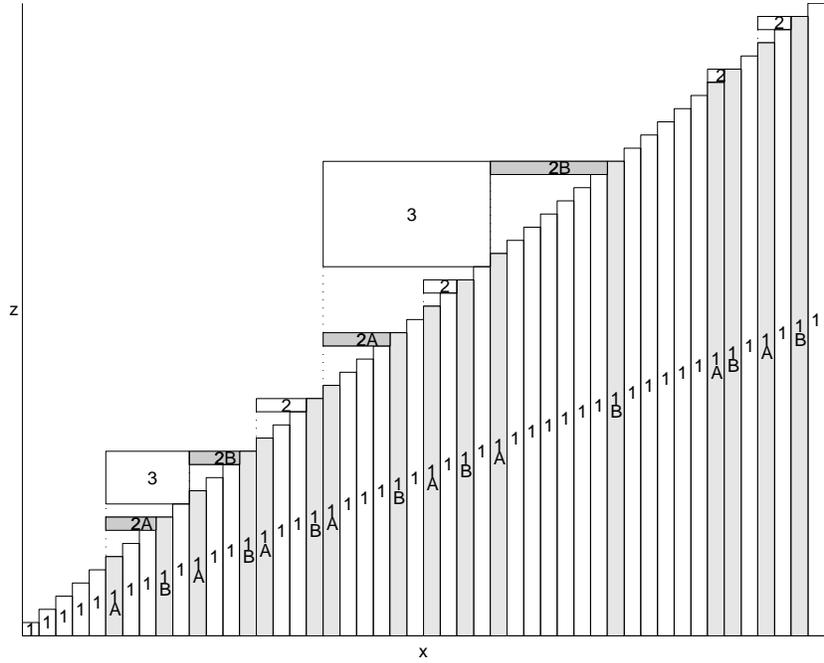


Figure 4: Arrangement of Type 3 blocks for a the 2-set consisting of the shaded 2-blocks. The shaded 2-blocks are taken in order and temporarily labelled A, B, A, B, \dots as shown. For each consecutive pair $2A, 2B$ of 2-blocks, a 3-block is placed with its right face in contact with both the $2B$ and the underlying $1A$, and with its left face vertically above the $2A$ and therefore also above the corresponding $1A$.

Hence this example shows that the chromatic number required to colour the adjacency graph of n cuboids can grow at least as fast as $\log \log n / \log \log \log n$.

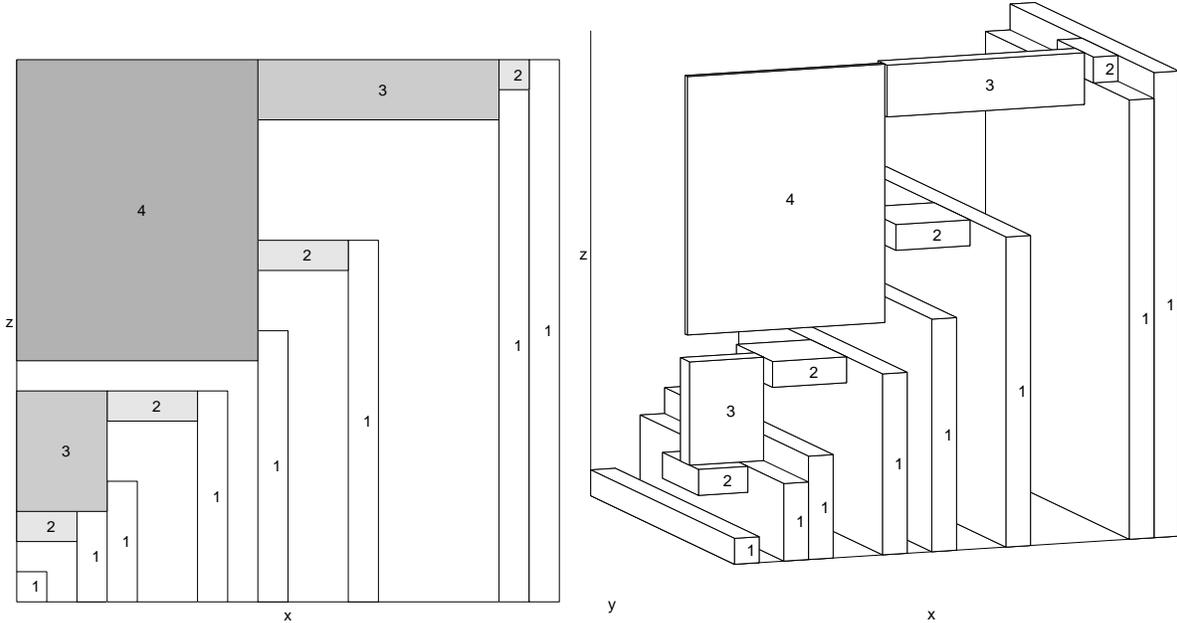


Figure 5: *An arrangement of blocks in a cross-section showing some blocks up to one of Type 4 in the (x, z) -plane (left), and how they might look in 3 dimensions (right).*

4 Upper Bound

If we let $\chi(n)$ denote the largest chromatic number for the adjacency graph of n blocks, there is a simple bound $\chi(n) \leq 4 \log_2 n + 1$ which can be proved inductively by showing that $\chi(n) \leq 4 + \chi(\lfloor n/2 \rfloor)$. For this, let the upper and lower limits of block i in the x -direction be $a_i < b_i$. Let the set of these $2n$ limits be arranged in order as $x_1 \leq x_2 \leq \dots \leq x_{2n}$, and let $t = x_n$. Then let n_A, n_B, n_C, n_D and n_E be the numbers of blocks in the following classes:

$$(A) : a_i < b_i < t; \quad (B) : a_i < b_i = t; \quad (C) : a_i < t < b_i; \quad (D) : t = a_i < b_i; \quad (E) : t < a_i < b_i. \quad (3)$$

Certainly either $n_A + n_B \leq n/2$ or $n_D + n_E \leq n/2$. We shall assume the former and it will be clear that a similar proof applies to the latter. The (A) and (B) blocks can then be coloured with $\chi(\lfloor n/2 \rfloor)$ colours. The blocks in classes (C) and (D) form a planar graph, which is the adjacency graph of the rectangles in which these blocks intersect the plane $x = t + \epsilon$ (for small enough ϵ), so they can be coloured with 4 colours, which we choose different from those used for (A) and (B). Finally since the blocks (E) have $x_n < a_i < b_i$ there are at most n possible upper and lower x -limits for these blocks, and so $n_E \leq n/2$. But the colours that were used on blocks (A) and (B) can be reused on (E), so we have $\chi(n) \leq 4 + \chi(\lfloor n/2 \rfloor)$ as required.

5 Conclusion

Stimulated by the channel assignment problem for WLAN in office blocks, we have shown that an unbounded number of colours may be required for the adjacency graph of cuboids. There is, however, a gap between our lower and upper bounds. An interesting open problem is to obtain tight bounds.

Another interesting problem is that of efficiently colouring office block graphs using a small number of colours. This leads us to the design of approximation algorithms for office block colouring. Thomassen [1] proved that every planar graph can be represented as the adjacency graph of cuboids. This implies that there is no approximation algorithm with guarantee better than $\frac{4}{3}$ unless $P = NP$. There is an $O(\log n)$ -approximation algorithm since our upper bound technique can be implemented in polynomial time. What is the exact approximability of this problem?

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