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Peer interactions in a computer lab: reflections on results of a case study involving web-based dynamic geometry sketches

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Abstract

A case study, originally set up to identify and describe some benefits and limitations of using dynamic web-based geometry sketches, provided an opportunity to examine peer interactions in a lab. Since classes were held in a computer lab, teachers and pairs faced the challenges of working and communicating in a lab environment.

Research has shown that particular teacher interventions provide motivation for the consideration of new ideas, and help uncover misunderstandings that may interfere with student progress [Towers, J. (1999). *In what ways do teachers interventions interact with and occasion the growth of students' mathematical understanding*. Doctoral Dissertation, University of British Columbia, Unpublished]. Examples of student discourse presented here suggest that certain peer interactions act in similar ways—helping propel students towards new understanding. On the other hand, they also show that some peer interactions, although superficially similar to teacher interventions, may hamper student progress.

Keywords: Peer learning; Interventions; Computers; Mathematics education; Dynamic geometry

Secondary school mathematics students in a computer lab often spend more class time interacting with one another than with their teacher. Many mathematics education researchers have examined the nature of pupil–pupil communication (Cobb, 1995; Sfard & Kieran, 2001; Yackel, Cobb, & Wood, 1991) while others have studied computer-supported learning (cf. Balacheff & Kaput, 1996; Chazan, 1990; Clements & Battista, 1989; Laborde, 2000). Since scheduled time in a computer lab setting is now an integral part of many mathematics programs, especially for dynamic geometry experiences, it is important to investigate

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the nature of pupil–pupil communication in relation to computer-supported activities in a computer lab setting. Using the results, mathematics educators can develop specific approaches to promote and support learning in this special environment.

A case study, originally set up to identify and describe some benefits and limitations of using pre-constructed dynamic web-based sketches to develop geometric reasoning, provided an unexpected but rich opportunity to examine the nature of peer interactions in a computer lab environment. It is hoped that the reflections offered in this article may inform educators is similar, and promote further discussion on peer communication in computer labs.

1. Introduction

There is evidence from many researchers that collaboration and discussion have a beneficial impact on mathematics learning (cf. Cobb, 1995; Collins, Brown, & Duguid, 1989; Gordon Calvert, 2001; Schoenfeld, 1994). De Kerckhove (1996) suggests that when people work together there is energy generated by the fact that some people 'do not know'. This 'not knowing' is important for the rest of the group because it helps them begin working towards solving the problem. From a practical standpoint, teachers often do not have a chance to listen to each individual, but in pairs or groups, students can help one another and correct errors and misconceptions (Collins, Brown, & Duguid, 1989). Gordon Calvert notes, "The continuous relationship between the participants and the mathematical world gives rise to new actions and experiences, new questions and concerns, and new explanations in a recursive cycle" (Gordon Calvert, 2001, p. 141).

Nevertheless, a common concern of teachers is that students who work in groups to develop and share results may just be sharing ignorance. These worries are echoed by Sfard and Kieran (2001) who, in regard to their study of the communication between two students, report: "we could not escape the impression that not much mathematical learning occurred during the long hours the boys spent together" (p. 71). Sfard and Kieran do point out that a recent study by Lavy of collaboration in a computer-supported environment, which used similar methods of analysis, revealed learning progress that might not have occurred if students had worked separately (see Sfard & Kieran, 2001, p. 71); however, Sfard and Kieran's research reminds us that group work is not a panacea and that teachers must assume an active monitoring role, intervening as necessary to guide the learning process (Cobb, 1995).

In 1986, Yerushalmy and Houde noted with reference to The Geometric Supposer (Schwartz & Yerushalmy, 1986):

> The pedagogy we used most closely resembles the teaching ordinarily found in science classes where the primary focus is on the scientific process of collecting data, conjecturing and finding counterexamples or generalizations . . .. Students spent the majority of class time discussing and doing geometry rather than listening to a teacher talk about it.
>
> (Quoted in Fey, 1989, p. 245)

Such a methodology requires a teacher who is a facilitator of the inquiry process rather than a transmitter of knowledge (Ball, 2000; Chazan & Ball, 1999; Collins, Brown, & Newman, 1989). But how do teachers manage facilitation in a lab? In an ideal world, the computer lab environment could enable the mathematics teacher to interact frequently and effectively with many students. While moving around the class, the teacher could see what a student is responding to and (if students are in pairs) she could hear what the

students say about the mathematics. Thus, teachers could readily identify areas of need and intervene to correct misinterpretations, to strengthen concepts, and to encourage extended thinking.

In reality, facilitating can be a challenge in a lab–classroom where the physical configuration often makes it difficult for teachers to move quickly and easily between students, and blocked sightlines sometimes hamper students from attending to explanations. It is here that pupil–pupil interactions take on particular significance.

This paper examines selected interactions that took place between pairs of students in senior mathematics classes in a lab environment, and addresses two questions: what is the nature of pupil–pupil interactions in a lab environment, and how do these interactions affect student development of understanding in mathematics?

2. Literature

In 1999, Towers researched the nature and effect of mathematics teachers' interventions with individual students. She built on work of Pirie and Kieren (1994), whose theory for the growth of mathematical understanding describes how a student can progress through stages from not knowing to knowing. Pirie and Kieren's model includes eight modes of understanding: primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, and inventizing. These modes can be thought of as nested rings. Rather than progressing through stages in a linear way, a student in Pirie and Kieren's model moves back and forth between the rings. Once past a particular landmark (e.g., property noticing), a student can proceed without the necessity of referencing earlier stages; however, at any time, the model provides for a student to return to an inner layer—to "fold back".

In her research, Towers categorized 12 "intervention strategies" that teachers use with individual students: (1) managing—carrying out administrative and disciplinary work, (2) checking—asking if the student understands, (3) enculturating—introducing students to terminology and processes used in the mathematics community, (4) reinforcing—stressing an idea, (5) clue-giving—deliberately pointing the student to the correct answer or path, (6) anticipating—trying to prevent the student from making a mistake, (7) blocking—stopping a student from following a particular path, (8) inviting—suggesting "a new and potentially fruitful avenue of exploration" (p. 201), (9) modelling—providing an example of the teacher's own approach, (10) rug-pulling—deliberately introducing a puzzling idea, (11) retreating—walking away and allowing the student to think further, and (12) praising.

Towers also noticed that in extended interventions (e.g., when students needed more than a brief comment) teachers used three identifiable teaching "styles": (a) showing and telling—the teacher provided extensive information but without checking that the student was following the explanation, (b) leading—the teacher worked through steps towards a particular goal, stopping frequently to check that the student was following, and (c) shepherding—the teacher used "subtle nudging, coaxing, and prompting" (p. 202) to help a student move towards understanding.

Towers found that the shepherding style and the strategies of rug-pulling and inviting consistently contributed to the growth of understanding in mathematics as evidenced by student movement towards the outer ring of the Pirie–Kieren model. Her work is of interest here because her categories provided an initial framework for an analysis of the pupil–pupil interactions in the study.

2.1. Peer learning

Substantial research has shown that peer learning, i.e., learning with and from peers, is effective (cf. Cohen, 1994; Good, Mulryan, & McCaslin, 1992), although problems have been observed. Some difficulties with regard to low achievers have been linked by researchers, such as Cohen (1994), to differences in status (academic or social). For example, students of higher academic ability may take control of a discussion, cutting off low achieving students from interacting with the group. Since interactions are correlated to status and are also predictors of learning gains (Cohen, 1994), equalizing status is an important consideration in group learning. Research suggests that one effective method involves teachers intervening to support the confidence of all group members; another involves choosing tasks that are non-conventional and ill-structured (Cohen, 1994). Such tasks require multiple abilities and rely less on prior knowledge, thus allowing a greater range of students to interact. However, Mulryan (see Gabriele & Montecinos, 2001) warns that choosing unfamiliar or ill-structured tasks may exclude activities closely related to the curriculum, which could, in turn, limit opportunities for group work.

Another source of difficulty with regard to peer learning relates to the nature of mathematics; in a mathematical activity students must often focus on ideas, symbols, and objects to interpret and solve problems. Depending on the student and the relative difficulty of the problem, this requires time for individual thinking—a process that can be interrupted by the demands of interpersonal communication (Sfard & Kieran, 2001).

With regard to the studies on peer learning it is important to point out some similarities and differences vis à vis the present study.

- Organization: The students in the study worked in pairs rather than small groups.
- Familiarity of tasks:
  - No study students had used geometry software. Thus, high and low achieving students were beginners in using JavaSketchpad to examine geometric concepts.
  - Dynamic geometry software is a visualizing tool. Since school mathematics has traditionally relied on numeric or symbolic reasoning, visual reasoning tasks are often unfamiliar and challenging to students at all achievement levels.
- Prior knowledge and links to curriculum:
  - The knowledge of geometry concepts and terminology did differentiate learners to some degree; however, in general, such knowledge was sparse. This is not uncommon in secondary school mathematics since geometry is often neglected in favor of numeracy and algebra (cf. Lehrer & Chazan, 1998; Whiteley, 1999).
  - Although the activities were designed to encourage exploration and discussion, the topics were part of the regular program.

2.2. Dynamic geometry

Dynamic geometry involves exploration of geometric relationships by observing geometric configurations in motion. These configurations can be constructed using programs such as The Geometer's Sketchpad (Jackiw, 1991), Cabri Géomètre (Baulac, Bellemain, & Laborde, 1992) or Cinderella (Richter-Gebert & Kortenkamp, 1999).

Unlike objects in draw programs, figures constructed in a dynamic geometry environment retain constructed relationships under the operation of dragging. For example, if two lines are constructed to be perpendicular, they will remain so when either line is dragged to a new position. Any onscreen measurements that are affected by the movement automatically update to new values.

In the study, students manipulated pre-constructed sketches created with The Geometer's Sketchpad, then converted to HTML format with JavaSketchpad (Jackiw, 1998). JavaSketches can be viewed and manipulated through a Java-compatible web browser, such as Internet Explorer or Netscape Navigator. These interactive models can include action buttons to hide or show details, and to move or to animate objects; points on the sketches can be dragged, and related onscreen measurements update automatically, however, elements cannot be constructed or deleted.

3. The study

The study involved three grade 12 advanced mathematics classes (A, B, and C) from two urban co-educational schools, each with a population of approximately 1000 students from low to middle class backgrounds. Both schools were multicultural, although a large percentage of the students in school 2 (classes B and C) were of Asian descent. The 69 students in the study were between 17 and 18 years of age.

Three 75-min sessions were held with class A, and four 45-min sessions with each of classes B and C. Students worked in pairs, and in each class several pairs were studied in more depth by audiotaping or videotaping their activities.

The topics for the tasks were chosen from a section of the curriculum on deductive proof involving triangles and parallel lines. Although the students had done introductory work on deductive geometry related to congruence and parallelism in grade 10 and on similarity in grade 11, none had worked with dynamic geometry software.

Four web-based dynamic geometry sketches were prepared for the exploration tasks. They included facilities for hide/show, movement, and animation where appropriate. A labsheet for each activity included directions for opening and manipulating the sketch, a description of the problem to be investigated, instructions for exploration, questions to focus student attention, and space for written work.

The study used multiple sources of information—observation field notes, videotaping/audiotaping of selected student pairs, a student questionnaire, and interviews with the teachers. The raw data were analysed by coding, developing categories, describing relationships, and applying simple statistical tests where appropriate.

3.1. Participants

In each class students chose their own partners. Table 1 summarises information about the makeup of the pairs by gender and by class.

Seven pairs of students were videotaped and six pairs were audiotaped. Pairs to be taped were chosen by each teacher to represent the range of achievement levels within the class, but the students in any particular pair were not necessarily at the same achievement level (see Table 2). Before the study sessions each class spent one lesson reviewing geometry terms and concepts from earlier grades. For class A, the

Table 1
Number of pairs by gender and by class

Type of pair	Class A	Class B	Class C
Girl/girl	5	4	4
Boy/boy	6	6	2
Boy/girl	1	1	4

achievement levels shown in Table 2 reflect the results of a geometry quiz on review work. For classes B and C, levels reflect overall math achievement as reported by the teachers.

3.2. Tasks

Problems chosen for the study tasks were similar in difficulty to those in the students' text. They involved triangles and quadrilaterals, and were based on theorems the students had studied. Task materials – pre-constructed sketches and accompanying student labsheets – were prepared according to the following guidelines:

1. The interactive diagrams supported investigation from a transformational perspective as well as from a straightforward application of congruency theorems. That is, objects could be rotated, reflected, and/or translated to examine relationships.
2. The design of each sketch addressed areas that the study teachers had identified in pre-session discussions as problems for their students. For example, color was used to highlight small triangles or pairs of triangles in order to help students pick out small triangles within a larger diagram; motion buttons were created to separate and join overlapping shapes in order to help students learn to mentally visualize relationships; capabilities were provided to remove detail in order to help students focus on the whole shape; shapes were included that could be reflected or rotated in order to help students understand that rotated or reflected copies are congruent to the original.

Table 2
Taped pairs by class, achievement level, and taping method

Class	Names (aliases)	Achievement (based on teacher assessment and/or quiz results)	Taping method
A	Sue and Paul	Excellent/excellent	Video
A	Pat and Dave	Average/weak	Video
A	Joe and Bob	Average/average	Audio
A	Barb and Clara	Very good/very good	Audio
B	Beth and Kim	Very good/average	Video
B	Ray and Owen	Good/good	Video
B	Doug and Sal	Average/average	Audio
B	Lou and Rob	Very good/weak	Audio
C	Sarah and Earl	Good/average	Video
C	Lily and Fran	Very good/average	Video
C	Jan and Pam	Very good/Good	Video
C	Katy and Bea	Average/average	Audio
C	Tara and Mary	Very good/average	Audio

3. Instructions and questions were intended to: focus student attention on details in the sketch; encourage students to explain their thinking; help students move through an investigation by examining the evidence in the onscreen model, checking hypotheses, then considering other possibilities; encourage students to write a proof.

In this article the tasks for the sessions will be referred to as Tasks 1–4. A brief description of each is provided here to help the reader follow the discussion.

3.2.1. Task 1 overview

This sketch (see Fig. 1) introduced students to JavaSketchpad, and addressed student difficulties with overlapping figures and selection of triangles. The problem could be viewed as an application of reflection. Since ΔABC and ΔFCB are reflections of one another, dragging point A causes point F to move in the opposite direction. In the sketch, there were action buttons to show the perpendicular bisector through H [note that point G – the intersection of the bisector with AC – will only appear if the triangles are overlapping], to separate triangles ABC and FCB, and to reflect triangle ABC in a mirror line. An additional button, "Match FCB and A′B′C′" caused triangle FCB to move on top of the reflection of triangle ABC, demonstrating congruency. A reset button moved triangle FCB to its original position. In addition, the "Show Given Information" button controlled the display of markings that indicated the equality of AB

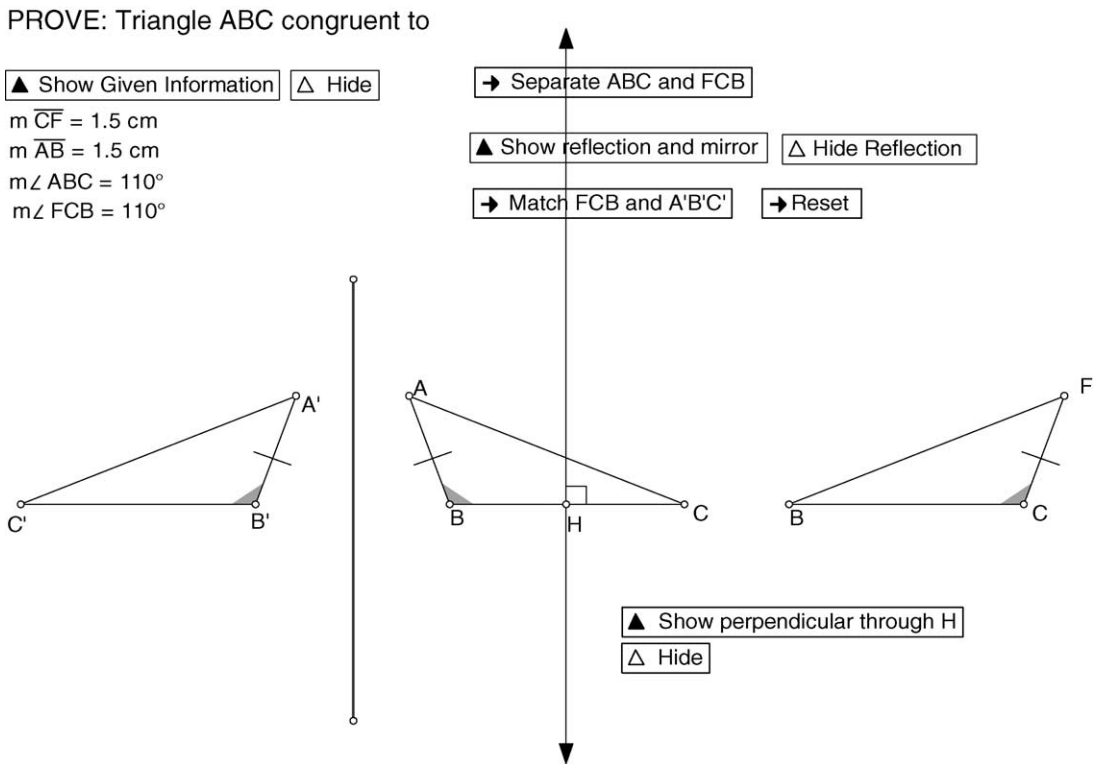

Fig. 1. Sketch for Task 1. View on selecting: "Show Given Information," "Separate ABC and FCB," "Show reflection and mirror," and "Show perpendicular through H."

and FC, and ∠ABC and ∠FCB, as well as the measures of these lengths and angles. Measurements (which updated) were included to help students notice that the two triangles remain congruent.

To prove that triangle ABC is congruent to triangle FCB, students could use a straightforward application of SAS (side, angle, and side) congruency after noting that BC is a common side. Alternatively, they could argue that ΔABC is congruent to ΔFCB because when the triangle is flipped over FC will lie on AB, CB will lie on BC and ∠FCB will lie on ∠ABC (i.e., students could use the idea of reflection.)

Students were encouraged to notice details and explore the mathematics; the labsheet directed them to "observe the diagram", to "drag point A", and to "explain the meaning of the tick marks and the angle shading". In addition, students were asked (a) to interpret the meaning of the image: "How can the information provided by these images be used to explain why ΔABC is congruent to ΔFCB?", (b) to use deductive reasoning: "What additional information can you deduce about point H from the diagram?", and (c) to extend their experience: "Find another pair of congruent triangles in the figure".

3.2.2. Task 2 overview

The second sketch also presented a problem that could be also viewed as an application of a reflection (see Fig. 2). The sketch was designed to address student difficulties with overlapping triangles, selection of triangles, and two-step proofs. The following were used to help students notice details: the four chosen pairs of congruent triangles were shaded in four different colors; given equal angles were shaded red; information could be toggled off and on to allow details to stand out; triangle pairs could be separated; measurements for the given angles and lengths were displayed; measurements were updated as the sketch was dragged. In addition, to help students pick out a shape within the larger diagram, overlapping figures

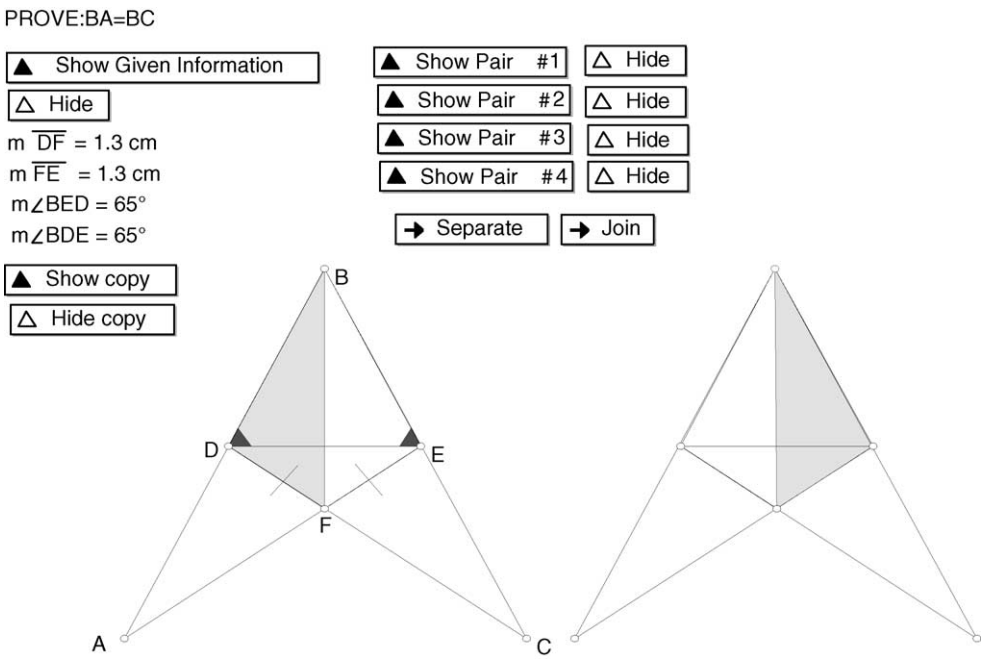

Fig. 2. Sketch for Task 2. View on selecting: "Show Given Information," "Show Copy," and "Show Pair #1."

could be separated, color was added to emphasise the shapes, and color was used to overlay angles and sides within the shape.

To prove that BA = BC, students could prove ΔABE congruent to ΔCBD by ASA (angle, side, angle) congruency, or by various other routes; however, in all cases students needed to deduce at least one piece of information from the given information before developing the proof (i.e., all options involved at least two steps). For example, in the case just given, students needed to deduce that ∠DEF = ∠EDF by the isosceles triangle theorem, and that ∠BED + ∠DEF = ∠BDE + ∠EDF before employing ASA.

All chosen pairs of triangles were reflections, and congruency could be established or not established by considering what would happen if one member of the pair was flipped over, e.g., if ΔBDF was reflected in BF, BE would lie on BD (equal sides of an isosceles triangle), EF on DF (given equal) and BF on BF (common sides). However, for pair #2, ΔBFA and ΔBFC, only the behaviour of BF could be predicted.

The labsheet encouraged students to:

- Explore: drag each red point. . ..
- Notice: . . . observe the measurements.
- Interpret: write two additional facts that you know and explain why they are true.
- Deduce: if you proved the pair congruent, how would this help you prove BA = BC?
- Extend: what is an alternative explanation for the congruency of triangle ABC and triangle FCB?

3.2.3. Task 3 overview

Task 3 (see Fig. 3) required students to prove that triangle AMD is congruent to triangle CNB. Students could either apply properties of parallel lines or investigate the problem using a rotation. The triangles to be proven congruent were colored to attract student attention. When quadrilateral ABCD was dragged, AD and BC appeared to remain equal and parallel, as did AB and DC. When the "Show Given Information" button was used, students were able to deduce that ABCD was indeed a parallelogram since opposite sides were marked equal.

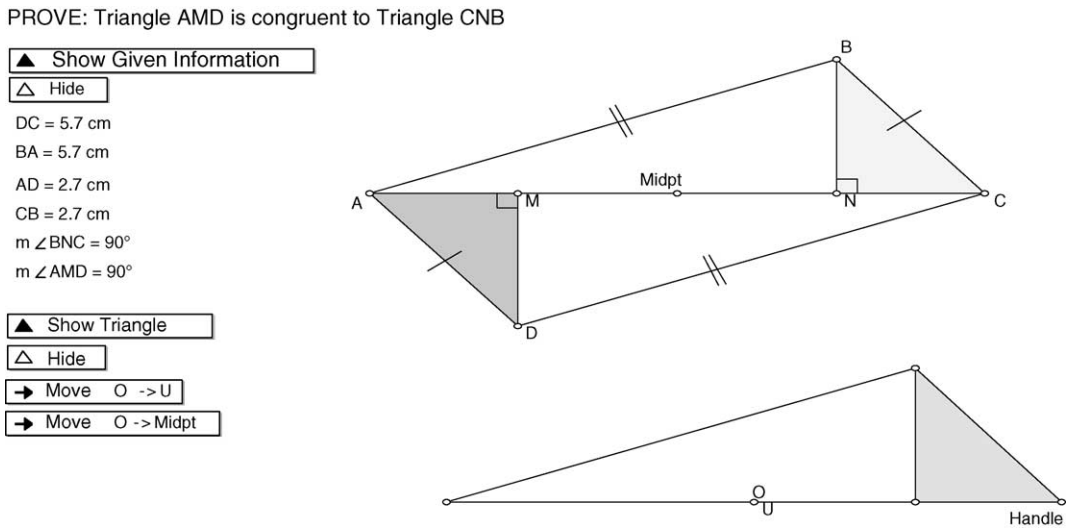

Fig. 3. Sketch for Task 3. View on selecting: "Show Given Information," "Show Triangle," and "Move O → U."

To prove that triangles AMD and BNC are congruent, students could use ASA congruency: AD = BC (given), ∠DAM = ∠BCN (parallel line law), and ∠MDA = ∠NBC (∠AMD = ∠CNB = 90°, so the remaining angles in the triangles are equal). They could also make use of the extra triangle that was provided. This triangle could be placed on top of ΔABC to demonstrate congruency. As ΔABC changed shape, the extra triangle changed as well. The extra triangle could also be rotated about point O (which lined up with the midpoint of AC), to lie on ΔADC. This demonstrated that AC split the parallelogram into two congruent triangles. The students could deduce further that, since a triangle can only have one altitude from a particular vertex, DM must equal BN.

In the accompanying labsheet students were encouraged to explore by being asked to "Drag . . ." and "Rotate . . ." the diagram. Questions were also included to help them notice, interpret, make deductions, and extend their understanding. For example:

- What do you notice about the new triangle?
- What is the relationship between the new triangle and ΔADC?
- Use your conclusions . . . to prove that ΔAMD is congruent to ΔCNB.
- How can the information provided by these images be used to explain why DM = BN?

3.2.4. Task 4 overview

This sketch (see Fig. 4) was designed to allow students to investigate the question: "When do the diagonals of a parallelogram right bisect one another?" The sketch included parallelogram ABCD, with diagonals AC and BD. The opposite sides were marked with arrows, the traditional markings for parallel lines. Measurements of the sides, diagonals, and semi-diagonals could be toggled on or off using show/hide buttons. Since it can be frustrating to drag an angle until the measurement is precisely 90°, a line, perpendicular to AC, could be activated by the button "Show Perpendicular". Thus, students could drag the diagram until BD was aligned with the perpendicular.

This task was undertaken after a short class session on what it means to conjecture. It gave the students, not a statement to prove, but the chance to decide what that statement should be. Since the question involved the term "right bisector", the first section of the labsheet asked questions about the meaning of "bisect", "right bisect", and "right bisect one another". Students were then asked to drag the diagram and to conjecture a response to the original question. Once students had formed a conjecture, they were asked to develop a proof of that conjecture and to outline an alternate proof.

3.3. Data coding and analysis

A qualitative analysis computer program, ATLAS. ti – The knowledge Workbench (ATLAS. ti, 1997), was used to codify, annotate, and sort text segments of the transcripts. Using the software, families of codes were created, then analysed to uncover relationships between student responses and the stimuli offered by teacher, student partner, labsheet or sketch.

In the first run-through, the transcripts were codified according to what was happening at the time. Some examples of these first codes were: students on task, pointing to screen, dragging diagram, using motion button, excited, reading, appealing for help, student states theorem, student restates known information, deducting from visual, and erroneous conclusion. The results were a rather disconnected series of observations.

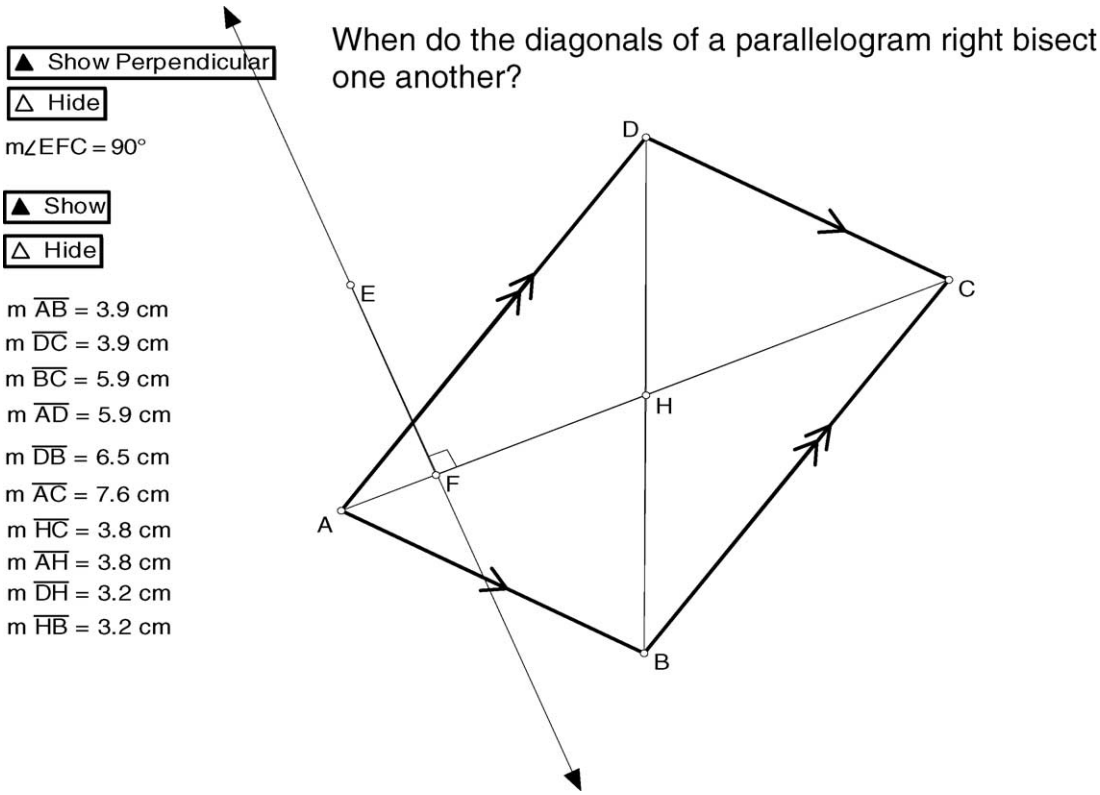

Fig. 4. Sketch for Task 4. View on selecting: “Show Perpendicular,” and “Show.”

On the second run-through each of these annotated activities was examined with regard to underlying motivation. Some of the codes at this stage were: checking understanding, modelling thinking, reinforcing ideas, posing inviting questions, color used as reference, and students use redo capability.

It was apparent to the researcher as the work progressed that many of the codes that described teacher interactions with taped pairs were similar to Towers’ (1999) styles and strategies. Nevertheless, while some of the coding was beginning to highlight themes connected to the teacher’s role, most of the transcripts contained very little teacher–student discourse. There was more discussion between student partners.

At this stage new codes were developed to describe pupil–pupil interactions: students correct one another, working at cross-purposes, struggle. In addition, an analysis table for each labsheet was created to allow an investigation of the relationship between student comments and particular labsheet questions or sketch features (see Fig. 5). The first column includes the instructions, statements, and questions from the original labsheet. The second column sets forth the intended purpose for each statement or question. The third column describes the role that the researcher had envisioned for the sketch in relation to the statement or question. In choosing entries for the columns, Towers’ category labels were used initially in order to facilitate comparisons to the transcripts; additional descriptions were developed as required. The transcripts were then re-examined to determine student responses to the statements. In addition to new labsheet and sketch codes which were inserted at appropriate questions and statements, new codes for student responses emerged, such as: hypothesises, struggles to remember, answers, is confused, concludes.

Instructions –Task 2	Labsheet	Sketch
1. Open Jsketch3.htm	Managing	
2. Drag each red point and observe the diagram	Focusing attention - visual	Drawing attention - color, movement
3. Click Show Given Information.	Focusing attention - detail	Drawing attention - hide/show
4. Drag points A and B again and observe the measurements.	Focusing attention - detail	Drawing attention - movement
Write two additional facts that you know and explain why they are true	Prompting action Checking for understanding	Providing shared image
5. There are several possible paths to prove that BA = BC.	Inviting exploration	Providing alternate paths
♦ Click Show Copy and Separate. (Either can be moved so they don't overlap).	Enculturating - program	Drawing attention - hide/show; Drawing attention - movement

Fig. 5. Portion of labsheet question analysis for Task 2.

These represented not just the outward action but also the inner thinking of the students as they worked with their labsheets and sketches.

4. Results

The transcripts show that students actively engaged with their partners throughout the sessions. They pointed out details, corrected one another, kept each other on task, and argued over interpretations. In fact, the role of 'the partner' had a far greater impact on the learning environment than the researcher expected.

Towers' intervention categories provided an initial framework for examining the conversations of the student partners. Analysis revealed that many student "interactions" could be thought of more accurately as "interventions" in that they attempted to change or affect the knowledge of the partner, i.e., pupils were not merely sharing information. Tables 3 and 4 summarize general observations from the study that show how students used in modified form some of the intervention techniques that Towers identified. The following examples provide a closer look at particular pupil–pupil interactions.

Table 3
Intervention styles as used by peers

Category	Observation
Leading	Some students went through a step-by-step explanation for their partner, checking for understanding along the way.
Showing and telling	Students frequently told their partners particular information, although it was sometimes inaccurate.
Shepherding	Most students did not use this intervention to help their partner understand; however, the give and take in which some students engaged as they worked towards understanding shared some elements with this style.

Table 4
Intervention strategies as used by peers

Category	Observation
Checking	Students checked for shared understanding; however, if both students misunderstood, checking did not lead to correction of error. When students proceeded without checking they sometimes worked at cross-purposes and became frustrated.
Reinforcing	Students often repeated a theorem or finding for shared understanding.
Inviting	Students sometimes played to explore a new direction—to see 'what if' (i.e., they "invited" themselves).
Enculturating	Students sometimes corrected one another with respect to terminology although occasionally the correction was wrong. Student discussion revealed that without adequate enculturation it is difficult for students to communicate meaningfully about mathematics.
Blocking	A student partner sometimes used blocking to keep the pair focused, or to cut off discussion. e.g., "Anyway, we are getting kind of carried away. All right, just leave it. All right, what do we see?" (Barb and Clara)
Modelling	This intervention was used to powerful effect by some students.
Praising	Students sometimes praised themselves – or could be heard giving a happy yelp – after successfully working something out.
Rug-pulling	Not used. Students in general do not have a deep enough knowledge base to use rug-pulling with their partner.

4.1. Styles

A few study students explicitly assumed a teaching role. In the next excerpt, Barb is confused by the need to mentally separate nested angles. Clara intervenes using a style that is reminiscent of *leading* (i.e., telling, but stopping to check for understanding).

Barb:	Angle A equals angle C.
Clara:	No. Angle A1 equals angle C1.
Barb:	Angle A equals angle C.
Clara:	You can't do that because this is angle C too.
Barb:	Oh yeah. Angle A1 equals angle C1. Would that be?
Clara:	Because of alternate parallel line theorem.
Barb:	Hold on. Hold on.
Clara:	Ok, we'll just say angle A1 equals angle C1.
Barb:	So what? So just cause these are equal doesn't—Oh yeah. All right, all right.
Clara:	Ok?

(Barb and Clara, Task 3)

In this exchange, Barb shows that she understands when she says, "Oh yeah. All right, all right", but it has taken her some time and Clara is clearly ready to explain again if Barb doesn't quite get it.

Other students used the *showing and telling* style to help their partner. For example:

Sarah:	Angle B equals angle B since the angle is common.
Earl:	How do you know that?
Sarah:	We can put them together. [She proceeds to demonstrate.]

(Sarah and Earl, Task 2)

However, when students did not possess sufficient knowledge, they sometimes provided incorrect information. Here, Doug has a problem differentiating the part from the whole in a diagram. Sal argues, hesitates as if he knows that Doug is wrong, but then gives up.

Sal: So we have—what?
Doug: We have FD.
Sal: No, no that's one big triangle.
This splits the triangle in half, right? So, FD is a common? A common side. They have a common side here, right? But F's not—Oh. All right.

(Doug and Sal, Task 1)

Unfortunately, no teacher arrived at this point, so the students based their conclusions on this line of reasoning, unaware of the error.

These examples are not meant to discount the value of independent investigation. Pat and Dave as the next example shows clearly came to a new understanding of rotation through their interactions with one another and with the sketch.

Dave: How do you rotate it? You can't unless it's round. You can only rotate it.
Pat: Oh, it rotates on one point . . ..
Dave: Yeah so—so it stays in one point.
Pat: It goes in a circle. It goes around the midpoint.
Dave: Yeah, it goes in a circle.

(Pat and Dave, Task 3)

Each student furthered the conversation by noticing details, paying attention to one another's comments, and adding ideas.

In a similar example, Sue and Paul (two above-average students) can be seen on videotape gazing intently at a sketch as it moves, spending time to take in the detail, then working rapidly—correcting one another, giving feedback, questioning, even occasionally completing one another's sentences.

Paul: Ok, angle FCB is equal to angle ABC.
Sue: What, what, what? Angle FCB right?
Paul: Yeah. Angle B—line.
Sue: Line FC, right?
Paul: And angle BC is common
Sue: Right
Paul: Or not angle BC—line BC.
Paul: Ok, so we have side, angle, side. Therefore, triangle ABC is congruent to . . .
Sue: FCB.
Paul: FCB by . . .
Sue: Side, angle, side.

(Paul and Sue, Task 1)

These conversations typify a form of peer–peer 'teaching' that shares at least some characteristics with the *shepherding* style noted by Towers. Neither student in a particular pair possessed the knowledge of a teacher; however, each nudged the other towards deeper understanding of the problem by reflecting ideas back and forth.

4.2. Strategies

Checking that students understand, and *enculturating* are two common teacher intervention strategies. The following excerpts show that students engage in such interventions with one another—with varying degrees of success.

Pat: Bisect means cuts in half you know. Wait. DC–DB bisects AC [pointing—spreads fingers to span each segment] It means it cuts it in half, right?
Dave: Yeah. So that's one half—that's one half.

(Pat and Dave, Task 4)

Here, Pat provides a definition of 'bisect'—an enculturating intervention; then the partners double-check their understanding of the situation. In the following exchange, however, students think that bisector always signifies 'angle bisector'. The lack of a correct definition hampers the students' ability to effectively check their understanding.

Ray: Oh, Ok, angle GBH is equal to angle GCH because this is a right angle triangle. This dissects [sic] this. And these are equal. I'm not sure . . .
Owen: If you bisect it—wouldn't it be equal anyways because it's a bisector?
Ray: Depends on the angle though—on the measurements of the angle.

(Ray and Owen, Task 1)

Notably, these students did not appeal to the teacher for guidance; there was no explicit misunderstanding between them. However, there seems to be a need for an enculturating intervention; dragging the sketch could not help these students discover the definition of bisector. In a related case, Pat and Dave were confused about naming conventions (i.e., that single letters in the sketches denoted points) but they asked the teacher to mediate their disagreement and received correct information.

Dave: G is the midpoint.
Pat: No H is the midpoint. G is the line in the middle.
Dave: I'm going to write 'H is the line that crosses the midpoint'.
Pat: Is H the line or the point?
Dave: H is the line that crosses the midpoint.
Pat: H is a point, not a line.

(Pat and Dave, Task 1)

These examples suggest that students who work on mathematical tasks in a lab may need follow-up sessions at which they can discuss the mathematical ideas they have explored, and access to textbook glossaries or online definition banks to support their use of mathematical terminology.

The two intervention strategies that Towers found most significant for the development of student understanding were *rug-pulling* and *inviting*. Rug-pulling, that is, deliberately creating a situation that surprises and challenges, involves a level of sophistication that is beyond most secondary school students; however, it is possible to consider *inviting* as a peer intervention if we include the idea of students *inviting themselves* to follow an unexplored path—to *play*.

Playing was a common student response that focused student attention and provided motivation for further investigation. In this example, Sue and Paul jabbed at the screen as they referred to items, using the familiar geometry terms correctly; however, they did not just leave the sketch static. They continued to play with the sketch and explore new possibilities.

Paul: GBH is congruent to GCH because angle, angle, given, common, common
Sue: So wait a second. Side, side–no side, angle, side
Paul: Angle B and angle C are equal and then H is the midpoint . . .. It's the right bisector too.
(Paul and Sue, Task 1)

Some students played to investigate a particular question—and actually called it "play" as seen in this quote from Sarah:

Sarah: I feel obligated that there must be 3 pieces of information given. Let's play with it and maybe we'll see.
(Sarah and Earl, Task 2)

Others, as this excerpt from Doug and Sal's tape shows, played just to see what would happen.

Doug: [Serious, interested. While one wrote, the other played with the sketch.] You can really mess this one up. Look what I did!
(Doug and Sal, Task 2)

Inviting oneself or one's partner to take a particular tack is not such an unusual idea. Students must learn to explore in order to investigate any open-ended problem in the dynamic geometry environment. This includes learning what tools to select and how to use them, but it also involves asking and answering the question, "What should I do next?" If it is viewed in this light, *inviting* is an intervention that students need to be able to use.

In several cases, study students used the *modelling* intervention, that is, they communicated their method of thinking about an idea in addition to or instead of the idea itself. In the annotated example below, Paul models for Sue his method of seeing. This is quite a wonderful effort on his part. Unfortunately, Sue is so wrapped up in her attempt to make sense of the problem that she misses its power.

Sue:	Ok go ahead, separate them. Oh, it's different. Ok, we know that . . . Ok just a minute. If these two angles . . .	*Watches Paul manipulate sketch*
Paul:	You are working inside the thing again. Just look at—see the red part? Stare at the red parts. Blur out the black parts. No looking at the black parts—look at the triangle.	*Models how he visualises—uses color as reference*
Sue:	Ok DF equals . . .	*Sue is still focused on a part*
Paul:	No. You are looking at DF again. Don't look at DF.	*Tries again*
Sue:	Look. We did not even notice. Angle BED equals angle BDE. Therefore, this line equals this line, so this line would equal this line. You getting it?	*Focused on sketch, but still not aware of partner's concern*
Paul:	Mmmmm, you are looking at the triangle. Inside of the triangle—[she is unconvinced.] You are. BDE. That's not the triangle any more. The triangle's nullified. It's BAE–that's your triangle. What do you know about it? Nothing. Do you have 3 pieces of information? NO.	*He is frustrated but continues to try*

(Paul and Sue, Task 2)

In this example, Sue's need to concentrate on the problem has caused her to shut out Paul's attempts at communication. Although Sue was on the wrong track, this episode provides an example of a student ignoring peer talk in order to think independently about a problem—a tendency noted by Sfard and Kieran in their study of student talk (2001).

Another example of modelling is captured in an excerpt from the transcript of Lou and Rob. Here, Rob has suddenly grasped the idea that one can have equal angles even if the triangles are not the same size. He models his visualization of the idea aloud—probably for himself as well as for Lou.

Rob:	Yeah, yeah, yeah. You are right. You know why? Cause—imagine like a small triangle inside a big one. They could have the same angles.

(Lou and Rob, Task 2)

The interchange, seen against the backdrop of new experiences with dynamic geometry tools, exemplifies Cohen's proposal that the use of unfamiliar activities creates equal-status peers (Cohen, 1994). The snippet reveals that Rob, a weak student, has grasped an important concept and is eager and able to explain to his partner Lou, whose achievement level is 'very good'.

The excerpt continues:

Lou:	But the angles won't be the same . . . those angles won't be the same. Miss, if all the angles of a triangle are equal . . . can you use angle, angle, angle as a theorem?
Teacher:	No
Rob:	Why doesn't that work?
Teacher:	Because when the angles are equal then the triangles can be similar . . . not necessarily congruent. One can be bigger than the other one in length but still have the same angles.

(Lou and Rob, Task 2)

Lou recognizes that angle, angle, angle is not grounds for congruency; however, his decision to 'check' brings the teacher into the discussion. At that point, Rob, thanks to his earlier flash of understanding, is ready to listen.

As one might expect, not all study examples of peer interventions were positive. Although Paul usually worked well with his partner Sue, an excellent student, he occasionally demonstrated frustration if she did not follow his thinking about a problem. Here, he uses sarcasm to cut off conversation about a problem. Effectively it is a *blocking* intervention, which signals Paul's intention to move immediately to the next question.

Sue:	Given. Angle BED is equal to angle BDE—that's given. Something else that's given–
Paul:	No that's not given. Write two additional facts that you know.
Sue:	Ok that we know. Well, this has got to be—[pointing]—this looks equal . . .
Paul:	Oh yeah, that's good, let's just make assumptions.

(Paul and Sue, Task 2)

For this pair, gender may have played a part, although the transcripts indicate that Sue often took the lead in exploring the sketches. The brief outburst did not appear to affect the pair's ability to work together; Sue continued to be an active participant in all discussions.

5. Conclusion

The student pairs whose experiences form the basis for the reflections in this article experienced the conditions of an actual secondary school class. They worked amidst normal hubbub, were subject to class time constraints, and were forced to compete for the attention of the teacher. Since the class was in the computer lab, interrupted sightlines and awkward configurations of desks and equipment (which could not be changed) accentuated the dependency of students on their partners.

The evidence indicates that student interactions involved more than a sharing of information; students intervened with their partner to correct, inform, cut off conversation, initiate play, and communicate their vision. We know that teacher interventions can provide motivation for the consideration of new ideas, and help uncover misunderstandings that may interfere with student progress (Towers, 1999). The evidence presented here suggests that particular peer interventions such as modelling and inviting, might act in similar ways; however, it also reveals that the relative isolation of student pairs in a computer lab can set the stage for interventions that interfere with the development of mathematical understanding. Such interventions may provide incorrect information or, more significantly, derail a line of investigation.

Towers' use of *intervention* implies teacher action. In this paper, the use of the term *intervention* instead of *interaction* or *conversation* to describe examples of peer–peer communication conveys the idea that students can actively, if subconsciously, work to influence their peers. Further research is required to determine how to help students in computer lab environments use interventions such as modelling or inviting to further their own understanding and that of their peers.

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