

## Optimizing the profit from a complex cascade of hydroelectric stations with recirculating water.

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**Abstract.** In modern reversible hydroelectric power stations it is possible to reverse the turbine and pump water up from a downstream reservoir to an upstream one. This allows the use of the same volume of water repeatedly and was specifically developed for hydro-electric stations operating with insufficient water supply. Pumping water upstream is usually done at times of low demand for electricity, to build up reserves in order to be able to produce energy during peak hours, thus balancing the load and making a profit on the price difference.

In this paper, we consider a branched model for hydroelectric power stations interacting in a complex cascade arrangement. The goal of this study is to provide guidance in decision-making aimed at maximizing the profit. A detailed analysis is made of a simpler reservoir configuration, which indicates that even though the problem is nonlinear, a bang-bang type of control is optimal, where the power stations are operated at maximum rates of flow. Some simple relationships between price and timing of decisions are calculated directly. A numerical algorithm is also developed.

**Keywords.** optimization, hydroelectric power, branched cascade, reversible turbines, case study

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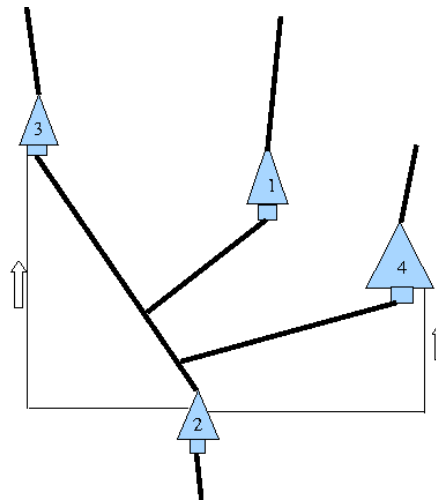
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## 1 Introduction

In modern reversible hydroelectric power stations it is possible to reverse the turbine to pump water up from a downstream reservoir to the upstream one. This is desirable when water supply is low as it allows reuse of the water. Water is usually pumped up at times of low demand and low price, to build up water reserves that can be used later to produce energy during times of high demand and high price, thus balancing the load and making a profit on the price difference. This, however, raises the question of when a turbine should be reversed to maximise profit. For a simple linear cascade composed of two hydroelectric stations that use the same stream the answer is reasonably straightforward. However, more complicated cascades, like the cascade depicted in Fig. 1, also exist and operate. Then there is also the question of which upstream reservoir is best to pump to.

The cascade that is schematically depicted in Fig. 1 was presented as a case study by the Portuguese electricity and gas transmission supply operator, Redes Energéticas Nacionais, S.A. (REN), to the 69th European Study Group held at the University of Coimbra in Portugal in April 2009. The problem proposed was focussed on profit maximization when operating such a system, and how to decide which upstream reservoir to pump to, when there is a choice.

This cascade is composed of four hydroelectric plants that are enumerated as shown in Fig. 1. For this particular system, cascades are formed by plants 2&3, 1&2, and 2&4. In cascades 2&3 and 2&4, water from the reservoir of plant 2 can be pumped back to plants 3 and 4, as indicated by the additional arrows in Figure. Pumping is done via reversible turbines at plants 3 and 4, respectively.



**Figure 1:** Schematic representation of a cascade of four hydro-electric stations.

Relevant main parts of a typical hydroelectric plant, or a power station, depicted in Figure 2, are: (i) the reservoir with its current water content (volume)  $V(t)$ , measured in  $m^3$  and represented by a triangle; and (ii) the turbine that produces electricity, represented by a rectangle. In a cascade of plants some turbines can be capable of operating in reverse, pumping water upstream. The flow

Optimizing the profit from a complex cascade of hydroelectric stations with recirculating water through a turbine at an instant  $t$  is denoted by  $q(t)$ .



**Figure 2:** Representation of a hydroelectric plant.

In a full model of a hydroelectric system, the incoming flows influence the water volumes, which determine the water levels, which in turn determine the heads, which influence the outgoing flows, which influence the reservoir levels as well as the energy sales, and these together with the price determine the profit. It is thus clear that in a full formulation the problem of optimizing the profit is a complex nonlinear stochastic problem. Similar or related problems have been studied for the past few decades. Even though various approaches to this problem have been developed, in general, due to complexity of the problem, these are mathematically complex and are usually perceived to be difficult to implement in practice. Very few operating hydroelectric sites actually use optimization methods for controlling the flows. Furthermore, certain aspects of the problem, like rain, evaporation, demand or pricing, are uncertain or stochastic by their nature. Thus, the methods available in the literature focus on deterministic models, where those aspects are only implicitly stochastic, as well as on stochastic models, where these uncertainties are explicitly taken into account. Some of the techniques involved include linear and nonlinear programming, network flow optimization, dynamic programming and stochastic optimal control. An extensive survey of methods for optimal operation of multi-reservoir systems can be found in the paper by Labadie [3] that was published in 2004.

A goal for this study was optimizing the profit from operating a complex cascade of hydroelectric plants, by developing an optimal strategy for shifting a turbine from one mode of operation to the other. This study focusses on the particular system of hydro-electric plants proposed by REN and depicted in Fig. 1. However, where possible, we also consider a more general system aiming to inform a general approach that may be applied to other systems.

## 2 Notation and Mathematical formulation

Here we define our notation and provide formulae for the specific case of a system of four hydroelectric power stations as depicted in Fig. 1. Below we use the same generic notation for all four plants, which are distinguished by subscripts  $i = 1, \dots, 4$ .

### Specified constants:

- initial and minimal water volumes in reservoirs,  $V_i^{in}$  and  $V_i^0$ , respectively;
- nominal, minimal and maximal water levels (metres above sea level) in reservoirs,  $Z_i^0, Z_i^{\min}$  and  $Z_i^{\max}$ , respectively;

- nominal flowrates and heads  $q_i^0, h_i^0$ ;
- other constants:  $\alpha_i, \beta_i, \zeta_i, \xi, \mu_i, \phi_i$ .

**Specified functions of time:**

- inflows from rivers  $I_i(t)$ ;
- the market price of 1 MWh  $\text{price}(t)$ .

**Calculated functions of time:**

- water volumes in reservoirs  $V_i(t)$ ;
- water levels in reservoirs  $Z_i(t)$ ;
- heads (differences in water levels)  $h_i(t)$ ;
- head losses  $\Delta h_i(t)$ .

**Sought-for functions of time:**

- flow rates  $q_i(t)$ .

Here we adopt the convention that pumping water upstream corresponds to  $q_i(t) < 0$ , and ‘turbinning’ (producing electricity) corresponds to  $q_i(t) > 0$ . Where appropriate, we also use the notation  $q^P$  and  $q^T$  for the rates of pumping and turbinning, respectively; then our convention is that  $q^P$  and  $q^T$  are non-negative.

The following expressions relate the elements of the system. They arise directly from the physics of water flow and electricity generation, and simply state that water volume is conserved, that water level varies according to flows in and out of a reservoir, that flowrate down a pipe varies as the square root of the height difference (pressure difference) driving the flow, and that the power produced is proportional to the height difference and flowrate (force times velocity).

The integrals  $\int_0^t f$  should be interpreted as  $\int_0^t f(u)du$ . Also, for the sake of simplicity, the dependence of functions on  $t$  is not always explicitly shown.

Equations

$$\begin{aligned}
 V_1(t) &= V_1^{in} + \int_0^t (I_1 - q_1), \\
 V_2(t) &= V_2^{in} + \int_0^t (I_2 - q_2 + q_1 + q_3 + q_4), \\
 V_3(t) &= V_3^{in} + \int_0^t (I_3 - q_3), \\
 V_4(t) &= V_4^{in} + \int_0^t (I_4 - q_4)
 \end{aligned} \tag{1}$$

describe the relationships of the water volumes in each reservoir, and follow from conservation of water volume.

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For instance, the most complicated volume function  $V_2$  is obtained as follows. The water volume in reservoir 2 as a function of time is determined by the initial volume  $V_2^{in}$  plus the cumulative volumes (integrals) of all inwards and outwards fluxes from the moment  $t = 0$  to the instant  $t$ . The inflows to the reservoir are comprised of the influx from an ‘external river’ (due to precipitation, for example) given by  $I_2(t)$ , and the outflows  $q_1, q_3, q_4$  from reservoirs 1, 3, and 4. The outflow from reservoir 2 is  $q_2$ . Note that  $q_3, q_4$  can be both positive and negative (for turbining and pumping, respectively), and that when negative these fluxes also cause reduction of water volume in reservoir 2. The formulae for volumes  $V_i, i = 1, 3, 4$ , are obtained analogously.

Water level is a function of reservoir volume, given by the formulae

$$\begin{aligned}
 Z_1(t) &= Z_1^0 + \alpha_1(V_1(t) - V_1^0)^{\beta_1}, \\
 Z_2(t) &= Z_2^0 + \alpha_2(V_2(t) - V_2^0)^{\beta_2}, \\
 Z_3(t) &= Z_3^0 + \alpha_3(V_3(t) - V_3^0)^{\beta_3}, \\
 Z_4(t) &= Z_4^0 + \alpha_4(V_4(t) - V_4^0)^{\beta_4}
 \end{aligned} \tag{2}$$

where the constants  $\alpha_i, \beta_i, Z_i^0, V_i^0$  are fitted to each reservoir.

The water head (the height difference) that drives each turbine is given by the difference in reservoir heights above and below the turbine as follows:

$$\begin{aligned}
 h_1(t) &= Z_1(t) - Z_2(t), \\
 h_2(t) &= Z_2(t) - \xi, \\
 h_3(t) &= Z_3(t) - Z_2(t), \\
 h_4(t) &= Z_4(t) - Z_2(t).
 \end{aligned} \tag{3}$$

The constant  $\xi$  for reservoir 2 is due to the fact that this reservoir is the lowest downstream in the chain of the reservoirs, and hence the outflow from this particular reservoir goes to the sea at atmospheric pressure rather than to another reservoir.

The water level  $Z$  of a reservoir has upper and lower limits that arise from technical or environmental considerations — for example, an empty reservoir has no water in it, and a full reservoir cannot go any higher in water level. These limits are given by the following formulae:

$$\begin{aligned}
 Z_1^{\min} &\leq Z_1(t) \leq Z_1^{\max}, \\
 Z_2^{\min} &\leq Z_2(t) \leq Z_2^{\max}, \\
 Z_3^{\min} &\leq Z_3(t) \leq Z_3^{\max}, \\
 Z_4^{\min} &\leq Z_4(t) \leq Z_4^{\max}.
 \end{aligned} \tag{4}$$

The flow through a turbine is limited by its maximal value  $q^0 \sqrt{\frac{h}{h^0}}$ , where  $q^0, h^0$  are constants that are specific to a turbine. The maximum pumping rate is given by  $q^0 - \zeta(h - h^0)$ , where the parameters are also machine dependent. The lower bounds for  $q_3$  and  $q_4$  are then obtained by

recalling that, with our definition,  $q$  is negative when pumping:

$$\begin{aligned}
 0 &\leq q_1(t) \leq q_1^0 \left( \frac{h_1(t)}{h_1^0} \right)^{1/2}, \\
 0 &\leq q_2(t) \leq q_2^0 \left( \frac{h_2(t)}{h_2^0} \right)^{1/2}, \\
 -q_3^0 + \zeta_3(h_3(t) - h_3^0) &\leq q_3(t) \leq q_3^0 \left( \frac{h_3(t)}{h_3^0} \right)^{1/2}, \\
 -q_4^0 + \zeta_4(h_4(t) - h_3^0) &\leq q_4(t) \leq q_4^0 \left( \frac{h_4(t)}{h_4^0} \right)^{1/2}.
 \end{aligned} \tag{5}$$

Finally, by varying the control variables  $q_i(t)$ , we can change the net profit. At a given moment, each plant either turbines producing revenue, or pumps water upstream originating expenses, or the system is shut and there is zero flow. We combine the formulae for the value of power output and the expense of pumping in the piecewise function  $r(t)$ :

$$r_i(t) := \begin{cases} 9.8q_i(h_i(t) - \Delta h_i^T(t))\mu_i^T(1 - \phi_i) & \text{if } q_i(t) \geq 0, \\ 9.8q_i(h_i(t) + \Delta h_i^P(t))\frac{1}{\mu_i^P(1 - \phi_i)} & \text{if } q_i(t) < 0. \end{cases} \tag{6}$$

Here

$$\Delta h_i^T(t) = \Delta h_i^{0T} \left( \frac{q_i(t)}{q_i^{0T}} \right)^2$$

represents friction losses when turbining, and

$$\Delta h_i^P(t) = \Delta h_i^{0P} \left( \frac{q_i(t)}{q_i^{0P}} \right)^2$$

is due to frictional losses when pumping; both these values are expressed as a head loss. The nominal values  $\Delta h_i^{0T}$ ,  $q_i^{0T}$ ,  $\Delta h_i^{0P}$  and  $q_i^{0P}$  are constants specific to each turbine; the parameters  $\mu_i^T \approx 0.95$  and  $1/\mu_i^P \approx 1.1$  represent efficiencies of turbines in electricity production mode ( $T$ ) and pumping mode ( $P$ ), respectively.

The market price of energy per megawatt is  $\text{price}(t)$ . The objective of the process is maximizing the profit, defined as

$$\text{Profit} = \int_0^1 \text{price}(t) \cdot \sum_{i=1}^4 r_i(t) dt.$$

The profit is to be maximized subject to the constraints listed in equations (1)–(6).

### 3 Energy Considerations

A simple approach to the question of whether to pump to the higher (perhaps smaller) reservoir, or to the lower (perhaps bigger) reservoir is to consider the cost benefit. Two cases are considered, one that there is plenty of water but limited time to pump or turbine it, and the other that there is limited water but plenty of time. Head losses are neglected.

### 3.1 Limited Water

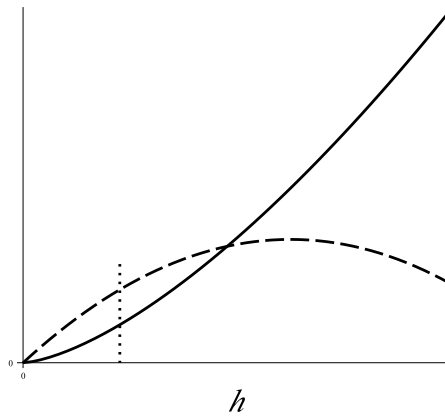
Here we consider the benefit of pumping a unit volume of water in unit time (so  $q = -1$ ) up through a head difference of  $h$  at a time when the cost is  $C$ , and then later turbining that unit volume (so that  $q = 1$ ) when the price of electricity is  $P$ . The profit is proportional to  $(P - C)h$  according to the formulae above. In this case, it is clearly better to pump to the higher reservoir, if there are no other considerations such as that reservoir being full. This corresponds to maximising the potential energy of the unit of water.

### 3.2 Limited Time

Now consider the benefit of pumping at maximum flowrate  $q^P$  (which depends on head) for a fixed small time when costs are low at the value  $C$ , then turbining water at maximum flowrate  $q^T$  for the same fixed small time when values are high at  $P$ . Then the net profit takes the form  $APq^T h - BCq^P h$ , with constant  $A, B$ . That is, net profit is

$$APh^{3/2} - BC(h - Dh^2),$$

where  $D$  is a constant. This means the net profit depends on the difference between a curve varying as  $h^{3/2}$  and a curve with the shape of an upside-down quadratic in  $h$ . Curves with these shapes are sketched in Fig. 3, and again indicate that it is advantageous (for large enough heads) to pump to the higher reservoir, if the choice has to be made and if other constraints do not prevent this.



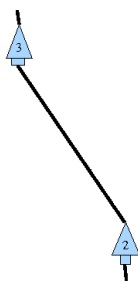
**Figure 3:** A sketch of two curves, the solid line of  $h^{3/2}$ , and the dashed line of  $h - h^2$ , to show the general shapes taken. The dotted vertical line shows where the maximum net loss is, to the right of which it is better to pump to the higher reservoir.

## 4 A Simple Model

We now focus on a simplified system by considering a linear cascade composed of a plant and a reservoir downstream (see Fig. 4). The purpose of this is to reduce the system to one that is amenable to some careful analysis, while still retaining interesting behaviour.

For the sake of simplicity, we assume there is no influx of water into the system from any external source, that is  $I_2 = I_3 = 0$ . Furthermore, we assume that the downstream reservoir (reservoir 2 in Fig. 4) has no turbine, and hence there is no flux from it. That is we assume that  $q_2 = 0$  always, and we will use notation  $q$  for  $q_3$ . We also prevent turbine 3 from pumping, so  $q \geq 0$ .

Note that the formulae  $Z = Z^0 + \alpha(V - V^0)^\beta$ , with  $\alpha, \beta > 0$ ,  $h = Z_3 - Z_2$ , imply that the functions  $Z_i$  are increasing with  $V_i$ , and hence bounds for  $Z_i$  translate to bounds for  $V_i$ . Further,  $h$  as a function of  $V_2, V_3$  is continuous, decreasing with  $V_2$ , and increasing with  $V_3$ .



**Figure 4:** Scheme of hydroelectric cascade with two reservoirs.

Thus, our model is constrained as follows:

$$V_3(t) = V_3^{\text{in}} + \int_0^t -q \geq 0, \quad (7)$$

$$V_2(t) = V_2^{\text{in}} + \int_0^t q \leq V_2^{\text{max}}. \quad (8)$$

There exists a continuous function  $h(x, y)$ , decreasing in  $x$  and increasing in  $y$  so that  $h(V_2(t), V_3(t))$  gives the head. We use the notation  $h(t) = h(V_2(t), V_3(t))$ . The bounds on the flow are given by

$$0 \leq q(t) \leq q_0 \sqrt{\frac{h(t)}{h_0}}, \quad (9)$$

and the simplified function to maximize is the profit

$$\text{Profit} = \int_0^1 p(t)q(t)h(t)dt.$$

Here  $p(t)$  is the price of electricity, and all functions are ultimately functions of time  $t$ , a parameter often suppressed for the sake of simplicity.

To solve the problem we are interested in, namely how to choose  $q(t)$  in order to maximize the profit, we consider how the profit changes when  $q$  is replaced by a new function  $\hat{q} = q + w$ . We say



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$w$  is the *change* (from  $q$  to  $\hat{q}$ ). The change  $w$  is said to be *feasible* if the constraints (7)–(9) still hold after its application. The volume functions associated with  $\hat{q}$  are

$$\hat{V}_3(t) = V_3(t) - \int_0^t w, \quad \hat{V}_2(t) = V_2(t) + \int_0^t w.$$

Here we use the notation  $\hat{h}(t) = h(\hat{V}_2(t), \hat{V}_3(t))$ , and hereafter the expression ‘for small ...’ will mean ‘there exists an  $\epsilon > 0$  such that for all ... smaller than  $\epsilon$  we have ...’.

**Lemma 1** *Assume  $q(t)$  is a continuous function, and  $t_0$  a point where conditions (7), (8) and (9) are satisfied with strict inequalities. Then there exist small continuous feasible changes  $w(t)$  with  $w(t_0) \neq 0$ . Such changes can be chosen of arbitrarily small support and such that  $\int_0^1 w = 0$ .*

**Proof.** Since by (9),  $q \geq 0$ , then  $V_3$  is non increasing, and  $V_2$  is nondecreasing. Hence (7), (8) are equivalent to  $V_3(1) \geq 0$  and  $V_2(1) \leq V_2^{\max}$ . By hypothesis,  $V_3(t_0) > 0$ ,  $V_2(t_0) < V_2^{\max}$  and  $0 < q(t_0) < q_0 \sqrt{\frac{h}{h_0}}$ . Continuity of the functions involved implies that there is an open  $t_0$ -centered interval  $I$  such that these inequalities still hold for all  $t \in I$  in place of  $t_0$ . Scaling the sin function (for which  $\int_0^{2\pi} \sin t \, dt = 0$ ), we get a continuous function  $w$  of arbitrarily small  $\|\cdot\|_\infty$ -norm, such that  $\text{supp}(w) \subseteq I$ , and  $\int_I w = \int_0^1 w = 0$ . The  $\|\cdot\|_\infty$ -continuity of integrals implies that, for all small enough  $w(t)$ ,  $\hat{V}_3(t)$ ,  $\hat{V}_2(t)$  satisfy inequalities (8) and (9), and that  $\hat{V}_i|_{I_c} = V_i|_{I_c}$ ,  $i = 2, 3$  holds. Therefore, we also have  $h|_{I_c} = \hat{h}|_{I_c}$ . The continuity of the head function in both variables guarantees that for small  $w(t)$ , the function  $\hat{h}(t)$  has only small deviation from  $h(t)$ .  $\square$

**Proposition 1** *Assume we subject  $q(t)$  to a change  $w(t)$  with  $\text{supp}(w) \subseteq I$  and  $\int_I w = \int_0^1 w = 0$ . Then the change in profit is given by*

$$\int_I p(t)(w\hat{h} + q(\hat{h} - h))dt.$$

**Proof.** We have  $\hat{q}\hat{h} = (q + w)(h + \hat{h} - h) = qh + q(\hat{h} - h) + w\hat{h}$ , and therefore

$$\begin{aligned} \widehat{\text{Profit}} &= \int_0^1 p(t)\hat{q}\hat{h} \\ &= \int_0^1 p(t)qh + \int_0^1 p(t)(q(\hat{h} - h) + w\hat{h}) \\ &= \text{Profit} + \int_I p(t)(w\hat{h} + q(\hat{h} - h))dt. \end{aligned}$$

The justification for the shortening of the interval of integration is that  $\text{supp}(w) \subseteq I$  and  $h|_{I_c} = \hat{h}|_{I_c}$  by a similar behaviour for the  $\hat{V}$  with regards to  $V$  (see proof of Lemma 1).  $\square$

We can take these calculations a step further. We recall an instance of Taylor’s theorem which states that if  $f$  is a twice differentiable function on a  $\underline{x}_0 = (x_0, y_0)$ -centered open ball in  $\mathbb{R}^2$  of radius larger than the norm of  $\underline{h} = (h_1, h_2)$ , then for some  $\vartheta$ ,  $0 \leq \vartheta \leq 1$ ,

$$f(\underline{x}_0 + \underline{h}) = f(\underline{x}_0) + \partial_1 f(\underline{x}_0)h_1 + \partial_2 f(\underline{x}_0)h_2 + \frac{1}{2} \sum_{j,k=1}^2 \partial_j \partial_k f(\underline{x}_0 + \vartheta \underline{h})h_j h_k.$$

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Here  $\partial_j$  denotes the derivative with regards to the  $j$ th variable,  $j = 1, 2$ , and underlined letters denote 2-tuples. Now, recalling that  $\hat{h}(t) = h(V_2(t) + \int_0^t w, V_3(t) - \int_0^t w)$ , fixing  $t$  and applying the Taylor formula to the 2-variable  $h$ , we get

$$\hat{h}(t) = h(t) + (\partial_1 h - \partial_2 h)(V_1, V_2) \int_0^t w + \left( \int_0^t w \right)^2 R(t),$$

where

$$R(t) = \frac{1}{2} \sum_{i,j=1}^2 \partial_i \partial_j h(V_2 + \vartheta, V_3 - \vartheta) (-1)^{i+j}.$$

Substituting this into the expression for Profit, and using some simplification of the notation, we find

$$\begin{aligned} \widehat{\text{Profit}} &= \text{Profit} + \int_I p(t) \left( \left( h + (\partial_1 h - \partial_2 h) \int_0^t w + \left( \int_0^t w \right)^2 R(t) \right) w + \right. \\ &\quad \left. \left( (\partial_1 h - \partial_2 h) \int_0^t w + \left( \int_0^t w \right)^2 R(t) \right) q \right). \end{aligned}$$

**Lemma 2** *Let  $I$  be an interval such that the function  $s$  is strictly monotone in  $I$ . Then there exists a differentiable function  $w : I \rightarrow \mathbb{R}$  such that*

1.  $\int_I w = 0$  and
2.  $\int_I s \cdot w > 0$ .

**Proof.** After rescaling we may take the interval to be  $[0, 2\pi]$ . Then function  $w(t) = \sin t$  satisfies (7). Furthermore,

$$\begin{aligned} \int_0^{2\pi} s(t) \sin t \, dt &= \int_0^\pi s(t) \sin t \, dt + \int_0^\pi s(t + \pi) \sin(t + \pi) \, dt \\ &= \int_0^\pi s(t) \sin t - s(t + \pi) \sin t \, dt \\ &= \int_0^\pi (s(t) - s(t + \pi)) \sin t \, dt \end{aligned}$$

holds. Moreover,  $\sin t \geq 0$  on  $[0, \pi]$ , and hence for strictly decreasing function  $s$   $s(t) > s(t + \pi)$  holds, and the integral is positive. If  $s$  is increasing, the same argument works by replacing  $\sin$  with  $-\sin$ .  $\square$

If  $h$  is constant, we have  $(\partial_1 h - \partial_2 h) \equiv 0$  and  $R(t) \equiv 0$ , and then this formula simplifies to

$$\widehat{\text{Profit}} = \text{Profit} + \int_I p(t) h(t) w(t).$$

In this case we can state the following corollary.

**Corollary 1** *If  $h$  is constant and there exists a moment  $t_0$  where all inequalities (7), (8), (9) are strictly satisfied, and in a neighbourhood of which  $p(t)$  is strictly increasing or decreasing, then the Profit can be increased.*

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This corollary can be translated as follows: if  $q$  is the current flow, and there is a moment  $t_0$  such that at this moment

- the reservoir 3 is nonempty, and the reservoir 2 is not full;
- the price is strictly dropping or increasing in some neighbourhood of  $t_0$ ; and
- the turbining is neither zero nor maximal,

then the flow  $q$  can be changed to one with a larger Profit.

Note that up to this point we have not considered the optimal solution. In fact, in the class of continuous functions there does not seem to exist an optimal solution; all solutions that can be improved upon. This suggests that the optimal solutions may be piecewise continuous. Since for a constant  $h$ , maximising  $\int_0^1 p(t)q(t)h(t) dt$  is equivalent to maximising  $\int_0^1 p(t)q(t) dt$ , we consider now this latter problem in more detail. We replace the conservation law of the previous problem, namely  $V_2 + V_3 = V_2^{\text{in}} + V_3^{\text{in}}$ , with  $\int_0^1 q = c$ . This can be interpreted, for example, as that in the interval  $[0, 1]$  the difference between turbining and pumping leads to a (possibly negative) net excess  $c$  in reservoir 3 without violating the volume inequality applying there. Therefore, the following theorem applies to a more general situation where pumping is allowed. From the proof an optimal method for choosing when to pump and when to turbine can be deduced.

**Theorem 1** *Let  $p : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$  be a piecewise strictly monotone continuous function, representing the price of electricity,  $p(t)$ . Then, if in the family of piecewise continuous functions  $\check{P}$  on  $[0, 1]$  there is an optimal solution to the problem*

$$\max \int_0^1 p(t)q(t),$$

*such that  $\int_0^1 q(t)dt = c$  and  $a \leq q(t) \leq b$  hold, then this solution can be found constructively.*

**Proof.** Since  $p : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$  is continuous, it is measurable and bounded; that is, for each  $e \in \mathbb{R}$ , the set  $E(e) = p^{-1}([e, \infty[)$  is a Lebesgue measurable set, and the real valued function  $\mathbb{R}_{\geq 0} \ni e \xrightarrow{l} \lambda(E(e))$  is decreasing, has  $l(0) = 1$ , and is of compact support. Piecewise monotonicity guarantees furthermore that  $l$  is continuous. Assume now the existence of a  $q \in \check{P}$  that is optimal among the feasible functions in  $\check{P}$ . If  $a = b$ , then  $q$  is necessarily constant and equal to  $a$ , and thus the unique feasible and hence optimal function in  $\check{P}$ . So we can henceforth assume  $a < b$ . Then there exists a unique  $\lambda_0 \in ]0, 1[$  such that  $c = \lambda_0 b + (1 - \lambda_0)a$ . By the properties of  $l$ , we find a unique  $e$ , such that  $l(e) = \lambda_0$  holds. From now on let  $E = E(e)$ . We claim that if there exists an optimal function  $q$ , it is given by

$$q(t) = \begin{cases} b & \text{if } t \in E, \\ a & \text{if } t \notin E. \end{cases}$$

To see this, note firstly that clearly  $q \in \check{P}$ ,  $q$  satisfies the bounds, and

$$\int_0^1 q(t)dt = \int_E q dt + \int_{E^c} q dt = \lambda(E)b + (1 - \lambda(E))a = \lambda_0 b + (1 - \lambda_0)a = c$$

holds. Hence  $q$  is feasible. Next we show that every feasible function  $\check{q} \in \check{P}$  with  $\check{q} \neq q$  is improvable (and hence not optimal). Take such a function  $\check{q}$ , and assume that  $\check{q}|_E = q|_E$  holds. Then there exists a  $t_0 \in E^c$  such that  $\check{q}(t_0) > q(t_0) = a$  holds. By piecewise continuity of  $\check{q}$ , we get that  $\int_{E^c} \check{q} dt > \int_{E^c} q dt$  holds. From here we infer that  $\int_0^1 \check{q} dt = \int_{E^c} \check{q} dt + \int_E \check{q} dt > \int_0^1 q dt = c$ , that is a contradiction. Therefore,  $\check{q}|_E \neq q|_E$  must hold. Similarly one can show that  $\check{q}|_{E^c} \neq q|_{E^c}$  holds.

Consequently, there exist  $t_0, t_1$  and intervals  $I_0, I_1$  of the same length, satisfying

$$t_0 \in \bar{I}_0, I_0 \subseteq E^c, t_1 \in \bar{I}_1, I_1 \subseteq E,$$

and  $\check{q}|_{I_0} > a, \check{q}|_{I_1} < b$ . Now define the function

$$\hat{q}(t) = \begin{cases} \check{q} & \text{if } t \notin I_0 \cup I_1, \\ \check{q} - \delta & \text{if } t \in I_0, \\ \check{q} + \delta & \text{if } t \in I_1. \end{cases}$$

For small enough  $\delta > 0$ ,  $\hat{q} \in \check{P}$  is feasible, and observing that  $p|_E > e. > p|_{E^c}$  holds almost everywhere (by strict monotonicity), we get

$$\int_0^1 p\hat{q} dt = \int_{[0,1]} p\check{q} dt + \delta \left( \int_{I_1} p dt - \int_{I_0} p dt \right) > \int_0^1 p\check{q} dt.$$

This shows that  $\hat{q}$  improves  $\check{q}$  and completes the proof.  $\square$

This Theorem claims, in the Corollary above, that if there exists an optimal control  $q(t)$ , then this optimal control is necessarily a bang-bang process: in order to maximize the profit it requires either pumping at the maximum possible rate, or turbining at the maximum flow. Note that the assumption that  $p(t)$  is strictly monotone is not very restrictive: price may be sampled and re-ordered to ensure that it is at least monotone if not strictly so.

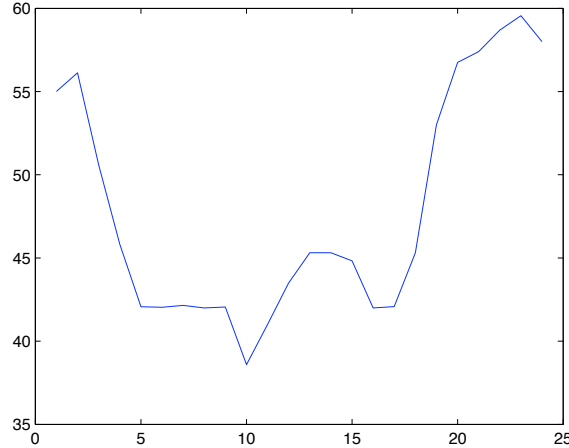
**Example 1** Assume that we have a price curve as shown in Fig. 5, and the constraints  $-1 \leq q \leq 2$  and  $\int_0^1 q dt = 1$ . The problem is maximizing  $\int_0^1 pq dt$  under these constraints.

**Solution:** From the equation  $\lambda_0 \cdot 2 + (1 - \lambda_0) \cdot (-1) = 1$  we find that  $\lambda_0 = 2/3$ . Draw a horizontal line  $l$  so that the points (a union of the intervals) in  $[0, 1]$  obtained by projecting the part of the curve lying above  $l$  has measure  $2/3$ . Turbine in the intervals formed by that set of points at rate 2 and pump at rate 1 in the remaining intervals.

The discrete analogue to the above theorem reads as follows.

**Theorem 2** Assume  $p_1, p_2, \dots, p_n$  are positive real numbers. Then the problem

$$\max \sum_i^n p_i q_i$$



**Figure 5:** *The price curve used in our example, showing the price of electricity during a winter's day.*

such that  $\sum_{i=1}^n q_i = c$  and  $a \leq q_i \leq b$  hold, has a solution such that all but at most one of the  $q_i$  are equal to  $a$  or  $b$ .

This theorem can be proved similarly to the previous theorem, or by using linear programming. Here the existence of an optimal solution can be readily guaranteed. In Section 6 we give an illustration of the theorem by means of a computational implementation.

## 5 More Complex Models

We now consider a complex cascade of  $n$  hydro-electric stations (Fig. 1). We assume that the result of previous Section can be extended to this more general case, so that to maximize profit a turbine must be operated at maximum flowrate for both regimes, turbining and pumping. That is, we assume that the control of the turbine and shifting from turbining to pumping is a bang-bang process.

This is a serious assumption, but is in the spirit of a heuristic approach to solving optimization problems that cannot be solved in any other way, even numerically [3], since the parameter space to be searched is prohibitively large. Note that bang-bang solutions are optimal when the objective function is linear in the control variable (here the flowrate), according to Pontryagin's minimum principle [1]. This corresponds to linearizing our objective function in the flowrates.

We also assume that the price of electricity is known in advance for the operational time interval.

We consider separately three possible cases, namely:

1. There is enough water and spare reservoir volume for continuous operation of all stations with water recirculation (*Large reservoirs case*).

2. There is not enough water for continuous operation of the stations with water recirculation; redistribution of water between these stations can be needed in this case (*Small reservoirs case*).
3. There is not enough water for continuous operation in stations with water recirculation, but there is a reserve of water in the reservoirs of “downstream only” stations; optimizing the usage of this water (with possible redistribution of this reserve for continuous operation of the stations with recirculation) is the question in this case (*Not enough water case*).

### 5.1 Case 1: Large Reservoirs

In this case, we assume that all reservoirs have enough water and enough empty space for continuous operation, that is, that the bounds on reservoir size never act to limit electricity production or pumping of water.

We consider operation of a system of  $n$  hydro-electric stations on the time interval  $[0, T]$  (where  $T$  may be 24 hours, or a week, or any other sensible period of time). We assume that for each of these stations the operational time interval  $[0, T]$  is divided into three operational states as follows:

1. on the interval  $T_i^T = [a_i, b_i]$  the  $i$ th station produces electric energy with the power  $P_i(t)$  releasing water downstream with the flow rate  $q_i^T(t)$ ;
2. on the interval  $T_i^P = [c_i, d_i]$  the  $i$ th station produces a reserve of water that can be used later, by pumping water upstream at the flow rate  $q_i^P(t)$ , and consuming energy to operate the turbine in pumping mode with the power  $C(t)$ ;
3. outside of these two intervals there is no flow. The intervals do not overlap, and without loss of generality we will assume the second interval follows the first.

For each station, profit is

$$\int_{a_i}^{b_i} p(t) P_i(t) dt - \int_{c_i}^{d_i} p(t) C_i(t) dt,$$

and the total profit of the system of stations over the operational interval is

$$U(a_i, b_i, c_i, d_i) = \sum_i \left( \int_{a_i}^{b_i} p(t) P_i(t) dt - \int_{c_i}^{d_i} p(t) C_i(t) dt \right).$$

Here,  $p(t)$  is the price of electricity that we assume known in advance, and the powers  $P(t)$  and  $C(t)$  are (dropping the subscripts)

$$P(t) = g\mu^T q^T(t) \cdot (h(t) - \Delta h^T)(1 - \phi)$$

and

$$C(t) = g \frac{1}{\mu^P} q^P(t) \cdot (h(t) + \Delta h^P),$$

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respectively. Here  $g$  is gravitational acceleration,  $g = 9.8 \text{ m.s}^{-2}$ ;  $h(t)$  is the head of a station (the difference between the levels of the upper and lower reservoirs),  $\Delta h^P$  and  $\Delta h^T$  represent the losses due to drag and friction, represented here as head losses,  $\Delta h^{P,T} = \Delta h_0^{P,T} (q^{P,T}(t)/q_0^{P,T})^2$  and  $\mu^{P,T}$  and  $\phi$  are constants.

In this subsection we assume that each station with water recirculation has sufficient reserve of water for continuous operation, and that the  $i$ th reservoir has an excess of water  $\Delta V_i$  over the time period (perhaps from precipitation), that is to be discarded from recirculation;  $\Delta V_i$  can be negative when we decide to store some extra water into the reservoir. That is, for each station with water recirculation we have the constraint

$$\int_{a_i}^{b_i} q_i^T(t) dt - \int_{c_i}^{d_i} q_i^P(t) dt = \Delta V_i. \quad (10)$$

Guided in a heuristic manner by the results of the previous section, and noting Labadie's comment [3] that heuristic methods are usually required, we assume that for both modes, pumping water and producing energy, operating the turbine at the maximum possible flow rate maximizes profit. Then we can formulate the problem of maximizing profit as an extremum problem with Lagrange multipliers and consider the functional

$$\begin{aligned} U(a_i, b_i, c_i, d_i) &= \sum_i \left( \int_{a_i}^{b_i} p(t) P_i(t) dt - \int_{c_i}^{d_i} p(t) C_i(t) dt \right) \\ &+ \sum_i \lambda_i \left( \int_{a_i}^{b_i} q_i^T(t) dt - \int_{c_i}^{d_i} q_i^P(t) dt - \Delta V_i \right). \end{aligned}$$

Extrema occur at critical points, which are either boundary values or obey the following set of equations:

$$\begin{aligned} \frac{\partial U}{\partial a_i} &= -p(a_i)P_i(a_i) - \lambda_i q_i^T(a_i) = 0, \\ \frac{\partial U}{\partial b_i} &= p(b_i)P_i(b_i) + \lambda_i q_i^T(b_i) = 0, \\ \frac{\partial U}{\partial c_i} &= p(c_i)C_i(c_i) + \lambda_i q_i^P(c_i) = 0, \\ \frac{\partial U}{\partial d_i} &= -p(d_i)C_i(d_i) - \lambda_i q_i^P(d_i) = 0. \end{aligned} \quad (11)$$

It is remarkable that in this case these equations are independent for each of the stations. According to (11), the equalities

$$\frac{p(a_i)P_i(a_i)}{q_i^T(a_i)} = \frac{p(b_i)P_i(b_i)}{q_i^T(b_i)} = \frac{p(c_i)C_i(c_i)}{q_i^P(c_i)} = \frac{p(d_i)C_i(d_i)}{q_i^P(d_i)} = -\lambda_i \quad (12)$$

hold for each reservoir. The values  $Q_i^T(t) = \frac{p(t)P_i(t)}{q_i^T(t)}$  and  $Q_i^P(t) = \frac{p(t)C_i(t)}{q_i^P(t)}$  have transparent meaning: the price of a unit volume of water that is flushed down or pumped up, respectively, for the  $i$ th reservoir at the moment  $t$ . The equalities (12) emphasize the fact that the power company really

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trades in water. We further refer to the values  $Q_i^P(t)$  and  $Q_i^T(t)$  as effective prices of water, for pumping and turbining respectively.

Note that for a given reservoir and at a given moment the prices of water will be different for turbining and pumping, that is  $Q_i^T(t) \neq Q_i^P(t)$ , and hence we apply the indexes  $T, P$  for turbining and pumping, respectively. This difference is mostly due to the fact that energy production is not a perfect process, and only a fraction of the potential energy that is stored by water can be turned into electric energy. In general,  $Q_i^T(t) < Q_i^P(t)$  holds. Indeed, if

$$P(t) = g\mu^T q^T(t) \cdot (h(t) - \Delta h^T)(1 - \phi)$$

and

$$C(t) = g\frac{1}{\mu^P} q^P(t) \cdot (h(t) + \Delta h^P),$$

then the effective water prices are

$$Q^T(t) = g\mu^T p(t) \cdot (h(t) - \Delta h^T)(1 - \phi)$$

and

$$Q^P(t) = g\frac{1}{\mu^P} p(t) \cdot (h(t) + \Delta h^P),$$

respectively. It is easy to see that even in the case  $\mu^T(1 - \phi) = \frac{1}{\mu^P} = \mu$ , the price difference is

$$\Delta Q = Q^P(t) - Q^T(t) = g\mu p(t) \cdot (\Delta h^P + \Delta h^T) > 0,$$

which is due to the head loss terms. Furthermore, due to different heads and differences in operating efficiencies, for two different stations the same volume of water can be turned into different amounts of electric energy. Therefore, for a given moment of time the prices  $Q_i^T(t)$  and  $Q_i^P(t)$  for different reservoirs are different.

The equalities (12) literally state that in order to maximize the profit, each process, turbining or pumping, must start and end at the moments when the effective price of water is equal. That is, the conditions

$$Q_i^T(a_i) = Q_i^T(b_i) = Q_i^P(c_i) = Q_i^P(d_i) \tag{13}$$

must hold for each reservoir.

The length of each regime, turbining,  $T_i^T = b_i - a_i$ , and pumping,  $T_i^P = d_i - c_i$ , can be found from the water balance constraint. By the Lagrangian minimax principle,  $n$  equalities

$$\frac{\partial U}{\partial \lambda_i} = \int_{a_i}^{b_i} q_i^T(t) dt - \int_{c_i}^{d_i} q_i^P(t) dt - \Delta V_i = 0$$

must hold. These conditions are equivalent to the water balance constraints (10) that hold in this case for each reservoir (and hence there are  $n$  constraints in total). These constraints are particularly simple if we assume that the flow rates  $q^T(t)$  and  $q^P(t)$  are constant in time. (Such



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an assumption appears to be accurate for reasonably short time intervals that are of order of a few hours.) Under this assumption,

$$\int_{a_i}^{b_i} q_i^T(t) dt = q_i^T(b_i - a_i), \quad \int_{c_i}^{d_i} q_i^P(t) dt = q_i^P(d_i - c_i),$$

and hence  $q_i^T T_i^T - q_i^P T_i^P = \Delta V_i$ . That is, when there is no excess of water to discard from circulation,  $\Delta V = 0$ , and the lengths of these two regimes are simply inversely proportional to the flow rates:

$$\frac{q_i^P}{q_i^T} = \frac{T_i^T}{T_i^P}.$$

We have to comment, however, that, due to the fact that  $Q_i^T(t) \neq Q_i^P(t)$  ( $Q_i^T(t) < Q_i^P(t)$ ), we cannot put  $T^T + T^P = T$  even if shifting a turbine from energy producing to pumping and back were possible. (Generally this shifting requires some time.) By (13),  $Q_i^T(b_i) = Q_i^P(c_i)$  must hold at the moments of the end of turbining mode,  $t = b_i$ , and the begin of pumping,  $t = c_i$ . But  $Q_i^T(b_i) < Q_i^P(b_i)$  and  $Q_i^T(c_i) < Q_i^P(c_i)$ . That is,  $b_i \neq c_i$ , and there is a time lag between stopping turbining at  $t = b_i$  and beginning pumping at  $t = c_i$ .

## 5.2 Case 2: Small Reservoirs

We assume now that the stock of water is not sufficient to maintain continuous operation of those stations that can recycle water. In this case the available water must be re-distributed between the reservoirs with the aim of maximizing profit. We assume that there is no excess of water, that is  $\Delta V = 0$ , and hence the water balance constraint is

$$\sum_i \left( \int_{a_i}^{b_i} q_i^T(t) dt - \int_{c_i}^{d_i} q_i^P(t) dt \right) = 0. \quad (14)$$

The problem is re-formulated accordingly as

$$\begin{aligned} U(a_i, b_i, c_i, d_i) &= \sum_i \left( \int_{a_i}^{b_i} p(t) P_i(t) dt - \int_{c_i}^{d_i} p(t) C_i(t) dt \right) \\ &+ \lambda \sum_i \left( \int_{a_i}^{b_i} q_i^T(t) dt - \int_{c_i}^{d_i} q_i^P(t) dt \right). \end{aligned}$$

As before, we are looking for the values of  $a_i, b_i, c_i, d_i$  that maximize this functional. Internal critical points occur when

$$\begin{aligned} \frac{\partial U}{\partial a_i} &= -p(a_i)P_i(a_i) - \lambda q_i^T(a_i) = 0, \\ \frac{\partial U}{\partial b_i} &= p(b_i)P_i(b_i) + \lambda q_i^T(b_i) = 0, \\ \frac{\partial U}{\partial c_i} &= p(c_i)C_i(c_i) + \lambda q_i^P(c_i) = 0, \\ \frac{\partial U}{\partial d_i} &= -p(d_i)C_i(d_i) - \lambda q_i^P(d_i) = 0. \end{aligned} \quad (15)$$

Hence,

$$\frac{p(a_i)P_i(a_i)}{q_i^T(a_i)} = \frac{p(b_i)P_i(b_i)}{q_i^T(b_i)} = \frac{p(c_i)C_i(c_i)}{q_i^P(c_i)} = \frac{p(d_i)C_i(d_i)}{q_i^P(d_i)} = -\lambda \quad (16)$$

hold for each reservoir.

Note that in contrast with the previous subsection, here  $\lambda$  is the same for all reservoirs with water recirculation. Using the idea of effective price of water from the previous subsection, we can rewrite (16) as

$$Q_i^T(a_i) = Q_i^T(b_i) = Q_i^P(c_i) = Q_i^P(d_i) = -\lambda.$$

The major difference with the earlier case is the fact that here for all stations with water recycling every mode of operating, that is turbining or pumping, should start at the same level of the effective price of water. We note that same price for different reservoirs does not mean the same moment of time, because the cost of pumping and the value of turbining may differ for different reservoirs. For example, if we have just two reservoirs  $i = 1, 2$  with water recycling, then

$$Q_1^T(a_1) = Q_2^T(a_2) = Q_1^T(b_1) = Q_2^T(b_2) = Q_1^P(c_1) = Q_2^P(c_2) = Q_1^P(d_1) = Q_2^P(d_2)$$

must hold. These equalities gives an idea how the total water stocks should be redistributed between the stations with water recirculating in the case when water stocks are insufficient to provide continuous operation of these stations. The condition

$$Q_1^T(a_1) = Q_2^T(a_2) = Q_1^T(b_1) = Q_2^T(b_2)$$

implies

$$\begin{aligned} \mu_1^T p(a_1) \cdot (h_1(a_1) - \Delta h_1^T) &= \mu_2^T p(a_2) \cdot (h_2(a_2) - \Delta h_2^T) = \\ &= \mu_1^T p(b_1) \cdot (h_1(b_1) - \Delta h_1^T) = \mu_2^T p(b_2) \cdot (h_2(b_2) - \Delta h_2^T). \end{aligned} \quad (17)$$

Assuming that both stations use the same technology and are similar in characteristics, and hence  $\mu_1^T \approx \mu_2^T$ , and neglecting by  $\Delta h_1^T, \Delta h_2^T$ , we obtain the conditions

$$p(a_1) \cdot h_1(a_1) = p(a_2) \cdot h_2(a_2)$$

and

$$p(b_1) \cdot h_1(b_1) = p(b_2) \cdot h_2(b_2).$$

That is, for two stations, switching between pumping and turbining should occur at times  $a_1, a_2$  (or  $c_1, c_2$ , respectively) such that the ratios of the energy prices at these times are inversely proportional to the ratios of the heads:

$$\frac{p(a_1)}{p(a_2)} = \frac{h_2(a_2)}{h_1(a_1)}, \quad \frac{p(b_1)}{p(b_2)} = \frac{h_2(b_2)}{h_1(b_1)}.$$

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The assumption  $\mu_1^T \approx \mu_2^T$  and the neglecting of  $\Delta h_1^T, \Delta h_2^T$  are not necessary here: it is easy to see from (17) that optimal times to switch between pumping and turbining satisfy

$$\frac{p(a_1)}{p(a_2)} = \frac{\mu_2^T \cdot (h_2(a_2) - \Delta h_2^T)}{\mu_1^T \cdot (h_1(a_1) - \Delta h_1^T)}, \quad \frac{p(b_1)}{p(b_2)} = \frac{\mu_2^T \cdot (h_2(b_2) - \Delta h_2^T)}{\mu_1^T \cdot (h_1(b_1) - \Delta h_1^T)}.$$

If we have more than two stations with water recirculating in the system, then these conditions must hold for every possible pair of stations.

These conditions also imply that a reservoir with a larger head should start producing electricity at a lower price; this suggests that such a reservoir can start earlier in a rising market, and continue longer in a falling price market, than a reservoir with a smaller head, echoing earlier simple analysis of the relative value of water in different reservoirs in section (3).

As earlier, the water balance constraint

$$\sum_i \int_{a_i}^{b_i} q_i^T(t) dt = \sum_i \int_{c_i}^{d_i} q_i^P(t) dt$$

must also hold in this case. This constraint can be obtained from the condition  $\frac{\partial U}{\partial \lambda} = 0$ .

### 5.3 Case 3: Not enough water

Let us assume now that the stations with water recirculation do not have sufficient reserve of water to maintain continuous operation, and that there are a few stations that are unable to pump water upwards. The latter have some reserve of water that can be used. The question is how this reserve can be used optimally. That is, should we flush this reserve down as soon as possible to provide enough water for continuous operation of a station with water recirculation, or should we apply a different strategy.

In this case it appears that if this reserve or a part of it can be re-distributed for the stations with water recirculation, then it makes sense to flush downstream the volume of water that is sufficient to provide uninterrupted operation of the stations with recirculation: the profit that was missed initially will be substantially replenished after a few recurrent operating cycles of the stations with water recycling.

## 6 Numerical Solutions

The numerical procedure and solutions presented in this section are intended to illustrate the theoretical ideas discussed in earlier sections.

The idea of reformulating the initial optimal control problem as a minimax problem can be further developed to construct a numerical procedure. In such a procedure, we can divide the whole time interval of a number of smaller discrete sub-intervals, assuming that the electricity price,  $p_j$ , is constant on each of these sub-intervals, and then consider a discrete-time minimax

problem. In this Section we indicate a numerical method that utilizes a discretized model of the problem.

We assume that the time period is divided into  $m$  subperiods, and that the electricity price is constant on each of these periods. Then the objective function is given by

$$U = \sum_{j=1}^m p_j \sum_{i=1}^4 r_{ij},$$

where

- price  $p_j$  is the market cost of electricity at the  $j$ th interval,  $j = 1, \dots, m$ ;
- power  $r_{ij} = \begin{cases} 9.8q_{ij}(h_{ij} - \Delta h_{ij})\mu_i(1 - \phi_i), & \text{if } q_{ij} \geq 0, i = 1, \dots, 4; \\ 9.8q_{ij}(h_{ij} + \Delta h_{ij})/\mu_i, & \text{if } q_{ij} < 0, i = 3, 4. \end{cases}$

The nominal point of each reservoir,  $(q_{i0}, h_{i0})$ , is assumed to be given. Parameters  $\mu_i$  and  $\phi_i$  are also known and are related to each power station's efficiency. The power  $r_{ij}$  depends on the head  $h_{ij}$  and the water flux  $q_{ij}$  in the  $i$ th reservoir on the  $j$ th time subinterval, where  $i = 1, \dots, 4$ ,  $j = 1, \dots, m$ . (Please note that here we again assume that the flux  $q$  is positive when the plant is turbining and negative when the plant is pumping water up, and hence we use the notation  $q$  rather than  $q^T$  and  $q^P$  for both regimes of operating.)

The flow rates are subject to constraints. Thus, for the particular system depicted in Fig. 1, the flow rates are limited by the following constraints:

$$\begin{aligned} 0 \leq q_{ij} &\leq q_{ij-1} \sqrt{\frac{h_{ij}}{h_{ij-1}}}, \quad i = 1, 2, j = 1, \dots, m. \\ -q_{ij-1} + \zeta_i(h_{ij} - h_{i0}) &\leq q_{ij} \leq q_{ij-1} \sqrt{\frac{h_{ij}}{h_{ij-1}}}, \quad i = 3, 4, j = 1, \dots, m. \end{aligned}$$

For the discrete-time model, the relationships between the heads, water levels, water volumes and the fluxes of the reservoirs for the particular cascade depicted in Fig. 1 are:

$$\begin{aligned} h_{ij} &= Z_{ij} - Z_{2j}, \quad i = 1, 3, 4, j = 1, \dots, m; \\ h_{2j} &= Z_{2j} - \xi, \quad j = 1, \dots, m; \\ V_{2j} &= V_{2j-1} + I_{2j} + q_{1j} + q_{3j} + q_{4j} - q_{2j}, \quad j = 1, \dots, m; \\ V_{ij} &= V_{ij-1} + I_{ij} - q_{ij}, \quad i = 1, 3, 4, j = 1, \dots, m; \\ Z_{ij} &= Z_{ij-1} + \alpha_i(V_{ij} - V_{ij-1})^{\beta_i}, \quad i = 1, \dots, 4, j = 1, \dots, m. \end{aligned}$$

Here  $\xi$  denotes sea level;  $I_{ij}$  denotes natural water influx into the  $i$ th reservoir on the  $j$ th subinterval;  $V_{ij}$  denotes the volume of water in the  $i$ th reservoir on the  $j$ th subinterval;  $\alpha_i$  and  $\beta_i$  are coefficients which depend on the characteristics of the reservoirs;  $i = 1, \dots, 4$ , and  $j = 1, \dots, m$ .

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The initial volumes  $V_{i0}$  are assumed to be known for each of the reservoirs. There are lower and upper bounds on the volume of water each reservoir can store,

$$V_i^{\min} \leq V_{ij} \leq V_i^{\max}, \quad i = 1, \dots, 4, \quad j = 1, \dots, m,$$

where  $V_i^{\min}$  and  $V_i^{\max}$  denote the bounds given for the  $i$ th reservoir.

In this formulation, the problem can, in principle, be solved numerically by a solver for non-linear optimization, or, if the objective function and the non-linear constraints are linearized, by a solver for linear programming.

## 6.1 Computational experiments

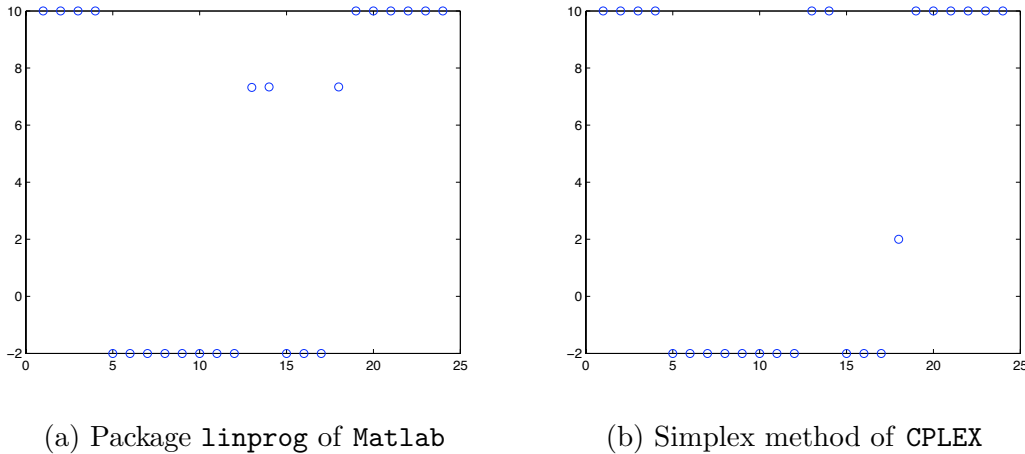
For the particular case of a single hydro-electric plant, with linear objective function and constraints, we have carried out numerical experiments using the numerical method that is described above. For computation we use a 24-hour time interval, divided into 24 subintervals, and the price of electricity given in Fig. 5. That is, we attempted to numerically find the set of constants  $q_j$  ( $j = 1, \dots, 24$ ) which maximize the objective function

$$\max \sum_{j=1}^{24} p_j q_j,$$

subject of constraints

$$\sum_{j=1}^{24} q_j = \theta, \quad -2 \leq q_j \leq 10.$$

Figure 6 shows the results of computations obtained for  $\theta = 100$  by a Matlab package for nonlinear optimisation `linprog` (Fig. a), and by the simplex method with CPLEX [2] (Fig. b).



**Figure 6:** *Solution of the linear problem with  $\theta = 100$*

When the simulation was repeated with  $\theta = 180$ , both CPLEX and Matlab gave the same bang-bang answer, with flow either fully pumping or fully turbining.

These comparatively simple tests illustrate nicely the theoretical results, suggesting that bang-bang values for  $q$  are optimal, with the exception of the Matlab optimization with  $\theta = 100$ , which did come up with a few intermediate flow values in its solution.

## 7 Conclusions

One of the key questions raised by REN at the Study Group was about deciding where to pump spare water, when there is a choice of upstream reservoirs to pump to, with different heads. The answer suggested by some simple analysis, as well as by a minimax approach, is that in general spare water should be pumped to the higher reservoir if possible, given the other constraints that may operate. This corresponds to maximizing the potential energy of the system of reservoirs.

Further analysis focussed on the question of how to decide what flowrates to use, to maximize profit. A theoretical analysis of a simple cascade of two reservoirs with a linear dependence of profit on flowrate indicates that all continuous solutions can be improved upon, suggesting that the optimal solution will not be continuous. Furthermore, if pricing is strictly monotone, it is shown for this simple cascade that any optimal solution if it exists can be constructed as a piecewise continuous bang-bang solution, with full flowrates used in either pumping or turbinning mode.

Further to this, a cascade of  $n$  reservoirs and turbines was considered, with flows assumed to be bang-bang. Maximizing on the times at which pumping and turbinning starts and stops, led to an insight into the importance of the effective price or value of a unit volume of water. Switching between pumping and turbinning should occur at times when these effective prices match. This, in combination with constraints on total flow volumes, can be used to determine the times to switch a turbine from one mode to the other.

However, the modelling and analysis presented here is very simple, is heuristic, and is deterministic. Precipitation, and electricity price and demand, are stochastic processes. Hence in general a nonlinear stochastic computer simulation approach is needed to provide definite scheduling of an operating cascade in practice, no matter whether it is branched or linear in structure. The forward problem, of determining the profit for a given schedule of flowrates and inputs and price/demand, is straightforward to compute. More challenging is the inverse problem, of determining the optimal set of flowrates to use.

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