The Rigidity of Periodic Frameworks as Graphs on a Torus

Elissa Ross

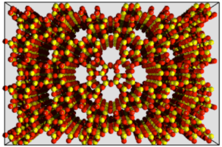
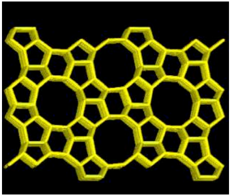
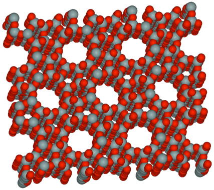
Fields Institute

October 17, 2011

Workshop on Rigidity and Symmetry

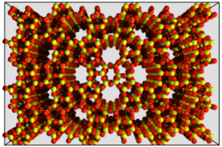
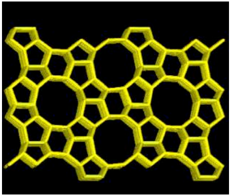
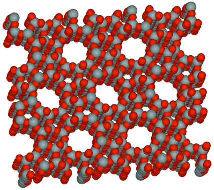
Zeolites

- Zeolites: Aluminosilicate minerals with "pore"-like structure

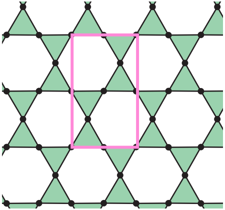
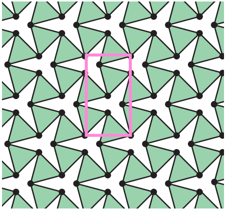




Zeolites

- Zeolites: Aluminosilicate minerals with "pore"-like structure

- Mathematical Zeolites: corner-sharing d-simplices, two at each corner

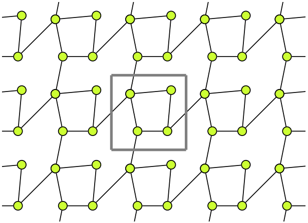



CLICK

Introduction

Problem

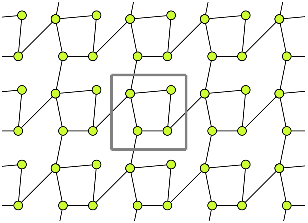
When is an infinite periodic framework rigid, so that its parts cannot be moved periodically with respect to one another?



Introduction

Problem

When is an infinite periodic framework rigid, so that its parts cannot be moved periodically with respect to one another?



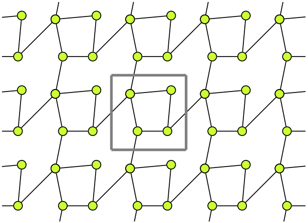
Other contributors to periodic rigidity:

- Whiteley (frame matroids, 1988)
- Connelly (sphere packings, 1988–)
- Guest & Hutchinson (zeolites, 2003–)
- Owen & Power (operator theory methods, 2009–)
- Borcea & Streinu (2010–)
- Malestein & Theran (2010–)

Introduction

Problem

When is an infinite periodic framework rigid, so that its parts cannot be moved periodically with respect to one another?



Other contributors to periodic rigidity:

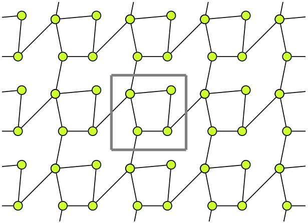
- Whiteley (frame matroids, 1988)
- Connelly (sphere packings, 1988–)
- Guest & Hutchinson (zeolites, 2003–)
- Owen & Power (operator theory methods, 2009–)
- Borcea & Streinu (2010–)
- Malestein & Theran (2010–)

OUTLINE

Introduction

Problem

When is an infinite periodic framework rigid, so that its parts cannot be moved periodically with respect to one another?



Other contributors to periodic rigidity:

- Whiteley (frame matroids, 1988)
- Connelly (sphere packings, 1988–)
- Guest & Hutchinson (zeolites, 2003–)
- Owen & Power (operator theory methods, 2009–)
- Borcea & Streinu (2010–)
- Malestein & Theran (2010–)

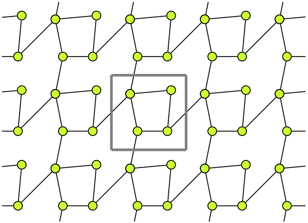
OUTLINE

1. Periodic frameworks as finite graphs on a torus: gain graphs

Introduction

Problem

When is an infinite periodic framework rigid, so that its parts cannot be moved periodically with respect to one another?



Other contributors to periodic rigidity:

- Whiteley (frame matroids, 1988)
- Connelly (sphere packings, 1988–)
- Guest & Hutchinson (zeolites, 2003–)
- Owen & Power (operator theory methods, 2009–)
- Borcea & Streinu (2010–)
- Malestein & Theran (2010–)

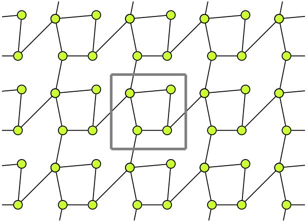
OUTLINE

1. Periodic frameworks as finite graphs on a torus: gain graphs
2. Frameworks on a fixed torus

Introduction

Problem

When is an infinite periodic framework rigid, so that its parts cannot be moved periodically with respect to one another?



Other contributors to periodic rigidity:

- Whiteley (frame matroids, 1988)
- Connelly (sphere packings, 1988–)
- Guest & Hutchinson (zeolites, 2003–)
- Owen & Power (operator theory methods, 2009–)
- Borcea & Streinu (2010–)
- Malestein & Theran (2010–)

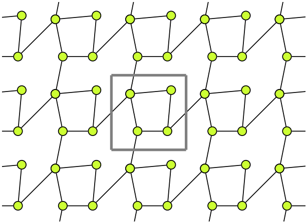
OUTLINE

1. Periodic frameworks as finite graphs on a torus: gain graphs
2. Frameworks on a fixed torus
   - 2-D: Henneberg & Laman results

Introduction

Problem

When is an infinite periodic framework rigid, so that its parts cannot be moved periodically with respect to one another?



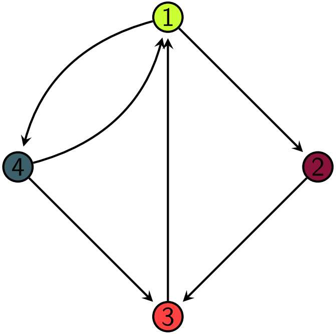
Other contributors to periodic rigidity:

- Whiteley (frame matroids, 1988)
- Connelly (sphere packings, 1988–)
- Guest & Hutchinson (zeolites, 2003–)
- Owen & Power (operator theory methods, 2009–)
- Borcea & Streinu (2010–)
- Malestein & Theran (2010–)

OUTLINE

1. Periodic frameworks as finite graphs on a torus: gain graphs
2. Frameworks on a fixed torus
   - 2-D: Henneberg & Laman results
   - n-D: necessary conditions

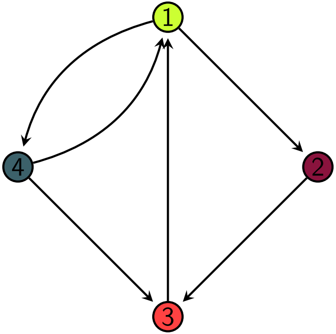
Gain Graphs



Gain Graph $\langle G, m \rangle$

- directed multigraph $G = (V, E)$

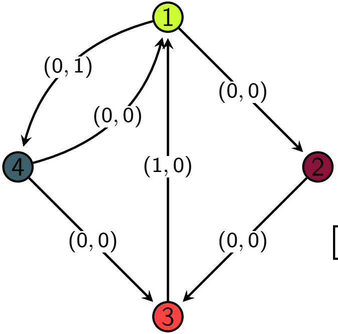
Gain Graphs



Gain Graph $\langle G, m\rangle$

- directed multigraph $G = (V, E)$
- *Gain assignment* $m : E^+ \longrightarrow \mathbb{Z}^d$
- *Gain group* $\mathbb{Z}^d$

Gain Graphs

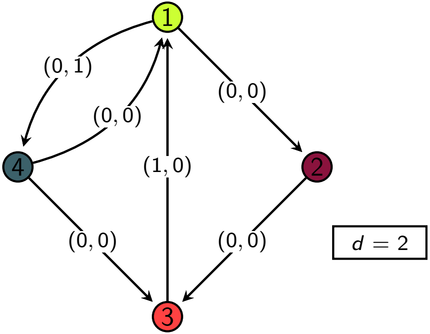


$\boxed{d = 2}$

Gain Graph $\langle G, m \rangle$

- directed multigraph $G = (V, E)$
- *Gain assignment* $m : E^+ \to \mathbb{Z}^d$
- *Gain group* $\mathbb{Z}^d$

Gain Graphs

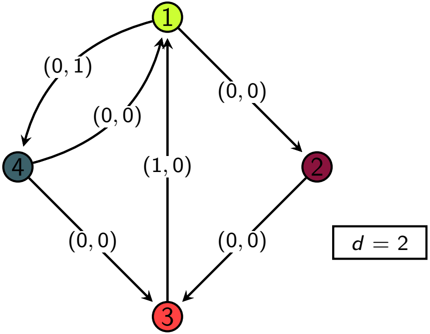


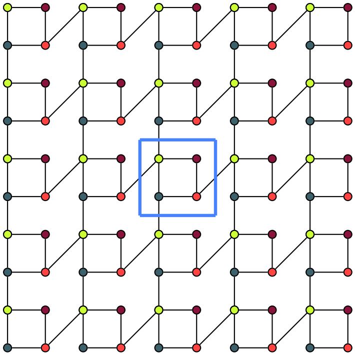
Gain Graph $\langle G, m \rangle$

- directed multigraph $G = (V, E)$
- *Gain assignment* $m : E^+ \to \mathbb{Z}^d$
- *Gain group* $\mathbb{Z}^d$

Derived Graph G^m

Gain Graphs





Gain Graph $\langle G, m \rangle$

- directed multigraph $G = (V, E)$
- *Gain assignment* $m : E^+ \rightarrow \mathbb{Z}^d$
- *Gain group* $\mathbb{Z}^d$

Derived Graph G^m

- Vertices: $V \times \mathbb{Z}^d$
- Edges: $E \times \mathbb{Z}^d$
- edges determined by gains

Periodic Orbit Frameworks on the Torus

Periodic Orbit Frameworks on the Torus

A *periodic framework* $(\langle \widetilde{G}, L\rangle, \widetilde{\mathbf{p}})$ is an infinite graph with periodic structure, placed periodically in $\mathbb{R}^d$.

Periodic Orbit Frameworks on the Torus

A *periodic framework* $(\langle \widetilde{G}, L\rangle, \widetilde{\mathbf{p}})$ is an infinite graph with periodic structure, placed periodically in $\mathbb{R}^d$.

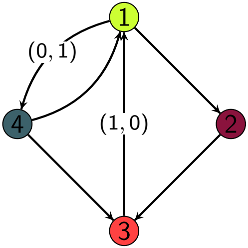
A *periodic orbit framework* $(\langle G, m\rangle, \mathbf{p})$ is:

Periodic Orbit Frameworks on the Torus

A *periodic framework* $(\langle \widetilde{G}, L\rangle, \widetilde{\mathbf{p}})$ is an infinite graph with periodic structure, placed periodically in $\mathbb{R}^d$.

A *periodic orbit framework* $(\langle G, m\rangle, \mathbf{p})$ is:

- a gain graph $\langle G, m\rangle$ with gain group $\mathbb{Z}^d$



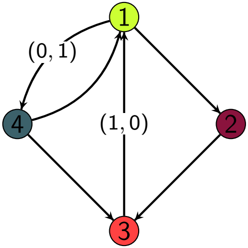
$\langle G, m\rangle$

Periodic Orbit Frameworks on the Torus

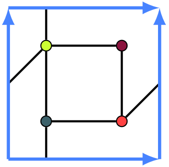
A *periodic framework* $(\langle \widetilde{G}, L\rangle, \widetilde{\mathbf{p}})$ is an infinite graph with periodic structure, placed periodically in $\mathbb{R}^d$.

A *periodic orbit framework* $(\langle G, m\rangle, \mathbf{p})$ is:

- a gain graph $\langle G, m\rangle$ with gain group $\mathbb{Z}^d$
- a map $\mathbf{p} : V \to \mathcal{T}_0^d = \mathbb{R}^d/\mathbb{Z}^d$



$\langle G, m\rangle$



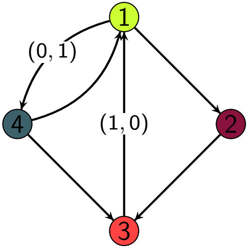
$(\langle G, m\rangle, \mathbf{p})$ on $\mathcal{T}_0^2$

Periodic Orbit Frameworks on the Torus

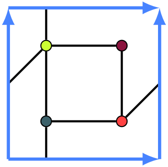
A *periodic framework* $(\langle \widetilde{G}, L\rangle, \widetilde{\mathbf{p}})$ is an infinite graph with periodic structure, placed periodically in $\mathbb{R}^d$.

A *periodic orbit framework* $(\langle G, m\rangle, \mathbf{p})$ is:

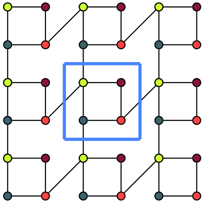
- a gain graph $\langle G, m\rangle$ with gain group $\mathbb{Z}^d$
- a map $\mathbf{p} : V \to \mathcal{T}_0^d = \mathbb{R}^d/\mathbb{Z}^d$



$\langle G, m\rangle$



$(\langle G, m\rangle, \mathbf{p})$ on $\mathcal{T}_0^2$



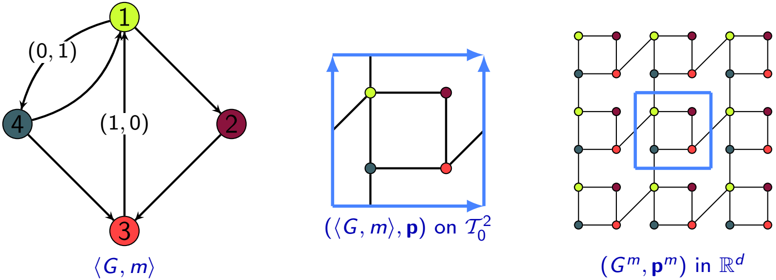
$(G^m, \mathbf{p}^m)$ in $\mathbb{R}^d$

Periodic Orbit Frameworks on the Torus

A *periodic framework* $(\langle \widetilde{G}, L\rangle, \widetilde{\mathbf{p}})$ is an infinite graph with periodic structure, placed periodically in $\mathbb{R}^d$.

A *periodic orbit framework* $(\langle G, m\rangle, \mathbf{p})$ is:

- a gain graph $\langle G, m\rangle$ with gain group $\mathbb{Z}^d$
- a map $\mathbf{p} : V \to \mathcal{T}_0^d = \mathbb{R}^d/\mathbb{Z}^d$



Theorem

Every periodic framework in $\mathbb{R}^d$ can be represented as a periodic orbit framework on a torus $\mathcal{T}_0^d$.

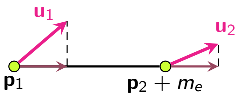
Infinitesimal Rigidity on the Fixed Torus

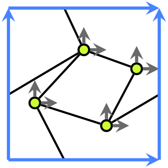
- Fixed torus: assume torus is of fixed shape and dimension

- Fixed torus: assume torus is of fixed shape and dimension
- An *infinitesimal motion* of $(\langle G, m\rangle, \mathbf{p})$ on $\mathcal{T}_0^d$ is a function $\mathbf{u} : V \to \mathbb{R}^d$ s.t.

$$(\mathbf{u}_i - \mathbf{u}_j) \cdot (\mathbf{p}_i - (\mathbf{p}_j + m_e)) = 0$$

for all $e = \{i, j, m_e\} \in E\langle G, m\rangle$



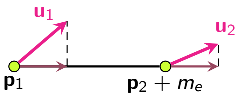


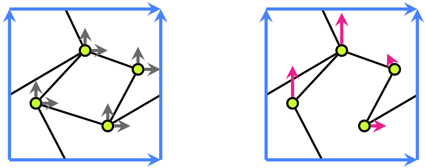
Infinitesimal Rigidity on the Fixed Torus

- Fixed torus: assume torus is of fixed shape and dimension
- An *infinitesimal motion* of $(\langle G, m\rangle, \mathbf{p})$ on $\mathcal{T}_0^d$ is a function $\mathbf{u} : V \to \mathbb{R}^d$ s.t.

$$(\mathbf{u}_i - \mathbf{u}_j) \cdot (\mathbf{p}_i - (\mathbf{p}_j + m_e)) = 0$$

for all $e = \{i, j, m_e\} \in E\langle G, m\rangle$





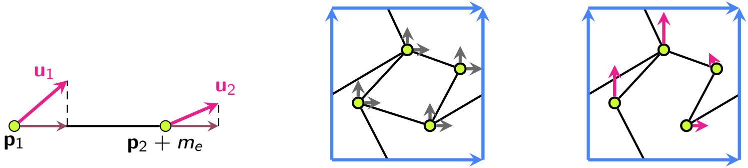
- $\mathbf{u}$ is *trivial* if it corresponds to a translation of $(\langle G, m\rangle, \mathbf{p})$ on $\mathcal{T}_0^2$

Infinitesimal Rigidity on the Fixed Torus

- Fixed torus: assume torus is of fixed shape and dimension
- An *infinitesimal motion* of $(\langle G, m\rangle, \mathbf{p})$ on $\mathcal{T}_0^d$ is a function $\mathbf{u} : V \to \mathbb{R}^d$ s.t.

$$(\mathbf{u}_i - \mathbf{u}_j) \cdot (\mathbf{p}_i - (\mathbf{p}_j + m_e)) = 0$$

for all $e = \{i, j, m_e\} \in E\langle G, m\rangle$



- $\mathbf{u}$ is *trivial* if it corresponds to a translation of $(\langle G, m\rangle, \mathbf{p})$ on $\mathcal{T}_0^2$
- $(\langle G, m\rangle, \mathbf{p})$ is *infinitesimally rigid* on $\mathcal{T}_0^d$ if the only infinitesimal motions of the framework are trivial.

Fixed Torus Rigidity Matrix

The *periodic rigidity matrix* $R_0(\langle G, m\rangle, \mathbf{p})$ is the $|E| \times d|V|$ matrix

$$\begin{bmatrix} \cdots & \mathbf{p}_i - (\mathbf{p}_j + m_e) & \cdots & (\mathbf{p}_j + m_e) - \mathbf{p}_i & \cdots \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{|V|} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Fixed Torus Rigidity Matrix

The *periodic rigidity matrix* $R_0(\langle G, m\rangle, \mathbf{p})$ is the $|E| \times d|V|$ matrix

$$\left[\begin{array}{ccccc} \cdots & \mathbf{p}_i - (\mathbf{p}_j + m_e) & \cdots & (\mathbf{p}_j + m_e) - \mathbf{p}_i & \cdots \end{array}\right] \left[\begin{array}{c} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{|V|} \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}\right]$$

- $(\langle G, m\rangle, \mathbf{p})$ infinitesimally rigid on $\mathcal{T}_0^d \iff rank R_0(\langle G, m\rangle, \mathbf{p}) = d|V| - d$.

Fixed Torus Rigidity Matrix

The *periodic rigidity matrix* $R_0(\langle G, m\rangle, \mathbf{p})$ is the $|E| \times d|V|$ matrix

$$\begin{bmatrix} \cdots & \mathbf{p}_i - (\mathbf{p}_j + m_e) & \cdots & (\mathbf{p}_j + m_e) - \mathbf{p}_i & \cdots \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{|V|} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

- $(\langle G, m\rangle, \mathbf{p})$ infinitesimally rigid on $\mathcal{T}_0^d \iff rank R_0(\langle G, m\rangle, \mathbf{p}) = d|V| - d$.
- *minimally rigid* = infinitesimally rigid and independent

Fixed Torus Rigidity Matrix

The *periodic rigidity matrix* $R_0(\langle G, m\rangle, \mathbf{p})$ is the $|E| \times d|V|$ matrix

$$\begin{bmatrix} \cdots & \mathbf{p}_i - (\mathbf{p}_j + m_e) & \cdots & (\mathbf{p}_j + m_e) - \mathbf{p}_i & \cdots \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{|V|} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

- $(\langle G, m\rangle, \mathbf{p})$ infinitesimally rigid on $\mathcal{T}_0^d \iff rank R_0(\langle G, m\rangle, \mathbf{p}) = d|V| - d$.
- *minimally rigid* = infinitesimally rigid and independent

Theorem (Periodic Maxwell Rule, Whiteley 1988)

If $(\langle G, m\rangle, \mathbf{p})$ is minimally rigid on the fixed torus $\mathcal{T}_0^d$, then $|E| = d|V| - d$.

Fixed Torus Rigidity Matrix

The *periodic rigidity matrix* $R_0(\langle G, m\rangle, \mathbf{p})$ is the $|E| \times d|V|$ matrix

$$\begin{bmatrix} \cdots & \mathbf{p}_i - (\mathbf{p}_j + m_e) & \cdots & (\mathbf{p}_j + m_e) - \mathbf{p}_i & \cdots \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{|V|} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

- $(\langle G, m\rangle, \mathbf{p})$ infinitesimally rigid on $\mathcal{T}_0^d \iff rank R_0(\langle G, m\rangle, \mathbf{p}) = d|V| - d$.
- *minimally rigid* = infinitesimally rigid and independent

Theorem (Periodic Maxwell Rule, Whiteley 1988)

If $(\langle G, m\rangle, \mathbf{p})$ is minimally rigid on the fixed torus $\mathcal{T}_0^d$, then $|E| = d|V| - d$.

- The rigidity properties of a *generic* framework are properties of the periodic orbit graph $\langle G, m\rangle$, NOT the configuration $\mathbf{p}$.

Periodic Inductive Constructions (2 dimensions)

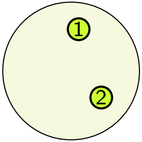
- Periodic inductive constructions preserve generic rigidity

Periodic Inductive Constructions (2 dimensions)

- Periodic inductive constructions preserve generic rigidity
- *Periodic vertex addition*: add a 2-valent vertex:

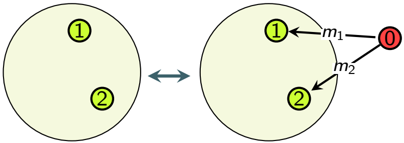
Periodic Inductive Constructions (2 dimensions)

- Periodic inductive constructions preserve generic rigidity
- *Periodic vertex addition*: add a 2-valent vertex:



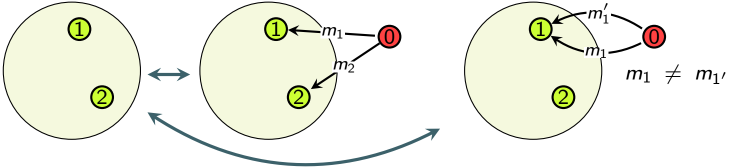
Periodic Inductive Constructions (2 dimensions)

- Periodic inductive constructions preserve generic rigidity
- *Periodic vertex addition*: add a 2-valent vertex:



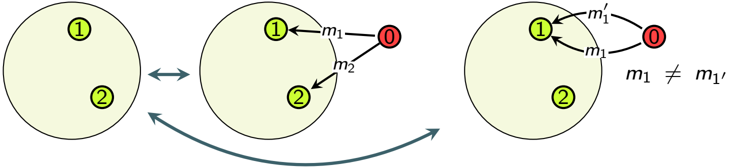
Periodic Inductive Constructions (2 dimensions)

- Periodic inductive constructions preserve generic rigidity
- *Periodic vertex addition*: add a 2-valent vertex:



Periodic Inductive Constructions (2 dimensions)

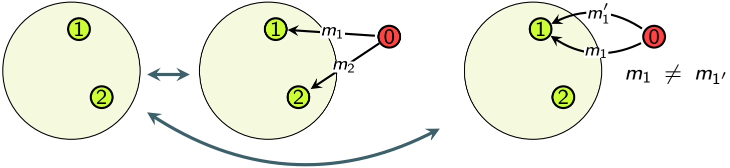
- Periodic inductive constructions preserve generic rigidity
- *Periodic vertex addition*: add a 2-valent vertex:



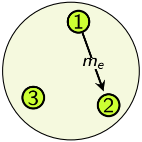
- *Periodic edge split*: Add a 3-valent vertex, delete an edge, preserve gain on deleted edge:

Periodic Inductive Constructions (2 dimensions)

- Periodic inductive constructions preserve generic rigidity
- *Periodic vertex addition*: add a 2-valent vertex:

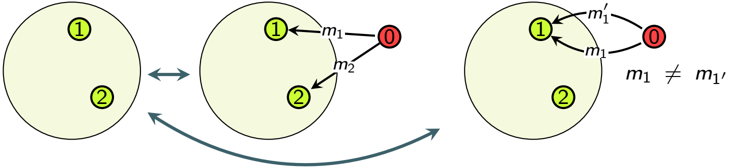


- *Periodic edge split*: Add a 3-valent vertex, delete an edge, preserve gain on deleted edge:

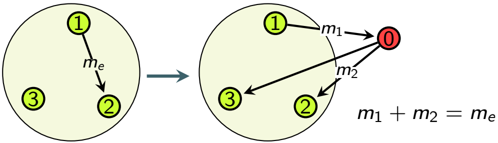


Periodic Inductive Constructions (2 dimensions)

- Periodic inductive constructions preserve generic rigidity
- *Periodic vertex addition*: add a 2-valent vertex:

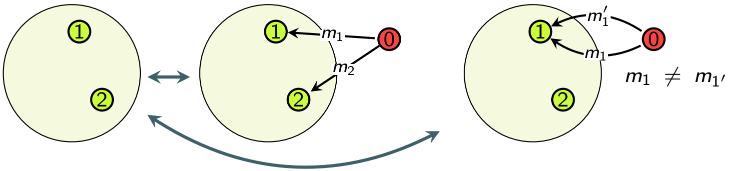


- *Periodic edge split*: Add a 3-valent vertex, delete an edge, preserve gain on deleted edge:

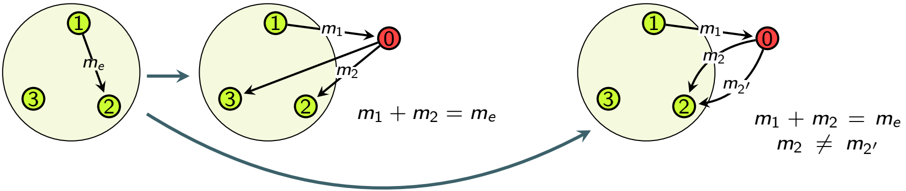


Periodic Inductive Constructions (2 dimensions)

- Periodic inductive constructions preserve generic rigidity
- *Periodic vertex addition*: add a 2-valent vertex:



- *Periodic edge split*: Add a 3-valent vertex, delete an edge, preserve gain on deleted edge:



Periodic Inductive Constructions

- These moves are really "local"

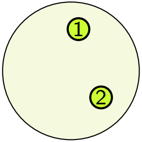
Periodic Inductive Constructions

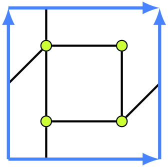
- These moves are really "local"
- easy to see that inductions hold for "some" choice of gains

Periodic Inductive Constructions

- These moves are really "local"
- easy to see that inductions hold for "some" choice of gains

Vertex additions:

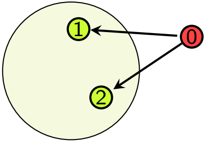


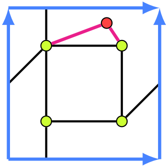


Periodic Inductive Constructions

- These moves are really "local"
- easy to see that inductions hold for "some" choice of gains

Vertex additions:

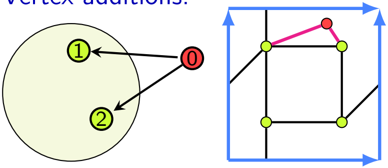


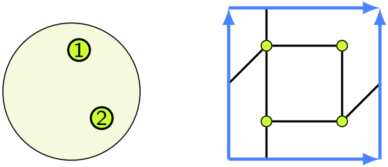


Periodic Inductive Constructions

- These moves are really “local”
- easy to see that inductions hold for “some” choice of gains

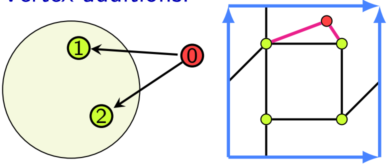
Vertex additions:

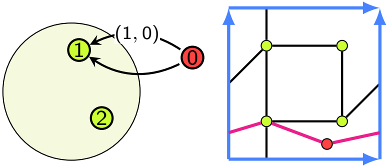




- These moves are really “local”
- easy to see that inductions hold for “some” choice of gains

Vertex additions:

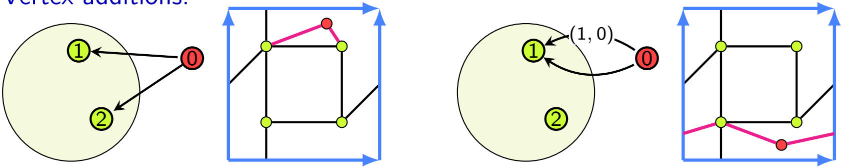




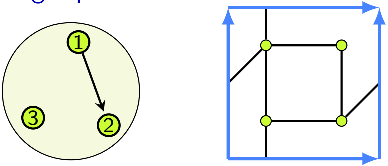
Periodic Inductive Constructions

- These moves are really “local”
- easy to see that inductions hold for “some” choice of gains

Vertex additions:



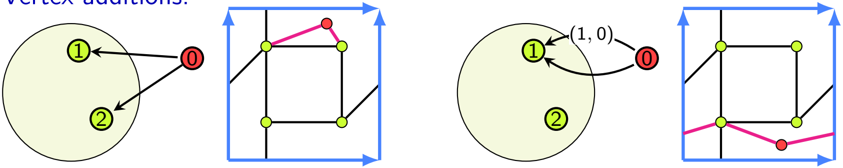
Edge splits:



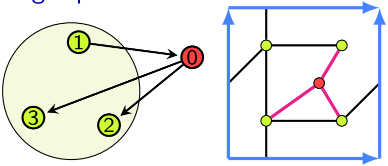
Periodic Inductive Constructions

- These moves are really "local"
- easy to see that inductions hold for "some" choice of gains

Vertex additions:



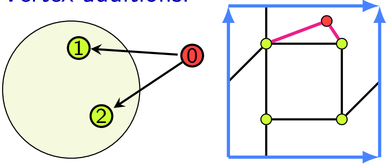
Edge splits:

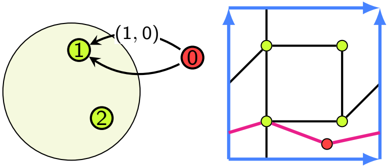


Periodic Inductive Constructions

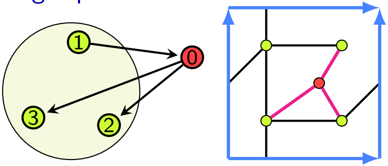
- These moves are really “local”
- easy to see that inductions hold for “some” choice of gains

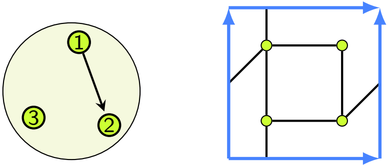
Vertex additions:





Edge splits:

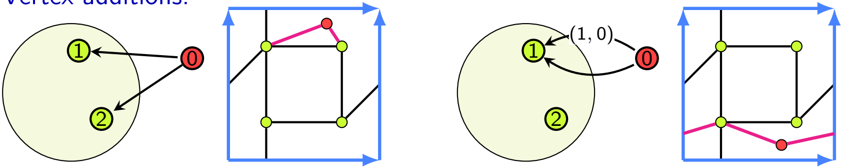




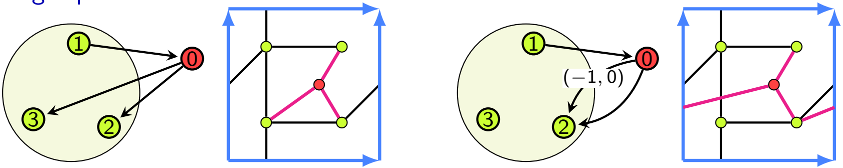
Periodic Inductive Constructions

- These moves are really "local"
- easy to see that inductions hold for "some" choice of gains

Vertex additions:



Edge splits:



Periodic Henneberg Theorem

Theorem (R., 2009)

A periodic orbit framework $(\langle G, m\rangle, \mathbf{p})$ with $|E| = 2|V| - 2$ is infinitesimally rigid on the fixed torus $\mathcal{T}_0^2$

$$\Longleftrightarrow$$

it can be constructed from a single vertex by a series of periodic vertex additions and edge splits.

Periodic Henneberg Theorem

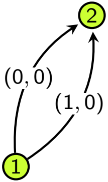
Theorem (R., 2009)

A periodic orbit framework $(\langle G, m \rangle, \mathbf{p})$ with $|E| = 2|V| - 2$ is infinitesimally rigid on the fixed torus $\mathcal{T}_0^2$

$$\Longleftrightarrow$$

it can be constructed from a single vertex by a series of periodic vertex additions and edge splits.

- Basic observation: the single vertex is rigid on the fixed torus



Periodic Henneberg Theorem

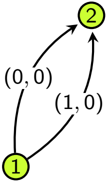
Theorem (R., 2009)

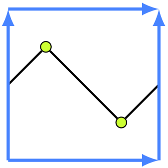
A periodic orbit framework $(\langle G, m\rangle, \mathbf{p})$ with $|E| = 2|V| - 2$ is infinitesimally rigid on the fixed torus $\mathcal{T}_0^2$

$$\Longleftrightarrow$$

it can be constructed from a single vertex by a series of periodic vertex additions and edge splits.

- Basic observation: the single vertex is rigid on the fixed torus
- Add a two-valent vertex, creating a non-zero cycle gain





Periodic Henneberg Theorem

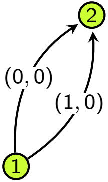
Theorem (R., 2009)

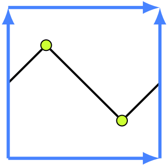
A periodic orbit framework $(\langle G, m\rangle, \mathbf{p})$ with $|E| = 2|V| - 2$ is infinitesimally rigid on the fixed torus $\mathcal{T}_0^2$

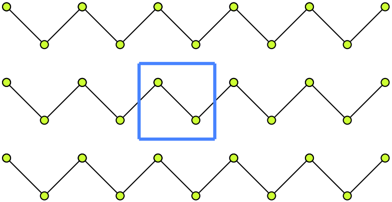
$$\iff$$

it can be constructed from a single vertex by a series of periodic vertex additions and edge splits.

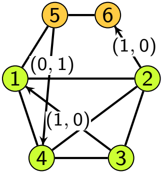
- Basic observation: the single vertex is rigid on the fixed torus
- Add a two-valent vertex, creating a non-zero cycle gain





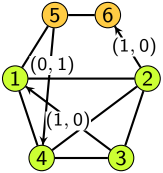


Periodic Laman Theorem

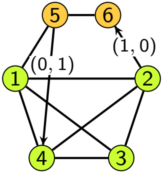
rigid: 

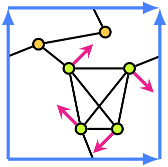
Periodic Laman Theorem

rigid:



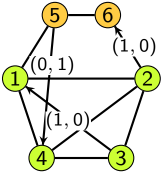
flexible:



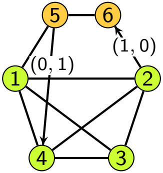


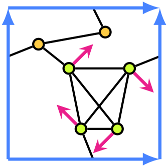
Periodic Laman Theorem

rigid:



flexible:

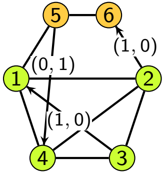




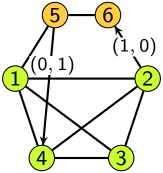
Constructive gain assignment for G on $\mathcal{T}_0^2$: Every subgraph $G' \subseteq G$ with $|E'| = 2|V'| - 2$ contains a cycle with non-trivial net gain.

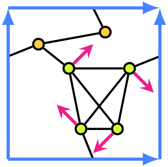
Periodic Laman Theorem

rigid:



flexible:



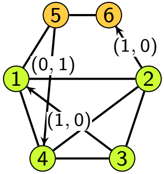


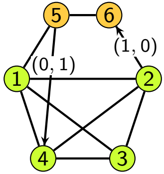
Constructive gain assignment for G on $\mathcal{T}_0^2$: Every subgraph $G' \subseteq G$ with $|E'| = 2|V'| - 2$ contains a cycle with non-trivial net gain.

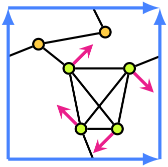
Theorem (R., 2009)

The periodic orbit framework $(\langle G, m \rangle, \mathbf{p})$ is generically minimally rigid on $\mathcal{T}_0^2$ if and only if $|E| = 2|V| - 2$, $|E'| \leq 2|V'| - 2$ for all $G' \subseteq G$, and $m : E^+ \to \mathbb{Z}^2$ is constructive.

Periodic Laman Theorem

rigid: 

flexible: 



Constructive gain assignment for G on $\mathcal{T}_0^2$: Every subgraph $G' \subseteq G$ with $|E'| = 2|V'| - 2$ contains a cycle with non-trivial net gain.

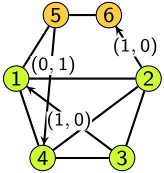
Theorem (R., 2009)

The periodic orbit framework $(\langle G, m\rangle, \mathbf{p})$ is generically minimally rigid on $\mathcal{T}_0^2$ if and only if $|E| = 2|V| - 2$, $|E'| \leq 2|V'| - 2$ for all $G' \subseteq G$, and $m : E^+ \to \mathbb{Z}^2$ is constructive.

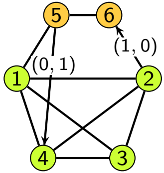
- Proof by induction

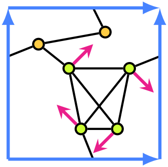
Periodic Laman Theorem

rigid:



flexible:





Constructive gain assignment for G on $\mathcal{T}_0^2$: Every subgraph $G' \subseteq G$ with $|E'| = 2|V'| - 2$ contains a cycle with non-trivial net gain.

Theorem (R., 2009)

The periodic orbit framework $(\langle G, m\rangle, \mathbf{p})$ *is generically minimally rigid on* $\mathcal{T}_0^2$ *if and only if* $|E| = 2|V| - 2$, $|E'| \leq 2|V'| - 2$ *for all* $G' \subseteq G$, *and* $m : E^+ \to \mathbb{Z}^2$ *is constructive.*

- Proof by induction
- Periodic Laman theorem on $\mathcal{T}_0^2$ naturally leads to an algorithm, based on the pebble game.

T-gain procedure

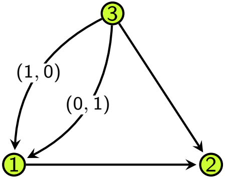
- T-gain procedure: tool to identify a fundamental system of cycles (Gross & Tucker, 1987)

T-gain procedure

- T-gain procedure: tool to identify a fundamental system of cycles (Gross & Tucker, 1987)
  - THEOREM: preserves rank of rigidity matrix

T-gain procedure

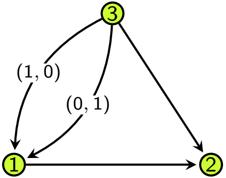
- T-gain procedure: tool to identify a fundamental system of cycles (Gross & Tucker, 1987)
  - THEOREM: preserves rank of rigidity matrix



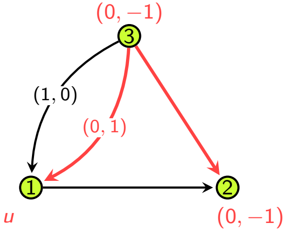
$\langle G, m \rangle$

T-gain procedure

- T-gain procedure: tool to identify a fundamental system of cycles (Gross & Tucker, 1987)
  - THEOREM: preserves rank of rigidity matrix

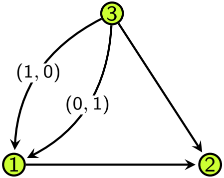


$\langle G, m \rangle$

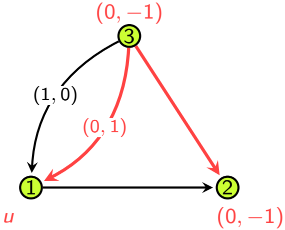


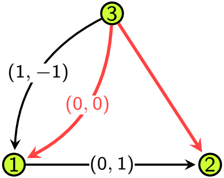
T-gain procedure

- T-gain procedure: tool to identify a fundamental system of cycles (Gross & Tucker, 1987)
  - THEOREM: preserves rank of rigidity matrix



$\langle G, m \rangle$

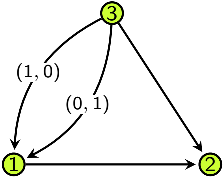




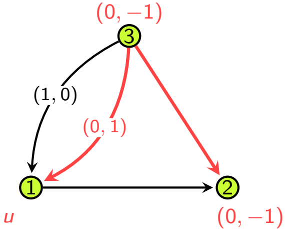
$\langle G, m_T \rangle$

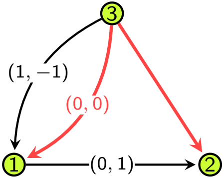
T-gain procedure

- T-gain procedure: tool to identify a fundamental system of cycles (Gross & Tucker, 1987)
  - THEOREM: preserves rank of rigidity matrix

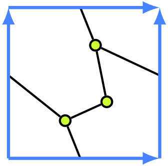


$\langle G, m \rangle$



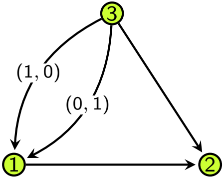


$\langle G, m_T \rangle$

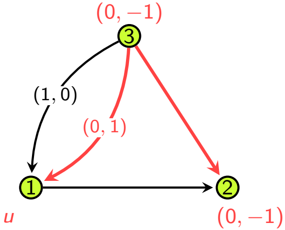


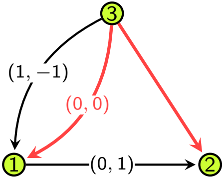
T-gain procedure

- T-gain procedure: tool to identify a fundamental system of cycles (Gross & Tucker, 1987)
  - THEOREM: preserves rank of rigidity matrix

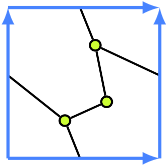


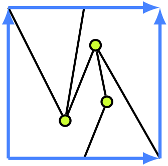
$\langle G, m \rangle$





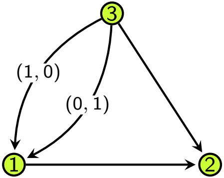
$\langle G, m_T \rangle$



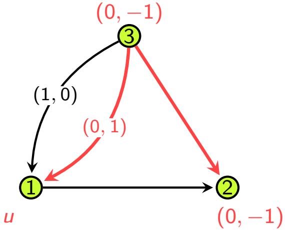


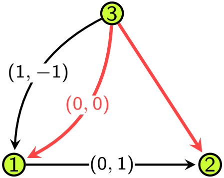
T-gain procedure

- T-gain procedure: tool to identify a fundamental system of cycles (Gross & Tucker, 1987)
  - THEOREM: preserves rank of rigidity matrix

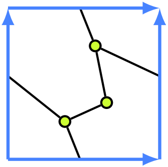


$\langle G, m \rangle$

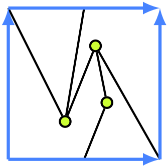




$\langle G, m_T \rangle$

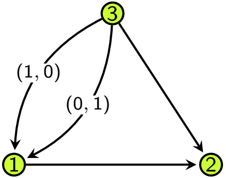


- graph isomorphism between G^m and G^{m_T}

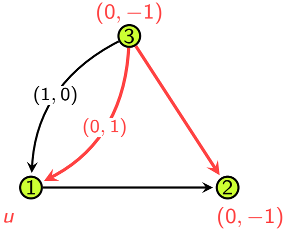


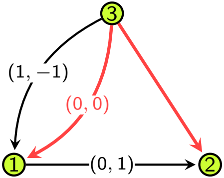
T-gain procedure

- T-gain procedure: tool to identify a fundamental system of cycles (Gross & Tucker, 1987)
  - THEOREM: preserves rank of rigidity matrix

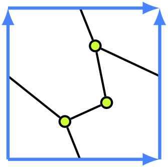


$\langle G, m \rangle$

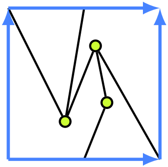




$\langle G, m_T \rangle$

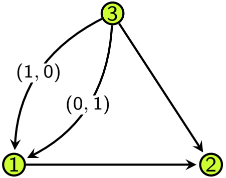


- graph isomorphism between G^m and G^{m_T}
- not an affine transformation, the graphs are homotopic

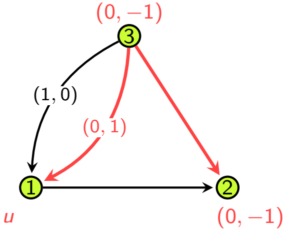


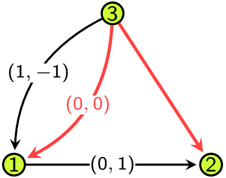
T-gain procedure

- T-gain procedure: tool to identify a fundamental system of cycles (Gross & Tucker, 1987)
  - THEOREM: preserves rank of rigidity matrix

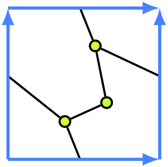


$\langle G, m \rangle$

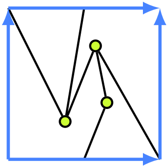




$\langle G, m_T \rangle$



- graph isomorphism between G^m and G^{m_T}
- not an affine transformation, the graphs are homotopic
- same rigidity properties



d-Dimensional Rigidity on the Fixed Torus

d-Dimensional Rigidity on the Fixed Torus

Theorem (R., 2011)

Let $(\langle G, m\rangle, \mathbf{p})$ be a minimally rigid framework on $\mathcal{T}_0^d$. Then $|E| = d|V| - d$, and for all subsets of edges $Y \subseteq E$,

$$|Y| \leq d|V(Y)| - \binom{d+1}{2} + \sum_{i=1}^{|\mathcal{M}_{\mathcal{C}}(Y)|} (d - i).$$

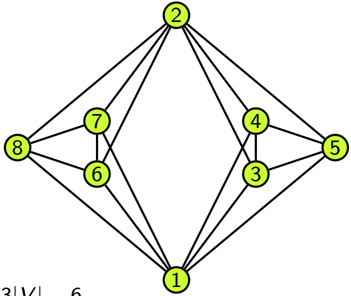
d-Dimensional Rigidity on the Fixed Torus

Theorem (R., 2011)

Let $(\langle G, m\rangle, \mathbf{p})$ be a minimally rigid framework on $\mathcal{T}_0^d$. Then $|E| = d|V| - d$, and for all subsets of edges $Y \subseteq E$,

$$|Y| \leq d|V(Y)| - \binom{d+1}{2} + \sum_{i=1}^{|\mathcal{M}_\mathcal{C}(Y)|} (d-i).$$

Periodic Bananas:



$|E| = 3|V| - 6$

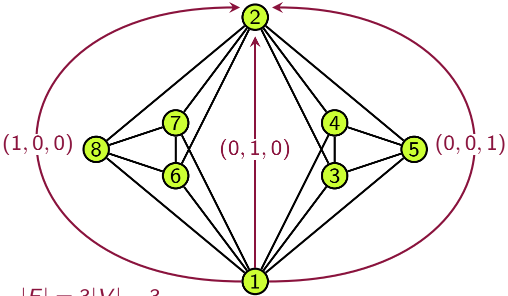
d-Dimensional Rigidity on the Fixed Torus

Theorem (R., 2011)

Let $(\langle G, m\rangle, \mathbf{p})$ be a minimally rigid framework on $\mathcal{T}_0^d$. Then $|E| = d|V| - d$, and for all subsets of edges $Y \subseteq E$,

$$|Y| \leq d|V(Y)| - \binom{d+1}{2} + \sum_{i=1}^{|\mathcal{M}_{\mathcal{C}}(Y)|} (d - i).$$

Periodic Bananas:



$|E| = 3|V| - 3$

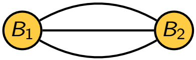
Bar-Body Frameworks for $d = 2$

	Finite	Periodic
Counts		
(Gain) Graph		
Framework		

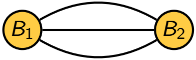
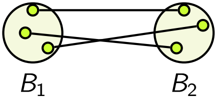
Bar-Body Frameworks for $d = 2$

	Finite	Periodic
Counts	$\lvert E\rvert = 3\lvert V\rvert - 3$	
(Gain) Graph		
Framework		

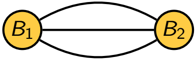
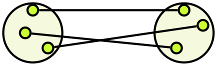
Bar-Body Frameworks for $d = 2$

	Finite	Periodic
Counts	$\lvert E\rvert = 3\lvert V\rvert - 3$	
(Gain) Graph	 $H = (V, E)$	
Framework		

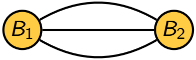
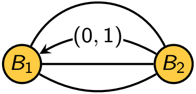
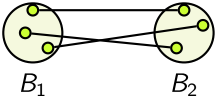
Bar-Body Frameworks for $d = 2$

	Finite	Periodic
Counts	$\lvert E\rvert = 3\lvert V\rvert - 3$	
(Gain) Graph	 $H = (V, E)$	
Framework	 (H, q) in $\mathbb{R}^2$	

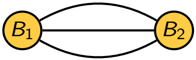
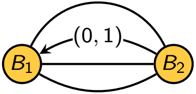
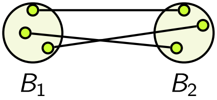
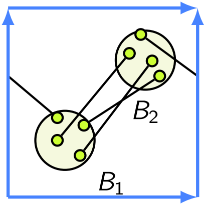
Bar-Body Frameworks for $d = 2$

	Finite	Periodic
Counts	$\lvert E\rvert = 3\lvert V\rvert - 3$	$\lvert E\rvert = 3\lvert V\rvert - 2$
(Gain) Graph	 $H = (V, E)$	
Framework	 B_1 B_2 (H, q) in $\mathbb{R}^2$	

Bar-Body Frameworks for $d = 2$

	Finite	Periodic
Counts	$\lvert E\rvert = 3\lvert V\rvert - 3$	$\lvert E\rvert = 3\lvert V\rvert - 2$
(Gain) Graph	 $H = (V, E)$	 $\langle H, m\rangle$
Framework	 (H, q) in $\mathbb{R}^2$	

Bar-Body Frameworks for $d = 2$

	Finite	Periodic
Counts	$\|E\| = 3\|V\| - 3$	$\|E\| = 3\|V\| - 2$
(Gain) Graph	 $H = (V, E)$	 $\langle H, m\rangle$
Framework	 (H, q) in $\mathbb{R}^2$	 $(\langle H, m\rangle, q)$ on $\mathcal{T}_0^2$

Bar-Body Frameworks for all d (Further Work)

Theorem (R., 2011)

Let $\langle H, m\rangle$ be a generically minimally rigid bar-body framework on $\mathcal{T}_0^d$, with $|E| = \binom{d+1}{2}|V| - d$. Then for all nonempty subsets $Y \subseteq E$ of edges,

$$|Y| \leq \binom{d+1}{2}|V(Y)| - \binom{d+1}{2} + \sum_{i=1}^{|\mathcal{M}_{\mathcal{C}}(Y)|} (d-i).$$

Bar-Body Frameworks for all d (Further Work)

Theorem (R., 2011)

Let $\langle H, m \rangle$ be a generically minimally rigid bar-body framework on $\mathcal{T}_0^d$, with $|E| = \binom{d+1}{2}|V| - d$. Then for all nonempty subsets $Y \subseteq E$ of edges,

$$|Y| \leq \binom{d+1}{2}|V(Y)| - \binom{d+1}{2} + \sum_{i=1}^{|\mathcal{M}_{\mathcal{C}}(Y)|} (d-i).$$

Conjecture: also sufficient

Other Questions

- Periodic frameworks with extra symmetry:

Other Questions

- Periodic frameworks with extra symmetry:
  - R., Schulze, Whiteley (2011) found necessary conditions for periodic frameworks for many symmetry groups in 2 and 3 dimensions, fixed and flexible torus

Other Questions

- Periodic frameworks with extra symmetry:
  - R., Schulze, Whiteley (2011) found necessary conditions for periodic frameworks for many symmetry groups in 2 and 3 dimensions, fixed and flexible torus
  - Malestein & Theran (2011) find sufficient conditions for periodic frameworks with symmetry given by rotations and translations in 2 dimensions, on the flexible torus

Other Questions

- Periodic frameworks with extra symmetry:
  - R., Schulze, Whiteley (2011) found necessary conditions for periodic frameworks for many symmetry groups in 2 and 3 dimensions, fixed and flexible torus
  - Malestein & Theran (2011) find sufficient conditions for periodic frameworks with symmetry given by rotations and translations in 2 dimensions, on the flexible torus
- Inductive characterizations of frameworks on the partially or fully flexible torus

Other Questions

- Periodic frameworks with extra symmetry:
  - R., Schulze, Whiteley (2011) found necessary conditions for periodic frameworks for many symmetry groups in 2 and 3 dimensions, fixed and flexible torus
  - Malestein & Theran (2011) find sufficient conditions for periodic frameworks with symmetry given by rotations and translations in 2 dimensions, on the flexible torus
- Inductive characterizations of frameworks on the partially or fully flexible torus
- Global rigidity for periodic frameworks

Other Questions

- Periodic frameworks with extra symmetry:
  - R., Schulze, Whiteley (2011) found necessary conditions for periodic frameworks for many symmetry groups in 2 and 3 dimensions, fixed and flexible torus
  - Malestein & Theran (2011) find sufficient conditions for periodic frameworks with symmetry given by rotations and translations in 2 dimensions, on the flexible torus
- Inductive characterizations of frameworks on the partially or fully flexible torus
- Global rigidity for periodic frameworks
- finite repetitive frameworks

Other Questions

- Periodic frameworks with extra symmetry:
    - R., Schulze, Whiteley (2011) found necessary conditions for periodic frameworks for many symmetry groups in 2 and 3 dimensions, fixed and flexible torus
    - Malestein & Theran (2011) find sufficient conditions for periodic frameworks with symmetry given by rotations and translations in 2 dimensions, on the flexible torus
- Inductive characterizations of frameworks on the partially or fully flexible torus
- Global rigidity for periodic frameworks
- finite repetitive frameworks
- Scaling: forced vs. incidental periodicity

The End

Thank you
Questions?