Fields Mathematics Education Journal

The (Relativity of the) Whole as a Fundamental Dimension in the Conceptualization of the Fraction[1]

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Abstract

In this article, we posit that a fundamental element in the learning and teaching of fractions is the (relativity of the) whole. This focus is grounded in a research project with elementary teachers, with whom we spent three sessions on fractions. In these sessions, the (relativity of the) whole, or what came to be called *the referent*, arose as a significant learning object concerning tasks involving fractions. Through outlining how teachers' meanings unfolded and evolved through the various tasks, we illustrate the complex ramifications underpinning the concept of the whole of a fraction. In addition, we further emphasize the importance played by the (relativity of the) whole in learning fractions as we outline the teachers' engagement with other subconcepts of the fraction concept.

Keywords: fractions, relativity of the whole, conceptual analysis

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In mathematics teaching and learning, fractions have long been seen as one of the most difficult concepts for children and teachers to understand (Ball, 1990; Boulet, 1998; Tzur, 1999; Charalambos & Pitta-Pantazi, 2007; Luo, Lo, & Leu, 2011; Pitkethly & Hunting, 1996; Schifter, 1998; Tobias, 2012). Along with the work of Kieren (1976) and Behr, Lesh, Post, and Silver (1983), these studies confirm the complexity of this concept, which is not a single but a multifaceted construct. Different interpretations of fractions, including part–whole, quotient, measure, ratio, and operator can be viewed as different perspectives for developing an understanding of rational numbers (Kieren, 1976). These interpretations form a basis for examining students' or teachers' understanding of fractions (see Wong & Evans, 2008, on measure interpretation; Noelting, 1980, on ratio interpretation; Mack, 2001, and Prediger & Schink, 2009, on part–whole interpretation).

This said, in most studies, one aspect that has rarely been considered but that we argue is of paramount importance in understanding the fraction concept – perhaps even vital to its understanding – is *the (relativity of the) whole*: what came to be called *the referent* by the teachers engaged in our research project. While most studies about fractions recognize the necessity of the referent to which the fraction refers, the referent in these studies is assumed fixed

[1] Preliminary ideas were developed in papers presented at CERME-8 (Bednarz & Proulx, 2013) and GDM-2012 (Bednarz, Proulx, Nadeau, & Sambote, 2012).

and determines the right answers to problems. However, through examining the various meanings attributed to fractions by teachers in our project, we questioned this assumption about a fixed referent and offered a different perspective about what is mathematically meant by a fraction, leading us to significantly question the process of teaching and learning fractions.

Below, we show how the (relativity of the) whole is of particular importance in working with fractions and explain what we mean by this concept. Following an explanation of the objectives and methodological considerations of our study, we analyze the multifaceted meanings about the whole that emerged in discussions by a group of in-service elementary teachers as they solved a variety of tasks involving fractions. We conclude with a discussion of the outcomes for teaching and learning fractions made possible through considering the concept of the (relativity of the) whole.

The (Relativity of the) Whole as a Fundamental Dimension in Understanding Fractions

The concept of a fraction being relative to a whole has frequently been alluded to by a number of researchers, particularly in studies involving teachers (e.g., Ball, 1990; Luo et al., 2011; Simon, 1993; Schifter, 1998; Prediger & Schink, 2009). Most of these studies have focused on operations on fractions. For example, in Simon's study (1993), prospective teachers were presented with a division-of-fraction problem: "How many cookies can be made with 35 cups of flour if one cookie needs 3/8 of a cup of flour?" Many of the teachers who arrived at a remainder of 1/3 (i.e., 93 and 1/3) defined this remainder as the remaining amount of flour (i.e., 1/3 of a cup of flour) instead of seeing it as 1/3 of what it took to make a cookie (i.e., 1/3 of 3/8 of a cup of flour). Similarly, Tobias (2012) explains that when her prospective teachers were asked to find "how many 1/4 pound servings of dough can be made from 1 7/8 pounds of dough" (p. 3), they answered 7 1/8, saying that 1/8 referred to the whole pound instead of viewing it in relation to a serving of dough (i.e., 7 servings plus ½ of a ¼ pound serving). This raises the significant issue of referring to the *appropriate whole* when discussing fractions in the part–whole interpretation.

Recently, Tobias (2012) addressed the concept of the whole of a fraction as an issue of language use. By highlighting the various difficulties that elementary teachers experience with fractions (e.g., Ball, 1990; Graeber, Tirosh & Glover, 1989; Simon, 1993), she focused on the importance, in both teaching and learning fractions, of conceptualizing the whole of the fraction to contextualize the situation to be solved, to understand what procedures to use, and to interpret the various solutions. She regards these elements as important because the difficulties that teachers experience in interpreting remainders correctly and in properly defining the whole to which these remainders refer affect their ability to understand, assess, and foster how their students think about fractions. She argued that a number of difficulties that learners experience with the concept of the whole arise out of language difficulties or from inappropriate use of language in defining wholes, giving the following example:

> In the context of subtraction, problems such as 3 – 2 can be stated as starting with three objects and taking away two of them. When the situation involves fractions, such as 3 – ½, it is incorrect to interpret this as starting with three objects and taking away half of them. (p. 2)

In the 3 – ½ problem, the ½ is not related to the 3, but to a unit of one. This difficulty (i.e., referring to the appropriate whole) can also be seen in how students convert a percentage to a decimal number. For example, in a context where a certain price, e.g., $50, has been reduced by 30%, and where the student must find the new price, it is not uncommon to see student answers such as 50 – 0.3 = 49.7. In this approach to the problem, the 30% is related to the number 50 and not to a unit of one or to the number 0.3 when converting 30% into decimals. Hence, 50 – 30% = 50 – 15 = 35 and not 50 – 30% = 50 – 0.3 = 49.7.[2] We can see in these examples the importance of understanding the whole to which we refer. For Tobias, incorrectly defining the whole for the fraction can lead to a misinterpretation of a problem and its answer:

> When asked to find how much *of the* pizza was eaten when 14 slices were eaten from two pizzas cut into 12 equal slices each, '*of the*' refers to a whole of two pizzas. Students incorrectly using the whole as one pizza will give solutions such as 14/12 *of the* pizza. This results in a solution describing that more was eaten than what was started with. (p. 2)

For Tobias, therefore, defining the appropriate whole for the fraction in question is fundamental to understanding and working with fractions successfully. Accordingly, she highlighted the importance of addressing language issues to define the whole to make sure that teachers use appropriate language to describe situations and to interpret learners' descriptions adequately. However, even if we agree with Tobias's argument that language issues are important, we believe that more important considerations than simply language issues are at play here in terms of learning and understanding the whole of a fraction. One of these considerations that we posit is related to the whole itself, which we call *the (relativity of the) whole*.

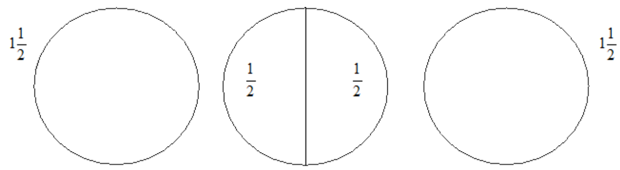
An Illustration of the (Relativity of the) Whole and Its Importance

To clarify what we mean by the (relativity of the) whole and to highlight its importance, we use a grade 4 classroom vignette, taken from an earlier study on fractions (Bednarz, 2000). In this vignette, students who had never been formally introduced to fractions were asked to solve the following problem: "Share three pizzas equally between two children." Three circled pizzas were

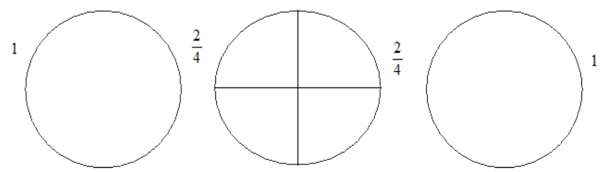
[2] In this sense, an expression like 30% = 0.3, frequently found in school textbooks, might be questioned.

drawn on the board for visual reference, and the students had to find as many ways as possible to solve the problem. Below is an extract of the classroom discussion that ensued between the teacher and her students.

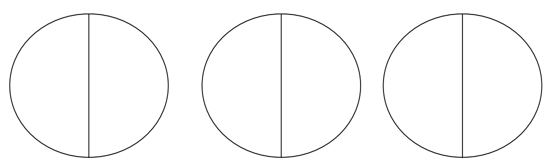
Marlene: One child has a pizza; the other one also has an entire pizza; then one child has half, and another has half. A pizza and a half each (she draws it on the board).



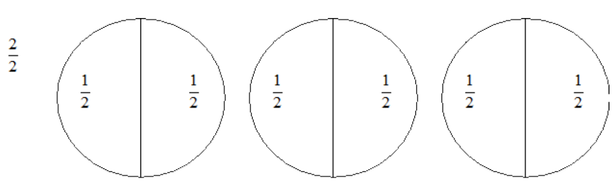
Teacher: OK, does it work? [Pupils: Yes!] Does anyone else have a solution?
Manon: There's one child who has one pizza; the other one has another pizza. I split the remaining one in four: One child gets two fourths, and the other one, two fourths. One child has a pizza and two fourths, and the other one, a pizza and two fourths.



Teacher: OK, another solution?
Veronique: I separated my pizzas like this: half of a pizza for a child, the other half for another child, another half for the first child, another half for the second, another half for the first, another half for the second, which means each child has three sixths.

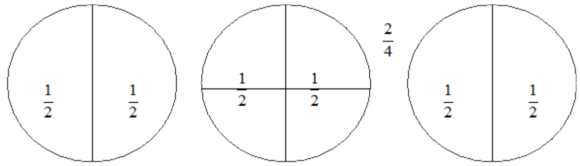


Teacher: What do you think? Is this right?
A student: No! Half plus half, that's two halves.
Teacher: Two halves (writing it on the board). Does this mean two over two?

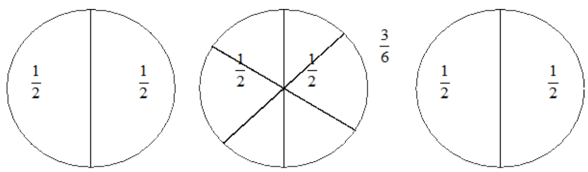


Martin: No, one over two and one over two; that's two over four.

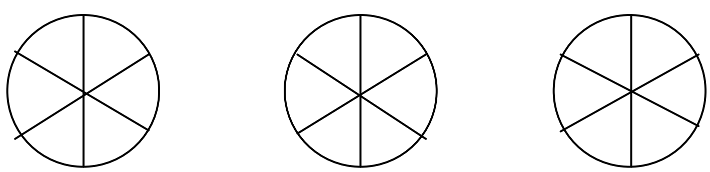
Teacher: You mean, I divided a pizza in four and I took two pieces.



Michel: No, it's not two over four [referring to Martin's solution]; it's one pizza and a half because it's two halves together: one pizza plus one half, that's one pizza and a half.
Veronique: Yes, but three sixths [returning to what she said previously]. That works…
Teacher: Does this mean we divided a pizza in six pieces to find something like that?



Veronique: No, in the end it's all the same.
Teacher: You mean we divided by the same thing and have the same parts of the pizza?
Veronique: We have the same result.
Teacher: OK! Let's go back to three sixths. How do I get this? What bothers me is the three sixths.
Gabrielle: Oh, I know! There's three because there's six in total: six pieces…
Teacher: Oh, as if one pizza was divided into six pieces. Alright, it's like the pizzas were divided into six pieces, right?



Gabrielle: No, we need to divide it in two. It makes three pieces each… All the pizzas, all of them together, are divided into six parts.
Another student: Does this happen by chance? Because, you know, there is one out of two: You take one pizza, and each pizza you take one out of two, and one out of two, and one out of two, and it makes three sixths.

Of interest in this vignette is not whether the teacher reacted appropriately to the students' answers and questions or even whether, based on the answers they gave, the students themselves really understood the concept of fractions. Of significance in the varied answers given is the possibility that *each of these answers might be valid and why*. When Manon says, “one and two fourths,” the whole she is referring to is one of the pizzas, her answer being one pizza and two fourths of one pizza. When Veronique says, “three sixths,” the whole she is referring to is the three pizzas in total, making her answer three sixths of all the pizzas. When Martin adds half and half, which gives him, not two over two, but two over four, is he wrong or could he be adding half of one pizza with half of one pizza, which gives him two of those halves out of four halves? When Michel adds half and half, he arrives at one and not two fourths, having taken a single pizza as the referent. For that matter, could the student’s question about the coincidence of a half plus a half plus a half giving three sixths be representative of a change in the whole opted for to represent the result of the operation?

For us, these are important questions that are not answerable by a right/wrong dichotomy. The consideration of *the (relativity of the) whole* for fractions is closely aligned with the view of Schifter (1998), who asks:

> “How can that piece of cake be 1/2 and 1/4 at the same time?” is the kind of question frequently heard. Hence, the importance of teachers repeatedly confronting apparently paradoxical results, which could only be resolved by sorting out the whole to which the fractions refer. (p. 67)

In the part–whole interpretation, fractions always need to be related to a whole, and this whole can change, which affords mathematical validity to what is offered. For example, the fraction ¼ cannot stand alone; it needs to be ¼ of something. It is in relation to these *somethings* that comparisons can be made. The question “Which is bigger: ½ or ¼?” can generate a variety of answers. If the same whole is considered for both fractions, ½ is bigger; if a different whole is considered for each fraction, then ½ can be bigger than, smaller than, or equivalent to ¼. By relating the fraction to various possible wholes, this *relativity of the whole* becomes of fundamental importance in mathematical reasoning about fractions and operations on fractions (Mack, 2001; Prediger & Schink, 2009; Schifter, 1998; Simon, 1993; Wong & Evans, 2008). In fact, the concept of the *relativity of the whole* is related to the underlying meaning of the concept of fraction itself.

Objectives

Although the studies cited above have reported on issues relating to the nature of the whole in teachers’ or students’ understanding of fractions, few have detailed (a) how this concept of the whole takes shape through the solving of different problems and (b) how it interacts with the meaning given to the fraction itself.

We are extending the fraction-teaching and -learning literature by documenting the multifaceted nature of this concept of the whole as illustrated by a group of practising elementary schoolteachers. We offer an analysis of the various meanings attached to this fundamental aspect of the fraction (i.e., the whole), as well as outlining the links between the meanings given to the concept of the whole and the ones given to the concept of the fraction itself. Because we consider these two concepts to be of central importance in better understanding the phenomenon of teaching and learning fractions, we argue for the paramount importance of *the (relativity of the) whole* in understanding the fraction concept and we offer an enriched perspective about what is mathematically meant by a fraction.

Methodological Considerations and Data Analysis

A group of 10 elementary schoolteachers (grades 4–6) participated in our two-year in-service professional development (PD) research project. Our project combined professional development and research concerns (Bednarz & Proulx, 2010) and is related to a teaching experiment methodology developed by Steffe (1983, 1991) and Steffe and Thompson (2000), which involves documenting learners' conceptualization of a concept over time. In this teaching experiment methodology, the researcher (who is also the teacher) builds models of the meanings developed in action by the students, challenging and continually restructuring these models based on the students' reasoning and actions enacted in new situations. This methodology helps to foster an understanding of how a concept is viewed and understood in all its complexity.

We had a similar goal in our research project: to document how elementary school teachers conceptualized a specific concept: in this case, fractions. As teacher-educator researchers ourselves, we designed and conducted the in-service sessions, participating in the development of the mathematical understandings that ensued, pushing teachers' explorations and understandings of the concept, and modeling their conceptual development (as did Steffe).

The project involved 15 day-long sessions held once a month over one and a half school years. We videotaped all the sessions as well as keeping a written research journal about significant events and reflections provoked by them.

Activities in the sessions covered "mathematical" tasks that teachers engaged with to explore various content. The mathematical themes worked on during the entire project were fractions, division, measurement (perimeter, area, volume) and decimal numbers. These tasks were based on content analyses (didactical, conceptual, epistemological; see Brousseau, 1998) inspired by the research literature and grounded in mathematics teaching and learning situations (see Proulx & Bednarz, 2010).

Tasks could take a number of forms: (a) analyzing and making sense of the students' actual solutions to a problem (chosen to explore a specific understanding, error, or conception); (b) anticipating students' responses to a problem; (c) exploring students' questions or answers to a problem; (d) looking at teaching vignettes about exploring mathematical content (e.g., the above vignette on pizzas); or (e) designing a problem around a specific concept that could be given to the students. These tasks addressed real mathematical events that might or actually did occur in the classroom, thereby illustrating the reality of the teachers' daily mathematical work (Bednarz & Proulx, 2009; Proulx & Bednarz, 2009, 2010). Teachers were invited to engage with these mathematical tasks in small groups followed by plenary discussions, and to explore, discuss, and elicit the mathematical ideas inherent in them.

In this paper, we focus on the first three-session block on fractions. Various problems involving the part–whole relationship in fractions as well as other fraction-related interpretations (ratio, measure) formed the basis of the interactions and discussions during these sessions. We examine eight of these problems that address the meaning of the part–whole in the context of partitioning (see problems A to H in the Appendix), focusing specifically on four of the problems to document the different meanings emerging from the teachers' analysis: A (John and Mary), B (Richard), C (ribbon), and D (pizza).

We adapted our data analysis procedures from the approach proposed by Powell, Francisco, and Maher (2003). The notes from our research journal helped us to focus on specific events of significance that occurred during the sessions. The first stage of our analysis, which took place at the end of the project, involved re-familiarizing ourselves with the sessions, including viewing the videotapes in their entirety to gain a sense of their content. The research team, composed of two researchers and two research assistants, was present at the viewing. Team members took notes to identify specific events of importance (and to compare them with the research journal notes) to ensure coherence in the data analysis procedures.

In the second stage of the analysis, in parallel with our research notes, we produced brief, time-coded descriptions of the content of each video. These were grouped under "events." In stage three, we reviewed the data (videotapes, time-coded notes, and supplementary materials) again to isolate possible *significant events*, leading us to stage four where we created more precise transcriptions of the data (verbal, gestures, drawings on the board, etc.), probing these data in detail to see if additional aspects could be added to the analyses of the previous stages. We analyzed each *event* in detail, relating it to previous and subsequent events to develop patterns concerning teachers' grasp of and insight into the concept of (relativity of the) whole. Our analysis led to the construction of a series of narratives about the events, which we outline below.

Analysis of the Different Meanings for the Concept of the Whole

The relevance of the (relativity of the) whole, which the teachers came to call the *referent* during the sessions, emerged over time as a significant issue to be considered in the teaching and learning of fractions (something noted in our research journal and confirmed by our analysis of the videotape transcripts). We present below the diverse meanings ascribed to the referent during the four tasks involving partitioning contexts (problems A to D). These various meanings assigned to the referent during the sessions were not developed linearly, but arose from the teachers' discussions of the various problems, and continued to be worked on in relation to other problems (something we discuss below in the section on interactions between the relativity of the whole and other aspects of fractions; see as well Bednarz & Proulx, 2011a).

Different Meanings for the Referent Enacted through Various Problems

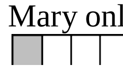
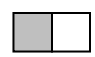
As the teachers explored and discussed the students' solutions for the first problem (problem A) about John and Mary (inspired by Hart, 1981), they posited a number of meanings for the referent. The analysis below presents these various meanings and shows how they arose out of the varied approaches to and understanding of students' solutions to the problem.

John and Mary each have pocket money. Mary has spent ¼ of her amount and John ½ of his. Who spent more money: John or Mary?

Student solution 1:
John because ½ is more than ¼.
E.g.: ½ of 16 = 8, but ¼ of 16 = 4

Student solution 2:
Mary ¼
John ½ = 2/4
John spent more and Mary less

Student solution 3:
I think it is John because he spent ½, which is half, and Mary only spent ¼.

Student solution 4:
Mary only spent a ¼ of her amount

John spent ½ of his amount so he spent more 

Ex.: they have \$8 each.
Mary spent 1/4 = \$2; she still has \$6
John spent ½ = \$4; he still has \$4

Student solution 5:
$\frac{1\times 2}{4\times 2}=\frac{2}{8}$ Mary spent less
$\frac{1\times 4}{2\times 4}=\frac{4}{8}$ John spent more

A referent absent from the solving process. In the initial explanations that some teachers in the group developed, there is no referent present in how teachers made sense of the students' solutions to the problem. This also highlights the fact that the fractions in the problem are considered as absolutes.

M.: We don't know how much money they [John and Mary] have, but it is not so important. This child [referring to the first student] understands the meaning of the fraction: that ½ is

more than ¼. He even gives an example to bolster his understanding: ½ of 16 is 8, ¼ of 16 is 4.[3]
M. [later on]: The meaning of the fraction, I would give him points for that. For me, because he understands the meaning of the fraction (that is, that ½ is more than ¼), he should get some points for that.

It is tempting in this example to say that fractions are treated as numbers, an important understanding in the learning process about fractions (i.e., in the transition from elementary to secondary school, and from natural to rational numbers). However, analysis of the transcript shows that these teachers did not explain fractions as numbers, but mainly took them for granted as a self-evident absolute (i.e., that ½ is simply a bigger fraction than ¼).

A contextual referent. Another meaning emerged in reaction to the previous one, where aspects of the referent were now considered. However, this new meaning, where the referent plays an important role for the answer to the problem, did not affect the concept of the fraction itself, but was mainly considered in relation to the context of the problem.

G. [reacting to M's earlier statement that not knowing how much money John and Mary had was not so important since the student understood the meaning of the fraction]: My students, on the contrary, could have said that we can't answer this question because we don't know how much money John and Mary have at the beginning. For example, if one had several million dollars and the other had very little money, the ¼ could be more [than the ½].
M.: You are right. The question should have stated that each had the same amount [of money] to start with. We ourselves solved [the problem by assuming] that each had the same amount, but you are right. If I have $100 and you have $10, even if you spend half, I stay richer. Our group [of teachers] did not see it like that when we made sense of the solutions.
A. [later on]: But he would have to get some points anyway since part of the answer is right!
M.: For [understanding] the meaning of the fraction, this student would get a good mark. For me, he understands the meaning of a fraction, that ½ is more than ¼.
S.: I would not have given him any points. He does not answer anything. If John is richer because he spent ½, he simply spent more. I don't see that the student understands the concept of the fraction. He only says that ½ is more than ¼, but there is no reasoning *per se*.
J.: [about the fourth solution] But if he does not mention that there is no referent, no amount of dollars, then we can't say that his solution is OK. If the child does not say that [his solution] depends on the beginning amount, I can't give him full marks. There could be different amounts of money involved. In a certain sense, he plays it safe here by assigning an amount at the beginning.

Here the context of the problem makes it necessary for the teachers to consider the referent (i.e., the amount of money at the beginning, which G. and J. both mention, as does S., a point of view

[3] We translated all transcripts from French to English.

accepted by M. when she says “You are right”). However, this contextual referent does not affect the meaning of the fraction itself (i.e., ½ remains bigger than ¼, as both M. and A. state). The referent thus remains independent of the fraction, only affecting the resulting amount in context while the fraction itself continues to be treated as an absolute.

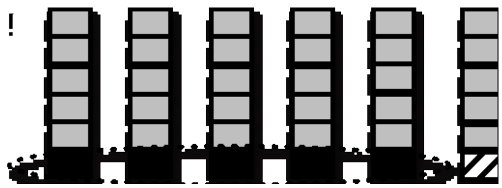
A fixed referent. One meaning put forth was to consider the role of the referent by underlining the importance of fixing it, that is, to determine it in advance. For teachers, this *a priori* fixing of the referent made it possible to operate on fractions and obtain a specific answer.

> M.: It would be necessary in the problem to say that each person has the same amount [of money]. Our group started with the assumption that each had the same amount. [She addresses the teacher educators-researchers:] Was not giving an amount to begin with done on purpose, like G. said? If we want to explore the meaning of fractions, then give children a referent and stop playing around.
> G.: I would ask the students, “What is missing? What other data should be added in the problem? So if John has one amount and Mary another amount, you could say this and that.”
> Mi.: We could also go further when there is missing data. I would present a situation to some students where [John and Mary have] the same amount [of money] and a situation to other students where there are different amounts [of money involved].
> S.: I think that in showing fractions in relation to a whole, we have to compare them to a unit. In the case of solution 3, it is obvious that there is not even a unit.

Even if there is a relationship linking the fraction to a whole (e.g., a contextual referent, as posited by G., Mi., M., and S.), this referent needs to be fixed, as expressed clearly by M., so that there is no ambiguity when operating on the fractions (i.e., so that considerations of the value of the referent are not part of the questioning anymore). This referent can vary from one situation to the next, as Mi. suggests, but it needs to be fixed for each of these situations (as one fixes the parameters of a function when treating a specific case). The comparison between two fractions (in this case, ½ and ¼) is related to a fixed whole, which has been determined in advance so that we do not have to consider it during the process of solving the problem. In this case also, the fraction continues to be treated as an absolute (½ is always bigger than ¼).

A relative referent. The concept of a fixed referent contrasts with that of a relative referent, which came up much later in the sessions. Here is some of the discussion that occurred regarding the ribbon problem (problem C, inspired by Schifter, 1998) and some solutions teachers had to make sense of. The various meanings assigned to the referent illustrated for the teachers the complexity of this new conceptualization, which required viewing the referent from a new point of view.

Mali has 6 meters of material. She wants to make ribbons of 5/6 meters for a school party. How many ribbons can she make and how much material will she be left with?



Student solution 1: 7 and 1/5

Student solution 2: We can make 7 ribbons

Student solution 3: 7 and 1/6 of a ribbon

Student solution 4: 7 and 1/7

S.: [discussing solution 3] This 1/6 as the remainder, is it OK?

A.: The remainder is 1/36. The whole is 36. You cannot say 7 and 1/6.

M.: Yes, you are right.

S.: You cannot say 1/6 because [the whole] is 6 metres, divided into 6 parts.

G.: If you look at 6 meters of material, altogether like the line that I drew representing the 6 metres, you have the 6 metres there. Here it is 5/6, so it's like if you took 5 parts always from 6 parts, and in the end the remainder is one thirty-sixth.

Mi.: But [student 3] says it is 1/6 of a ribbon.

M.: Yes, it's good because he says that it is "of a ribbon."

A.: And what is considered as the ribbon? It is 5/6 of a metre. It is not 1/6 of a ribbon. The ribbon measures 5/6 of a metre. No, [student 3's solution] does not work for me.

N.: For me, the answer is 7 ribbons and 1/6 of a metre [left over].

A.: [discussing solution 1] But it could be 1/5 of a ribbon.

Mi.: Yes, but the ribbon is 5/6 of a metre.

An.: But there are no fifths in this problem.

A.: I mean the ribbon [when created] is 5/5.

Mi.: No! 5/6.

A.: No, no, no. In the end, it becomes 5/5, the whole.

M.: Oh my God!

A.: And then the last piece [left over] remains a part of this whole; it remains 1/5.

Mi.: Yes, I understand what she means.

A.: If we say 1/5 of a ribbon, then the answer is correct.

S.: What is 1/5 of a ribbon? A ribbon is 5/6 [of a metre]. It makes 1/6 of a ribbon; you cannot say 1/5.

Mi.: It is because we always have a fraction in relation to a whole. A ribbon is 5/5. We have changed the whole!

For A. and Mi., the fraction here is both related to a whole and relative to that whole. The same piece of remaining ribbon can take on several values (e.g., 1/5, 1/6, 1/36), depending on which whole is considered, whereas this was not the case when the referent needed to be fixed and determined in advance, as in the earlier John and Mary problem.

In this problem, the referent is seen as variable, whereas in the previous problem, it was conceived of as parametric (i.e., the referent could be changed for different problems, but once it was determined for one problem, it had to remain fixed for that problem). This view of the referent is what appears central in the discussion about the pizza vignette (problem D) and potentially explains why different students arrived at different solutions: 3/6 of the pizzas, 1½ of one pizza, and so forth.

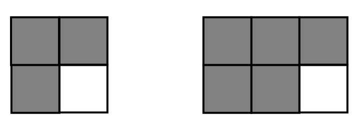
This sophisticated meaning about the referent (i.e., a relative referent) was developed, explained, and refined in the session through a consideration of the other tasks engaged with and the discussions that ensued. However, during this process, other interrelated meanings emerged. Using short excerpts, we present below three other meanings given to the referent through an exploration of various other problems.

A referent attached to a pictorial representation. For a number of tasks where students used pictures in their reasoning (e.g., problem C: the ribbons problem; problem A: John and Mary problem; see also problem F), the referent became closely linked (in how some teachers made sense of it) to a pictorial representation, which became the whole for these teachers. In this case, in the partitioning of the whole into parts, the part considered needed to be completely included in the whole (i.e., in the picture of the whole). This concept of inclusion considers the parts as elements of the whole, a meaning that is linked to the concept of a fraction as a comparison between the part (i.e., a number of pieces) and the whole in which it is included (i.e., the total number of pieces into which the picture is partitioned).

This interpretation first appeared in the Richard problem (problem B, inspired by Hill et al., 2007), which involved a comparison between 5/6 and ¾.

Richard is asked to make a drawing for comparing ¾ and 5/6.

He draws the following:



and explains that ¾ and 5/6 are equivalent.

What do you think of his answer?

A.: We know that it is not equivalent.

Researcher: Is it possible for ¾ to be equal to 5/6?

Many teachers: If the unit is not the same, then ¾ of what, or 5/6 of what?

So.: Not the same whole.

S.: Because here [implying that the student Richard is wrong], Richard has to compare fractions, and we know what the partitioning is. When you understand the partitioning of the fraction, you can make a drawing, divide it, for example, into 4 parts, and take 3 of the parts [showing how the 3 parts are included in the fourth]. We need to compare the same thing.

This pictorial representation of the referent, through interacting with other views of the referent, will appear clearer in some of the other problems discussed during the sessions. This “pictorial representation” view became an obstacle for some teachers, preventing them from understanding other teachers’ views about the referent, as this extract around the ribbon problem (problem C) shows.

In terms of the 7 and 1/5 solution in the ribbon problem, S. demonstrated her understanding of the whole attached to a pictorial representation in the way she explained why we cannot have 1/5 as an answer for the remaining part. The discussion below with another teacher (Mi), who “saw” the relativity of the whole, shows the differences in how the referent is perceived.

S.: It can’t be 1/5 of ribbon

Mi.: Yes it can, if you start with a ribbon that is 5/6.

S.: What is 1/5 of a ribbon? A ribbon is 5/6, so [a part] is 1/6 of ribbon; you can’t say 1/5.

Mi.: Because it is always the fraction in relation to a whole.

S.: We agree that the remaining piece is not inside the ribbon, so it can’t be 1/5.

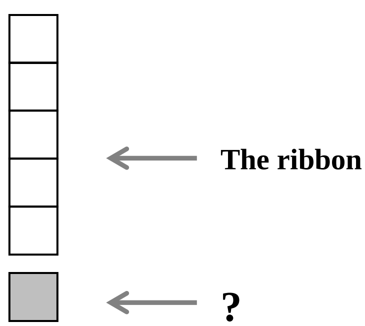
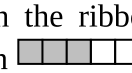


Figure 1. The remaining piece being outside of the whole.

The picture acting as the whole orients the meanings to be given to the referent and the fraction itself. The pieces considered must be “inside” and “part of” this whole. Here the referent is dependent on the drawing, based on seeing a fraction as a comparison between a part and the whole in which the part is included. In the ribbon problem, 1/5 is rejected as the remainder for the remaining piece of ribbon because it is not included in the drawing of the whole (i.e., the ribbon of 5/6).

This meaning about inclusion was referred to in the Richard problem and in the ribbon problem. It is also demonstrated in problem F (“All the fractions that can be seen in ”, inspired by Mason, 2008) where a rectangle is partitioned into five pieces and three pieces are shaded, the question being to find all possible fractions in the drawing. Such a conceptualization leads to seeing 3/5 or 2/5 as possible answers to the problem, but not fractions like 2/3 or 3/2.

A referent as a common denominator. Sometimes the referent is conceived by teachers as the common denominator, particularly in the case of comparison tasks, and becomes the basis for comparing fractions (as we see below in problem B: the Richard problem, which involves a comparison between 5/6 and 3/4).

> S.: Here we are in a case of comparison of fractions. We know what the partitioning is. We need to compare fractions with the same whole. We compare oranges with oranges, apples with apples.
> M.: But there are quarters and sixths, so it is apples and oranges.
> A.: We need to [recast the fractions as twelfths] to make it work.
> M.: We always say that apples compare with apples. This is what our referent is here. So you have to place fractions under the same denominator to compare them, so somehow this common denominator becomes the referent. We put the fractions into twelfths, so 3/4 becomes 9/12, and 5/6 becomes 10/12. And 10/12 is bigger than 9/12.

Of interest here is that by making the common denominator the referent, teachers are transforming the comparison of fractions to a comparison of natural numbers: 5/6 = 10/12 and 3/4 = 9/12, thus comparing 10 (twelfths) and 9 (twelfths). The transformation to a common partitioning (12) and the idea of equivalence underlying this process is not enacted (i.e., if I partition my whole in twice as many parts, to get 12 [the denominator], I need twice the number of parts [the numerator]). The reference is made to an object (a common denominator), which has to be the same for fractions to be compared (i.e., apples with apples, oranges with oranges). Teachers are working here, not on fractions (i.e., considering how a certain part relates to a whole) but on natural numbers, so it is no longer necessary to consider the partitioning.

A referent as a number linked to a fraction operator. In the ribbon problem (problem C), after the teachers decided that the remaining piece is 1/36 ("the whole is 36, you cannot say 7 and 1/6"), the following discussion took place about the remaining piece being 1/6.

> S.: You cannot say 1/6 because it is 6 metres, divided into 6 parts.
> G.: If you look at the number line that I drew representing the 6 metres, you have the 6 metres there. Here it is 5/6, so it's like if you take 5 parts always on 6 parts, and in the end there remains 1/36.
> A.: It is 5/6 of a metre. The ribbon is 5/6 of metre. No, 1/6 of the ribbon does not work.
> N.: For me, the answer is 7 ribbons plus 1/6 metre.
> N.: It means that [the remainder] is 1/36 of 6 metres of material. But it can be also 1/6 of one metre.
> S.: Because we can simplify the fraction: 1/36 x 6.

In the above discussion, the possibility of having 1/6 as the remaining piece is not easily accepted. Some teachers use the concept of the fraction as an operator to arrive at 1/6 of a metre: i.e., 1/36 of 6 metres = 1/36 x 6; the 6 cancels out, hence 1/6 x 1 (1/6 of 1 metre). They thus agree that the same piece of material can be assigned varied values (i.e., 1/36 of 6 metres or 1/6 of 1 metre) by working outside of the reference to a certain whole (i.e., either 6 metres of material or 1 metre of material). Thus, through *the operation of multiplication of a fraction by a natural number*, one can calculate and simplify to obtain the same value (that is, 1/6). In effect, it was not considering different wholes for the same piece of material that led to this conclusion, but rather multiplying and simplifying the numbers, with fractions here being considered as operators.

To sum up, diverse meanings for the referent, intertwined with diverse meanings about fractions itself, emerge from our analysis (see Table 1). These diverse meanings, as we illustrate above, are locally viable[4]. For example, (a) using a common denominator as a referent in the Richard problem enables teachers to give meaning to a comparison of fractions; (b) seeing the referent as a pictorial representation showing the part included in the whole allows teachers to give meaning to the remainder as 1/36 in the ribbon problem or to explore equivalences in the Richard problem; and (c) using the fraction as operator allows teachers to give meaning in the ribbon problem to the remaining piece of material as either 1/36 of 6 metres or 1/6 of 1 metre. These meanings also represent the many ways of making sense of the concept of a fraction in relation to a referent (see Table 1).

As the above examples show, these different meanings assigned to the referent interacted with one another, even within the same task and during the same session. For example, in the ribbon problem, acceptance of 1/5 as a viable interpretation of the remaining piece of ribbon provoked other meanings to be put forth and opposed: e.g., the referent associated with a pictorial representation leading to a rejection of the 1/5 remainder (ribbon problem C) because it was not part of the pictured whole; the referent as a number associated with an operator allowing the remaining piece of ribbon (ribbon problem C) to be given two possible values (1/6 and 1/36) without accepting other values like 1/5; a relative referent that opened several possible viable interpretations of the remaining piece in relation to different wholes.

[4] The notion of viability, grounded in constructivism, is fundamental in our interpretation of data, for giving meaning to the ways of approaching tasks by teachers (Bednarz & Proulx, 2011b). The model developed to give sense to teachers' reasoning is not one of inappropriate conceptions (of referent or fractions), but one of local knowledge that works under particular conditions: "[our conceptual structure, our actions] are viable as long as they do not clash with experience, as long as they remain tenable in the sense that they continue to do what we expect them to do" (Glasersfeld, 1987, pp. 5-6).

Table 1
Diverse meanings about the referent, associated with diverse meaning of the fraction

Diverse meanings for the referent…	***associated with***	**…diverse fraction sense**
Referent absent from the solving process	*…associated with…*	a fraction perceived as an absolute
Contextual referent	*… associated with…*	a fraction treated as an absolute, separated from the contextual referent
Fixed referent determined in advance	*… associated with…*	a fraction treated as an absolute, where the referent does not influence the fraction itself because the treatment of the fraction is independent of this referent (we fixed this referent, and we can then operate or reason without considering it)
Relative referent, which varies	*… associated with…*	a fraction that is relative to a whole, where the same piece can take many "values" in relation to different possible wholes
Referent as a pictorial representation where the part is included (i.e., a picture partitioned into a number of pieces)	*… associated with…*	a fraction seen as a number of pieces over the total number of pieces; the part (the number of pieces considered) has to be included in the (picture of) the whole
Referent as a common denominator	*… associated with…*	a fraction treated as a whole number (comparing 3 fifths to 4 fifths is like comparing 3 to 4)
Referent as a number linked to an operation on fractions	*… associated with…*	a fraction as an operator; the same piece can take different values through operating (1/36 x 6 = 1/6 x 1)

Interactions between the Relativity of the Whole and Other Aspects of Fractions

During the sessions on fractions, the concept of the (relativity of the) whole also played a role in relation to other aspects of fractions, particularly those aspects related to (a) the fraction as an operator and (b) the interpretation of the ratio.

Our analysis outlines an interaction in the ribbon problem (problem C) between an interpretation of the remaining piece of ribbon in terms of the relativity of the whole and an interpretation supported by the operator meaning of the fraction. In considering the remaining piece (see Figure 1) in relation to a whole that could vary, the teachers agreed that this remaining piece could take on a number of values (1/6, 1/36, 1/5, and even others like 1/12 or 1/18, etc.). However, for some teachers, the remaining piece could be given yet other values by seeing the fraction as an operator. For example, some teachers posited that the remaining piece of ribbon had a value of 1/36. But to arrive at this 1/6 of a metre, these teachers explained that 1/36 of 6 metres (i.e., 1/36 x 6) = 6/36 = 1/6, thus agreeing that the same piece of ribbon could have various values (1/36 of 6 metres or 1/6 of 1 metre) *by means of the multiplication of a fraction by* *a natural number,* that is, where fractions are considered as operators. In other words, the operator interpretation of a fraction disregarded the reference to a variable whole in order to give possible varying values to the remaining piece of ribbon.

Kieren (1976), in his conceptual analysis of the operator meaning of fraction, outlined the independence of this operator conceptualization from the part–whole interpretation:

> It should […] be noted that the fraction notion in this interpretation [rational numbers as operators or mapping] is based on the quantitative comparison of two sets or two objects; hence part-whole or class inclusion notions are not central to this interpretation. (p. 11)

Our analysis partially supports this claim about independence. In addition, it shows the possible effect of the operator meaning on the understanding of fractions in a given task where fractions are seen as absolutes, as numbers, as in the first meaning of the referent (i.e., referent absent from the problem-solving process) where the referent to the whole was absent. This limitation of the operator-view in understanding fractions is not without consequence, as was shown by the teachers who believed it was impossible in the ribbon example (problem C) to see the remaining piece of ribbon in terms of 1/5 of the ribbon. In fact, these teachers were surprised when we explained that the standard algorithm for dividing these fractions (the ribbon problem could be solved by a division of 6 by 5/6) would give 1/5 as the value of the remaining piece; this, in fact, is what their secondary-level teacher colleagues would have likely used to solve this problem. This surprise, and not the operator-meaning of fraction, led our teachers to try to make sense of the remaining piece of ribbon in various ways, which for many of them led to a view of the referent as relative (as in the fourth meaning of the referent: a relative referent).

Therefore, even if, as Behr et al. (1983) explain, the operator-meaning of fraction is useful in establishing equivalence between fractions (as was the case for these teachers in making sense of the remaining piece of ribbon as both 1/36 and 1/6), it illustrates the limiting effect of the operator lens for understanding issues related to the (relativity of the) whole, which is fundamental in understanding the concept of fraction. This difficulty is significant if one considers Kieren's (1976) argument that understanding rational numbers requires not only an understanding of each of the separate interpretations of fractions, but also of how these interpretations interrelate.

In the case of our teachers, the relationship was tenuous, and it was not clear how the operator-meaning of fraction could effectively be linked to the part–whole interpretation because the teachers' attention was focused on "operating" and not on the whole, leading to a "number" view of the fraction where the whole was not taken into account.

Another interpretation of fractions is that of the ratio, seen by Behr et al. (1983) as the most natural tool for promoting the concept of equivalence. In the third teacher session on fractions, we observed an interaction between the relativity of the whole and ratio in solving the orange juice tasks (inspired by Noelting, 1980). The teachers compared mixing two different proportions of water and orange juice to decide on the sweetness of the juice: three glasses of orange juice and four glasses of water compared with six glasses of orange juice and nine glasses of water. One teacher explained how she proceeded:

> A.: I have 3 parts orange juice and 3 parts water, and one part of water remains in the first case. In the second case, there are 3 parts water remaining. Thus this one [the second mixing] tastes less sweet.
> [Later, she explained again what she did to another teacher who questioned her.]
> A.: This one [the first mixing] has the equal quantities of water and juice with just one part water left over. Thus, it will taste more "orange" that the other one [the second mixing].

This reasoning is based on an interpretation of the two quantities of juice in terms of ratio: three parts orange juice and three parts water compared with six parts orange juice and six parts water, leading to the assertion "both taste the same." A. then focused on the remaining part of water (one part of water remaining versus three parts of water remaining) to make sense of the degree of sweetness of the juice in each case. Another teacher then provoked more discussion by asking a question about the remaining water:

> N.: Imagine that in the second case, it is not 3 but 2 parts water that remain. What would you say then? Will the juices have the same taste?

This question generated a discussion about various interpretations of fractions where the issue of the relativity of the whole is of concern.

> M.: Still sweeter on that side [the first mixing].
> N.: I think it would be the same. Can you put it into fractions?
> B.: Oh! It would be 6/8 and 3/4. Those are equivalent, no?\
> A.: But I don't understand. For me, both 6 for 6 and 3 for 3 represent one unit each. So I have one unit on this side and one on that side. There are 3 glasses of water remaining here [from the second mixing] and one on this side [from the first mixing].
> N.: No, but imagine there are only 2 glasses of water remaining on this side [the second mixing].
> Mi.: In relation to 2 units.
> A.: No, I do have one unit. My 6 over 6 is one unit, and my other 3 over 3 is one unit.
> Mi.: Yes, but if you compare the unit on this side [the first mixing] with the unit on that side [the second mixing], it is not the same unit. This is one unit [shows 3 glasses of orange juice with 3 glasses of water] whereas here you have 2 units [shows 6 glasses of orange juice for 6 glasses of water].
> N.: Look, I have placed my 3 glasses of orange here with my water and it gives me this [gestures to a small pitcher with her hand]. For this one [the second mixing with 2 remaining] it gives me this [gestures a second pitcher twice as tall as the first one]. I have a double amount. [Both mixtures taste the same now] but if I add another glass of water to each, they do not end up [tasting] the same.

Mi. and N. both approached the problem in terms of the part–whole meaning of a fraction, considering two different referents to explain the effect of pouring more water: one glass of water for one unit versus two glasses of water for two units; for them, each intervention had the same effect on the sweetness of its respective mixture. A., however, reasoned strictly in terms of equivalence of ratios, leading her to see three for three and six for six as the same ratio and both corresponding to one unit itself. For A., there was no longer a whole, since seeing it in terms of ratio somehow "erased" it; there was no interaction between her ratio view and the concept of a whole. The difficulty A. experienced is related to an acceptance that two wholes (unit of 6 over 6 and unit of 3 over 3) are not necessarily equivalent and also not necessarily of the value 1.

This relates to the problem reported in Mack's (2001) study, where students constantly referred back to the unit of one when multiplying two fractions (i.e., 3/4 x 2/3) because they had difficulties considering the new unit (i.e., 2/3). In our case, A. was focusing on *one* (i.e., what she called the unit) and not on the whole.[5] This shows, in fact, how working with ratios can make the referent of the fraction disappear: the whole became a one-unit measure independent of the quantities in question, thus making the referent whole disappear.

Therefore, to reiterate Kieren's (1976) argument that a complete understanding of rational numbers requires not only an understanding of each of the separate meanings of the fraction but also of how they interrelate, special attention must be paid to examining the whole through the ratio lens. Equivalence in terms of ratio is not necessarily the same as equivalence in terms of fractions related to a whole. The equivalence of ratios needs to be viewed in relation to what Behr et al. (1983) call a "comparative index" or an idea of magnitude, where the complete *amount* for each proportion (i.e., the total of the parts) is *not* the same. For example, 1:3 is in fact 4 parts in total, but is equivalent to 2:6 because for each 1 taken, this 1 is compared with 3, leading to have 2 to compare with 6; but the ratio 2:6 is in fact 8 parts in total.[6]

This is different when one considers equivalence of fractions in terms of taking a number of parts in relation to a whole but where the size of these parts matters: 2/6 is equivalent to 1/3 because with the same whole, one takes 1 part of a whole split in 3 parts for 1/3, and one takes twice the number of parts for 2/6, but from the same whole being split in parts half the initial size; this amounts to the same portion of the same whole – in other words, the same *total* – being considered. These kinds of interactions between various meanings are important, although complex, for developing what Kieren posits for understanding fractions. In the case of ratios, as in the case of the part–whole interpretation, the whole plays a role, albeit not the same role.

These two excerpts (about fraction-as-operator in the ribbon problem, and ratio in the orange juice task) illustrate how important the concept of (relativity of the) whole can be for developing an understanding of different aspects to be considered in learning fractions.

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[5] A link can be made here with Tobias (2012) in terms of language issues, where the word *unit* can often mean *one*.
[6] Obviously, all of this changes if one considers half-parts for 2:6, where the total for both 1:3 and 2:6 would be 4 parts. But this would not be a usual way of working with ratios.

Final Remarks and Discussion

This study discusses the multiple meanings assigned to the whole and to the multiple related meanings of fractions as revealed in various tasks. Far from representing a single picture, our results confirm the multifaceted character of fractions and of the (relativity of) the whole. Our results illustrate the complexity raised by Kieren (1976) and Behr et al. (1983), who introduced fractions as a complex amalgam of interrelated meanings, linking them to operations, equivalence, and problem-solving.

Our findings shed light on the possible influence of one interpretation on another [e.g., the relationship between ratio, operator, and (relativity of the) whole], but also on the complexity of the interpretation of the whole of the fraction itself. We see, in fact, how various conceptualizations of the whole and of the fraction impact (a) on the comparison of fractions (e.g., problem A – John and Mary problem), (b) in giving meaning to equivalence (e.g., problem B – Richard problem), or (c) in division and multiplication contexts and the interpretation of the result (e.g., problem C – ribbons). We also see how certain interpretations (e.g., the referent as a pictorial representation, or the referent as a number linked to an operator) can impact possible changes of view in how to approach the referent of the fraction, which is a flexible approach necessary to make sense of division and multiplication problems (e.g., problems C and G).

We can also understand how such conceptualizations can influence teaching situations such as in the pizza vignette (problem D) where the multiple answers given by the children in the discussion as well as the validity of and justification for these answers presented a challenge for the teachers. Difficulties with the concept of the relativity of the whole can constitute an obstacle in such teaching situations, restricting the teacher from fully contributing to the learning processes from which these answers arise. The various meanings about the (relativity of the) whole raise questions about a number of issues concerning the teaching and learning of fractions as well as the consideration of the referent.

Differentiating between a fixed whole and a relative whole is, as illustrated in our study, of major concern for understanding fractions. Novillis-Larson's (1980) studies, cited by Behr et al. (1983), illustrate the difficulties that students experience in perceiving the unit of reference (see also Hart, 1981; Tzur, 1999; Wong & Evans, 2008). In an experiment with grade 7 students involving tasks that required situating diverse fractions on varied number lines represented by one or two units of length and where the partitioning of each unit corresponded to the number associated with the denominator, Novillis-Larson pointed out that when a number line of length two units was involved, almost 25% of the sample used the whole line as the unit. Like the results of Mack (2001) and Prediger and Schink (2009) in the context of multiplication of fractions, Novillis-Larson's findings confirmed that the conceptualization of the unit, the referent, was fixed, i.e., associated to *one*. For example, children described by Novillis-Larson had difficulties in seeing that the same point could be associated with different values in relation to a whole that could vary. We find similar results about the multiplication of fractions (4/5 of 2/3) in Prediger and Schink's (2009) study, where students' understanding of the multiplication was constrained by the problem of changing referent wholes. Awareness about changing referent wholes as well as seeing the referent as something varying and not fixed is therefore an important issue.

In our study, the following task (as well as the ones presented above) shows the fundamental importance of the relativity of the whole. In this task (see Figure 2), we saw how teachers re-used the idea of (a relative) referent as they defined each point of the number line in relation to a single referent: e.g., point A is described as ½, but also as 1/6 (i.e., 1/6 of 3) or as ¼ (i.e., ¼ of 2).

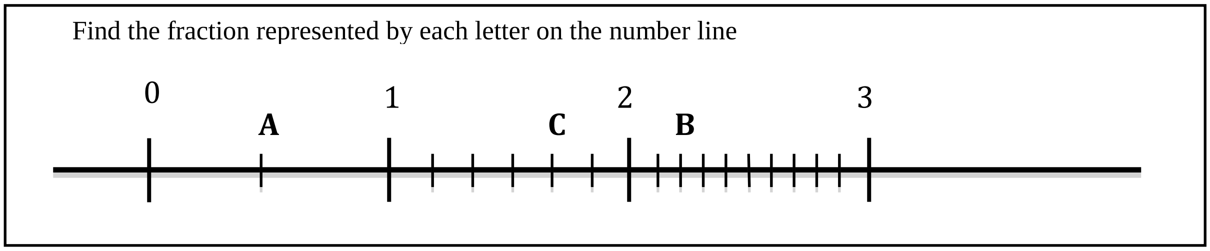


Figure 2. Number line task.

This changing view of the unit of reference gains importance for comparing fractions, for operations, and for contextual problem-solving, as shown above. Moreover, the relativity of the whole – this capacity to change perspective – has been shown to be crucial, as numerous studies have illustrated, for multiplication and division of fractions (Mack, 2001; Prediger & Schink, 2009; Schifter, 1998; Simon, 1993). Our argument here is that this concept is central even to the learning of fractions themselves.

Our data analysis from the various tasks shows how the (relativity of the) whole is a key element to consider in dealing with fractions. The main difference between our work and previous studies cited above is related to this *(relativity of the) whole*. While other studies have recognized the necessity of a changing referent, they have also posited that the referent of the fraction needs to be fixed, determining then the possible right answers to a problem. Our paper, however, questions this view of fractions. The concept of the relativity of the whole offers a wider and deeper perspective into what is meant by a fraction.

This concept of the (relativity of the) whole also raises significant questions about teaching and learning fractions and about the preparation of mathematics teachers. The pizza vignette at the beginning of the article amply illustrates (a) how the (relativity of the) whole can indeed be at the centre of a classroom exploration of a specific problem, (b) how diverse solutions can be reinterpreted in relation to the whole, and (c) how the (relativity of the) whole transforms or expands the validity of the answers given by students. The (relativity of the) whole is thus a key issue at play for understanding and exploring fractions.

Appendix
The eight problems

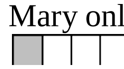
A: John and Mary problem (with different students' solutions)

John and Mary each have pocket money. Mary has spent ¼ of her amount and John ½ of his. Who spent more money, John or Mary?

Student solution 1:
John because ½ is more than ¼.
Ex.: ½ of 16 = 8 but ¼ of 16 = 4

Student solution 2:
Mary ¼
John ½ = 2/4
John spent more and Mary less

Student solution 3:
I think it is John because he spent ½, which is half, and Mary only spent ¼.

Student solution 4:
Mary only spent a ¼ of her amount

John spent the most, who spent ½ of his amount.

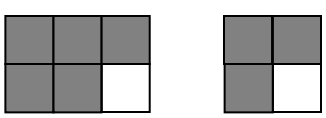
Student solution 5:
$\frac{1\times 2}{4\times 2}=\frac{2}{8}$ Mary spent less
$\frac{1\times 4}{2\times 4}=\frac{4}{8}$ John spent more

Ex.: they have $8 each.
Mary spent 1/4 = $2; she still has $6
John spent ½ = $4; he still has $4

B: Richard problem (inspired by Hill et al. 2007)

Richard is asked to make a drawing for comparing ¾ and 5/6.

He draws the following:

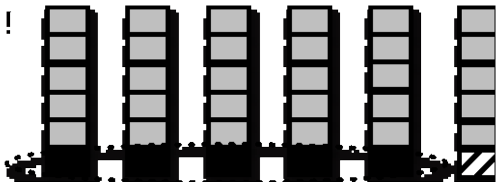


And explains that ¾ and 5/6 are equivalent.

What do you think of his answer?

C: Ribbons problem (with different students' solutions)

Mali has 6 meters of material. She wants to make ribbons of 5/6 meters for a school party. How many ribbons can she make and how much material will she be left with?



Student solution 1: 7 and 1/5

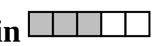
Student solution 2: We can make 7 ribbons

Student solution 3: 7 and 1/6 of a ribbon

Student solution 4: 7 and 1/7

D: Pizza teaching vignette (shown above, section 3)

E: Problem: How is it possible that when I add ½ it becomes the 1/3?

F: Problem: All the fractions that can be seen in  **.**

G: Problem: How many glasses of ¾ of liters can you fill with 4 litres? (with one student's solution given)

Student solution

¾ + ¾ = 6/4 = 1 ½

1 ½ + ¾ = 2 ¼ + ¾ = 3

3 + ¾ = 3 ¾

3 ¾ + ¼ = 4

5 glasses 1/4

H: Problem: Lea fills a glass of water to 5/7 of its capacity. If she pours 3/5 of it, what fraction does the water now occupy? (with one student's solution given)

Student solution

5/7 – 3/5 = 25/35 – 21/35 = 4/35

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