



Regular polygonal complexes in space

Daniel Pellicer

Egon Schulte

Polyhedra

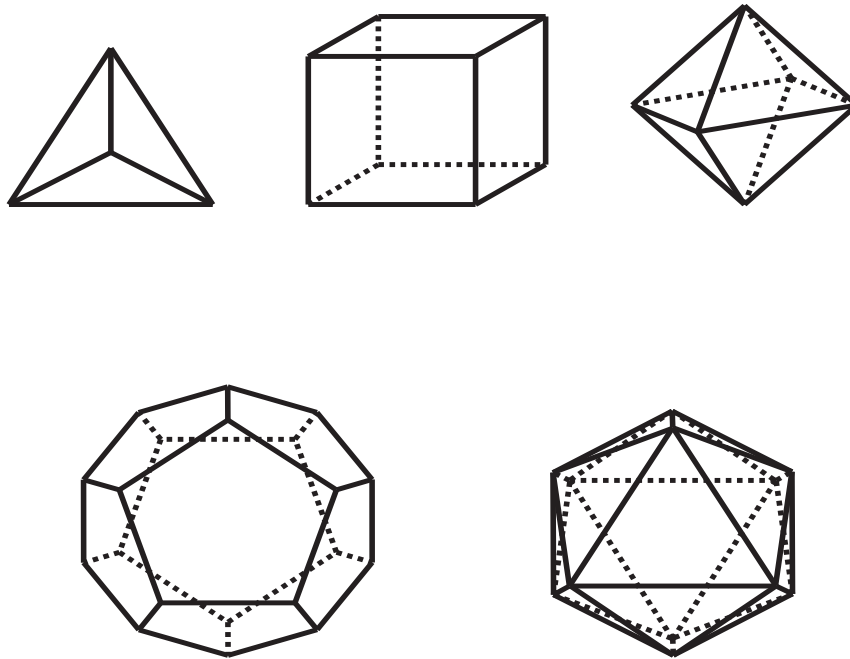
Three horizontal bars of varying lengths and colors (yellow, grey, black) extending from the left side of the slide.

- Search for symmetric structures in space.

Polyhedra

- Search for symmetric structures in space.

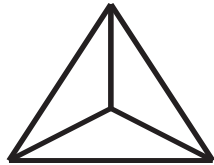
Convex polyhedra



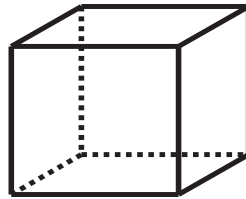
Polyhedra

- Search for symmetric structures in space.

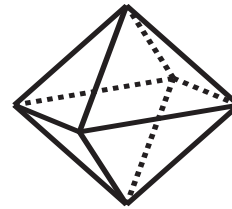
Convex polyhedra



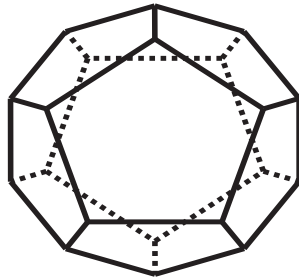
$\{3,3\}$



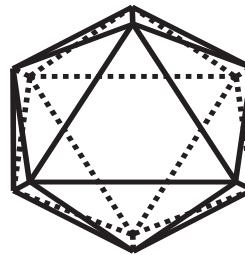
$\{4,3\}$



$\{3,4\}$

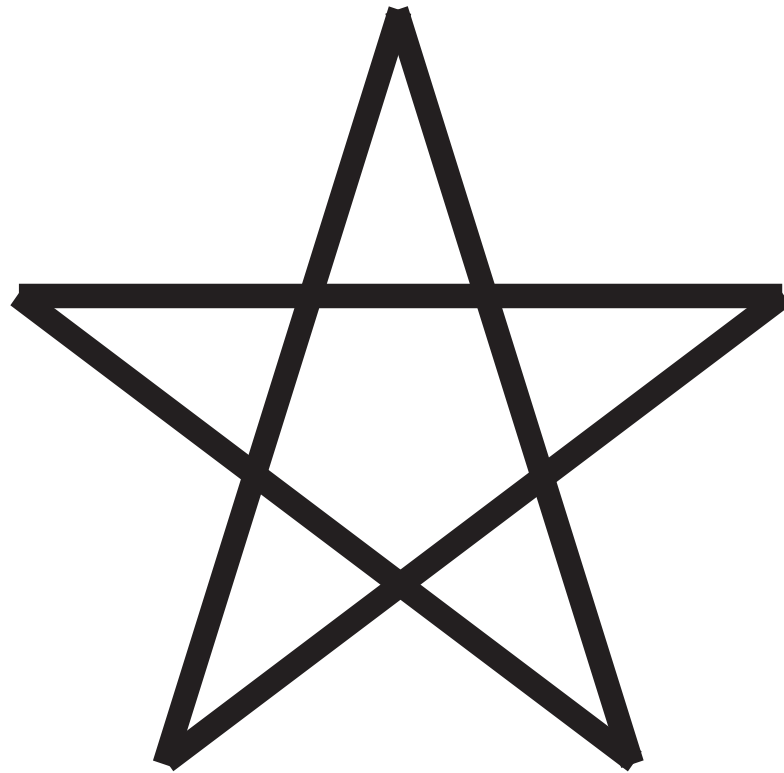


$\{5,3\}$

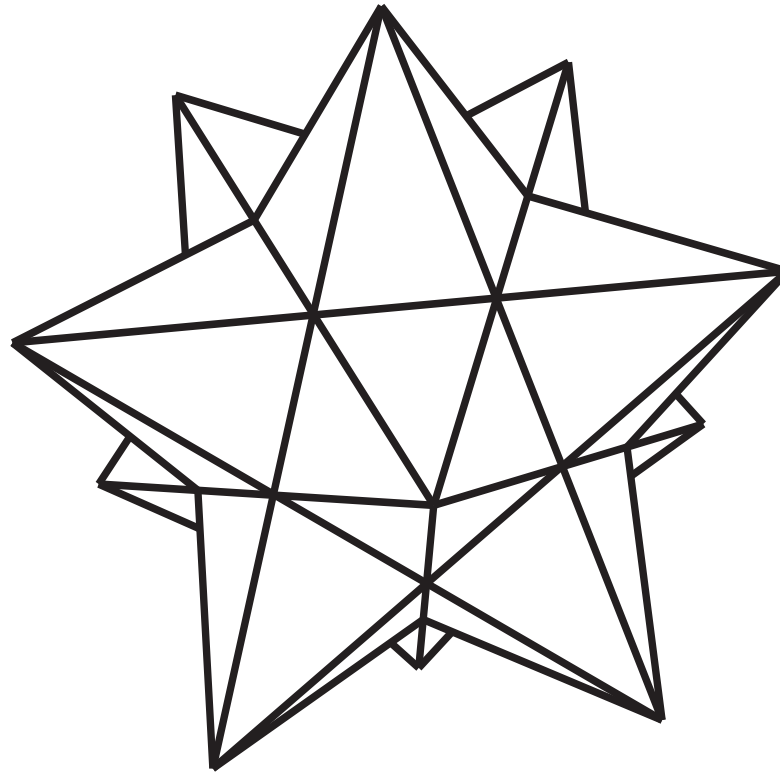


$\{3,5\}$

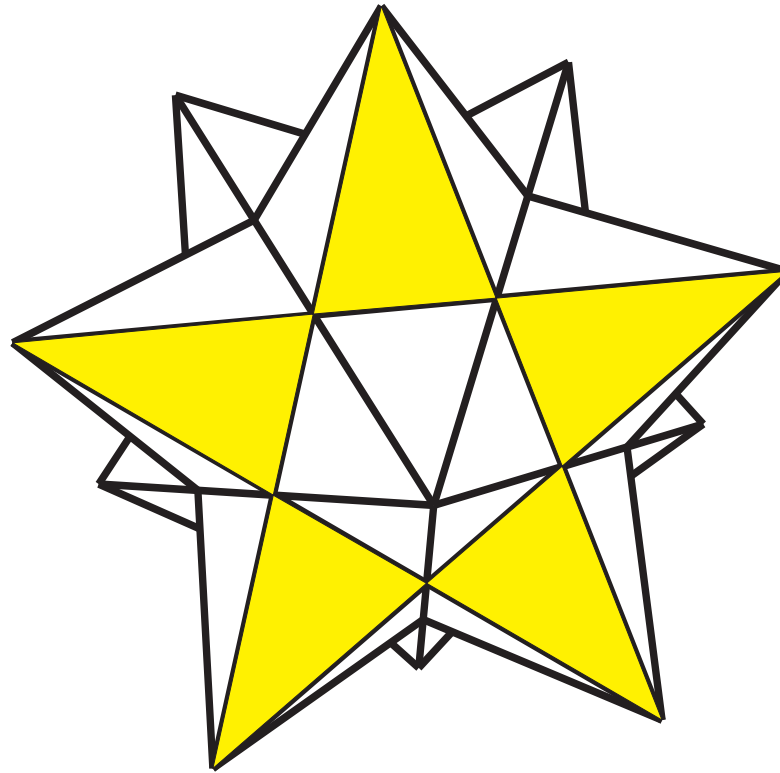
Kepler



Kepler



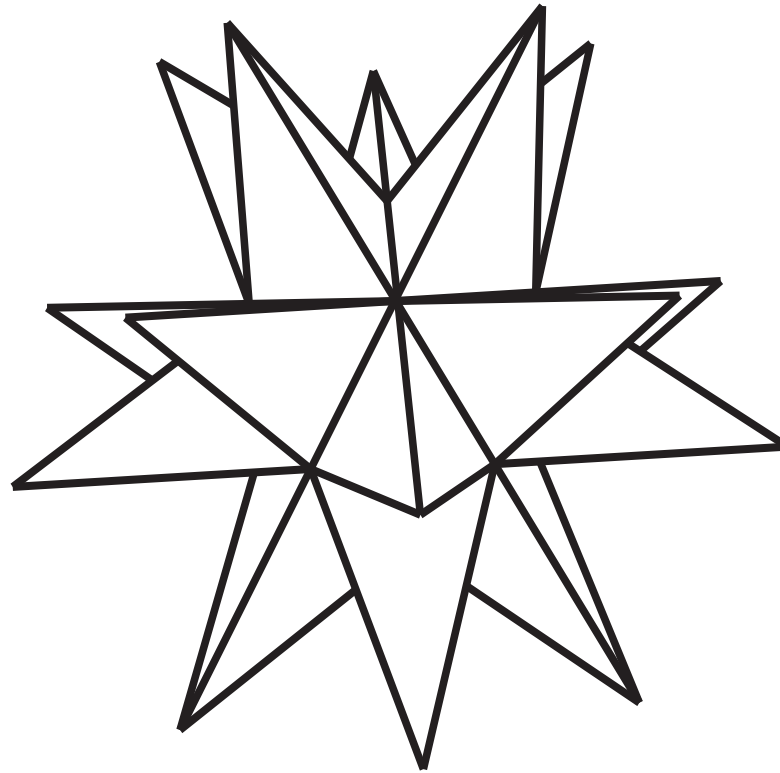
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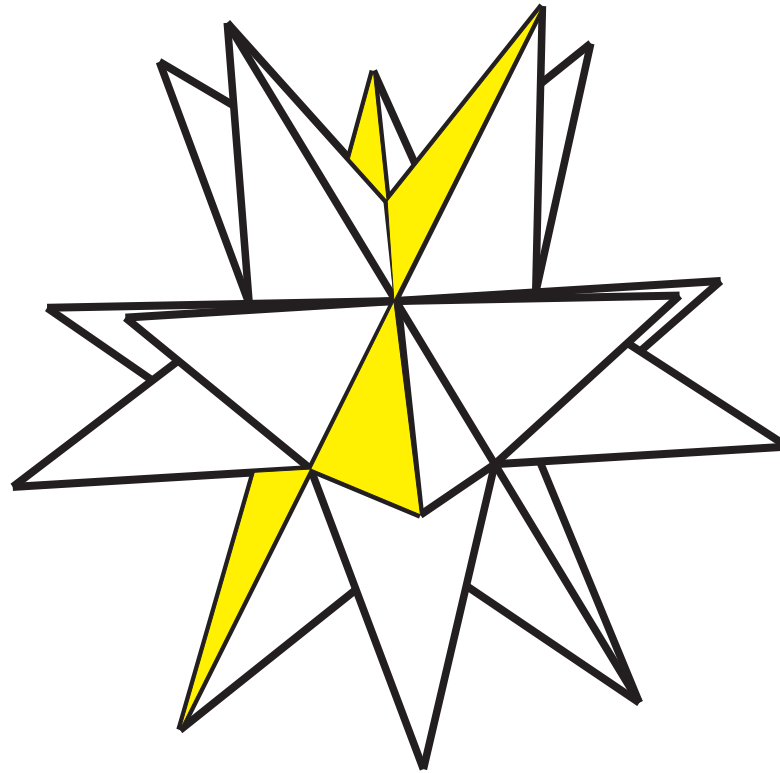
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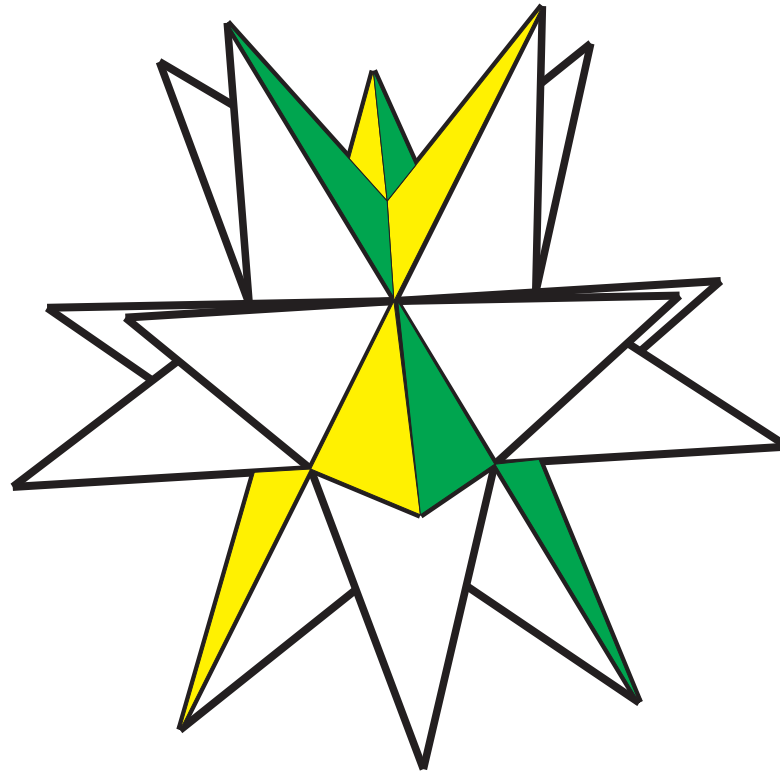
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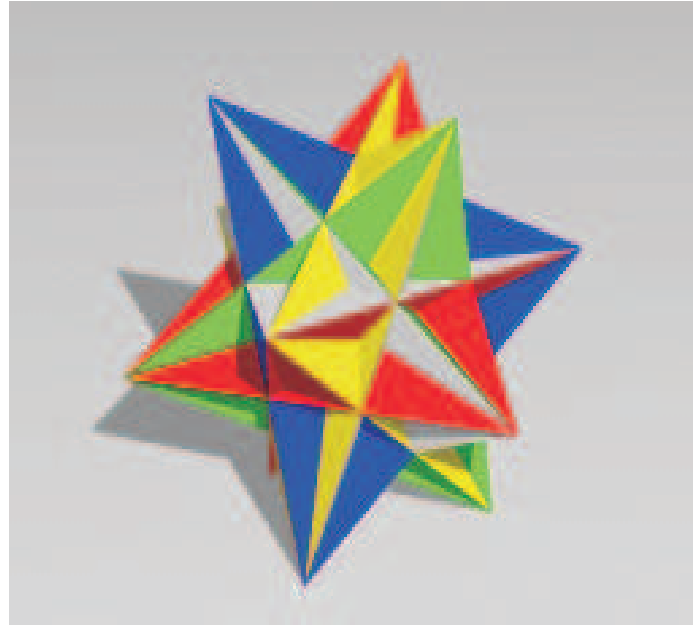
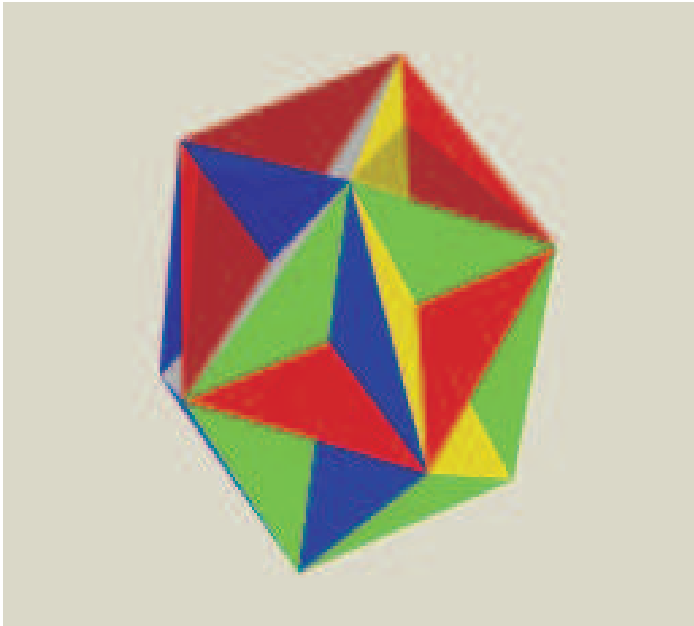
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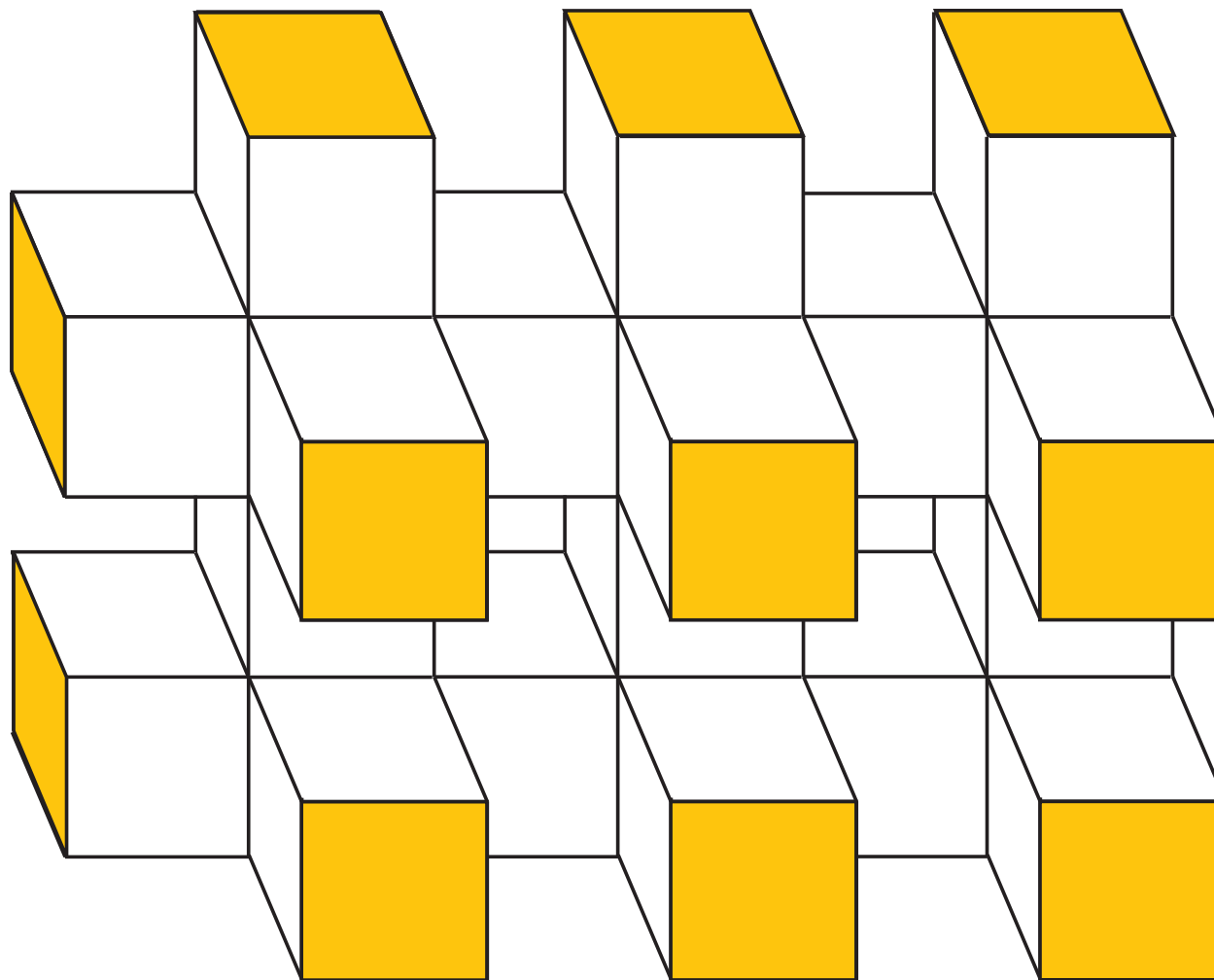
Kepler



Poinsot



Petrie, Coxeter



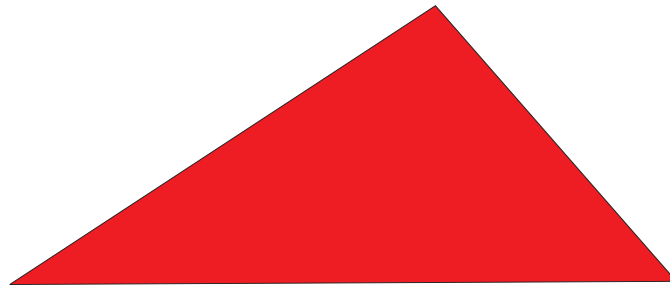
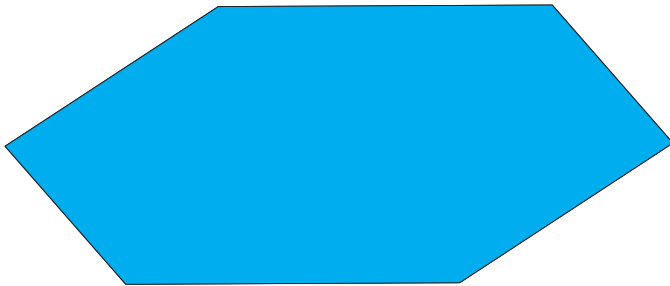
Combinatorial polyhedra

A decorative graphic consisting of three horizontal lines: a thick black line at the top, a thin grey line in the middle, and another thick black line at the bottom.

► polygon \longrightarrow 2-regular connected graph

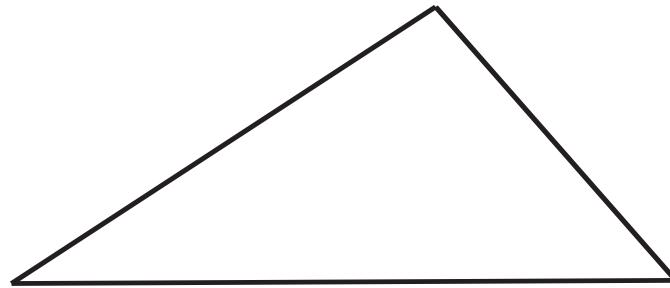
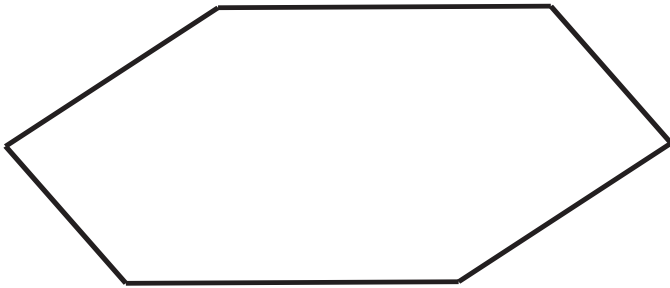
Combinatorial polyhedra

► polygon \longrightarrow 2-regular connected graph



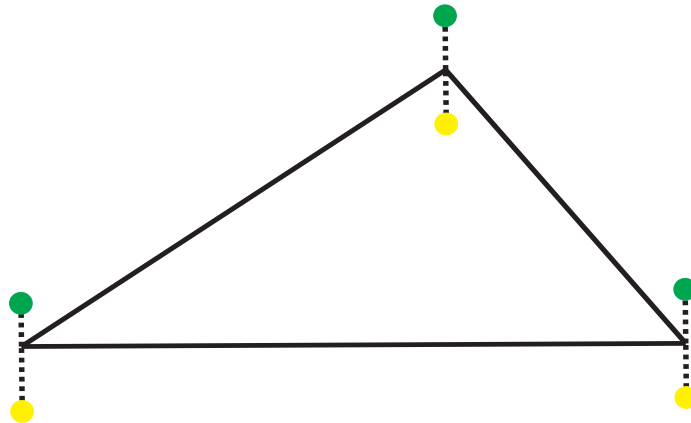
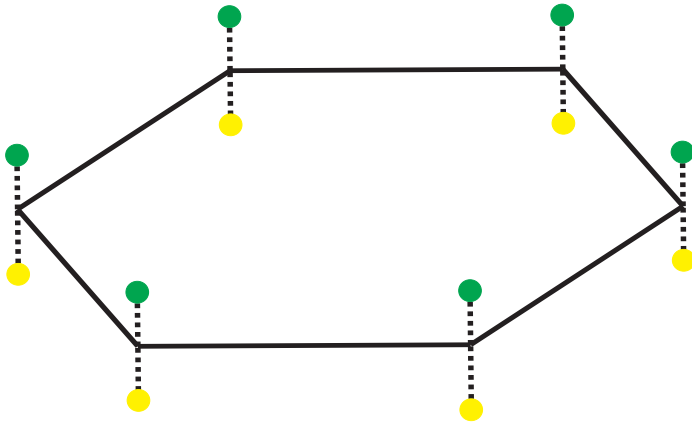
Combinatorial polyhedra

► polygon \longrightarrow 2-regular connected graph



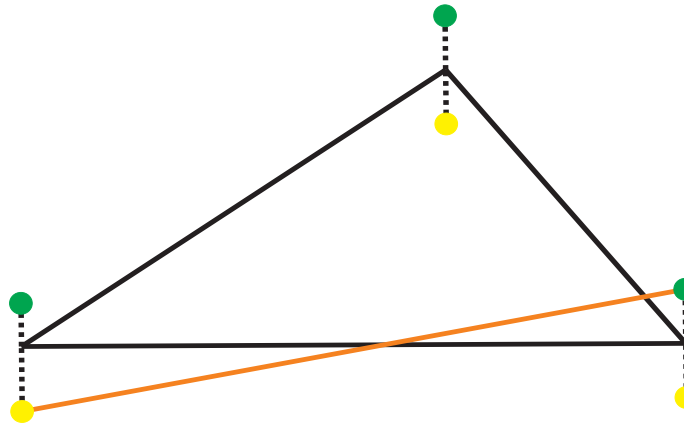
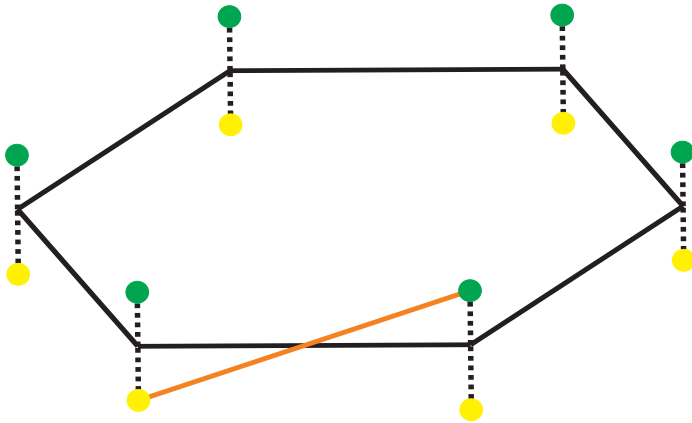
Combinatorial polyhedra

► polygon \longrightarrow 2-regular connected graph



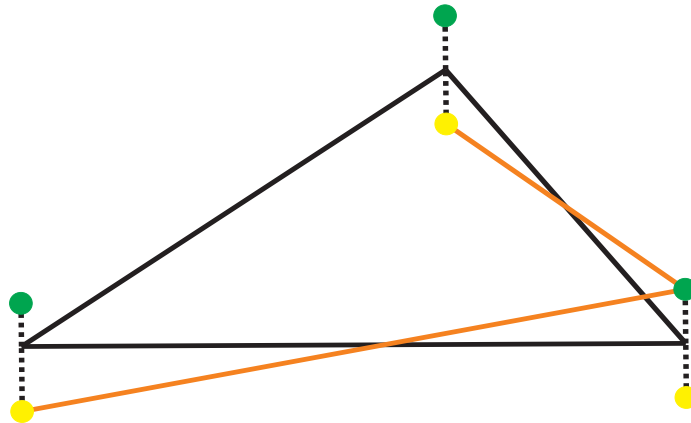
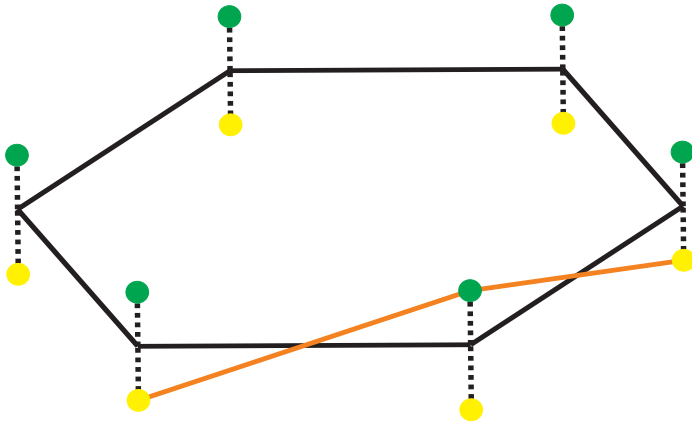
Combinatorial polyhedra

► polygon \longrightarrow 2-regular connected graph



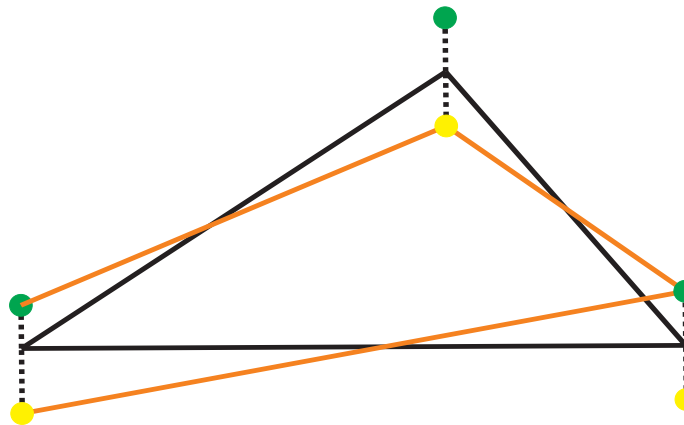
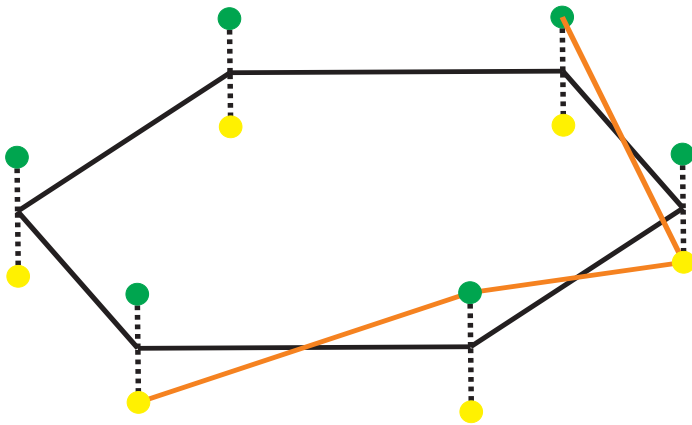
Combinatorial polyhedra

► polygon \longrightarrow 2-regular connected graph



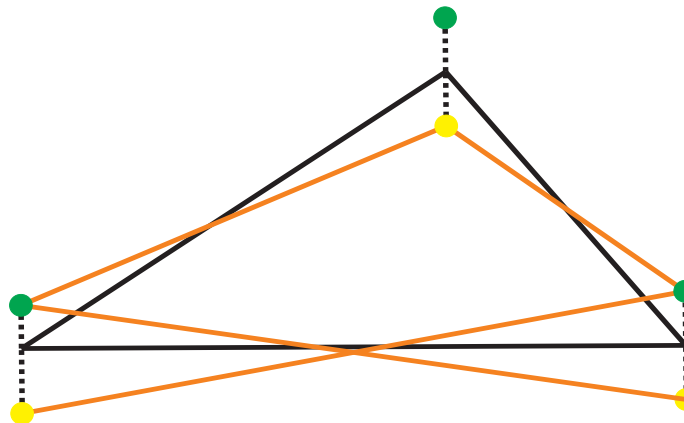
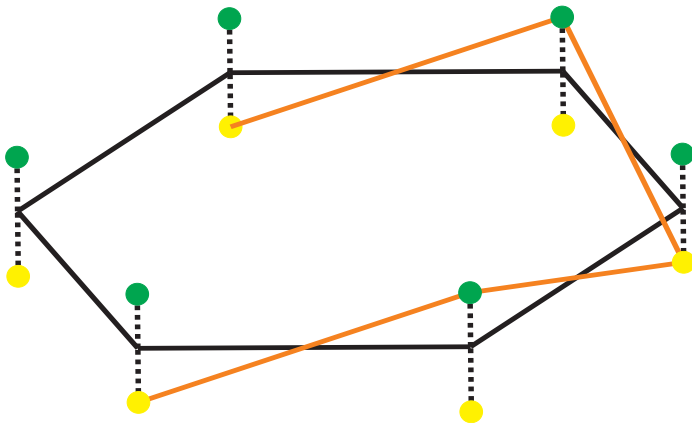
Combinatorial polyhedra

► polygon \longrightarrow 2-regular connected graph



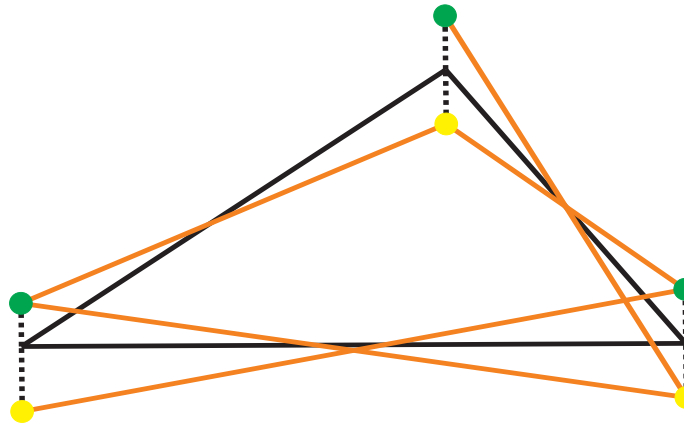
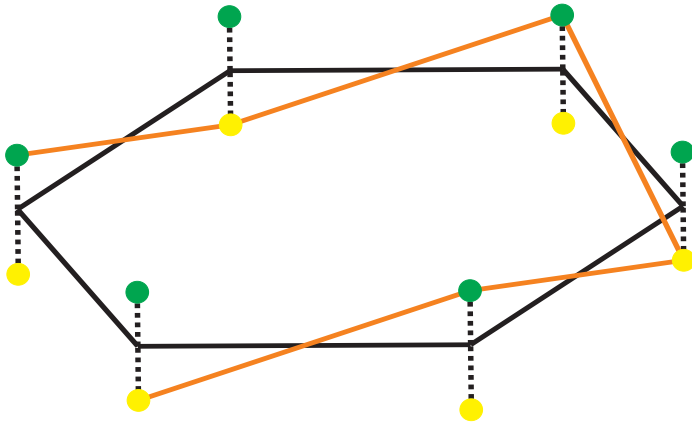
Combinatorial polyhedra

► polygon \longrightarrow 2-regular connected graph



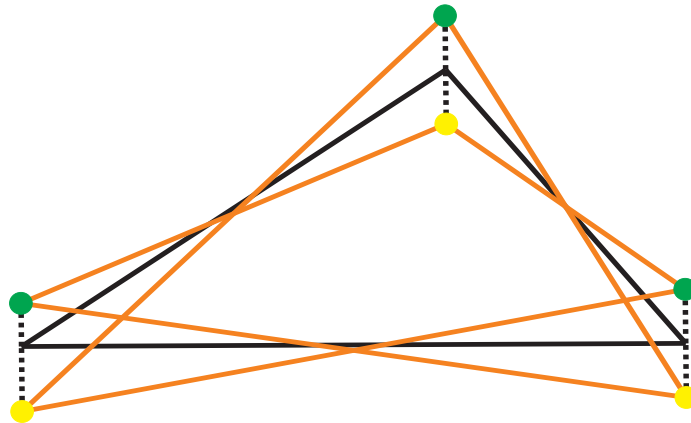
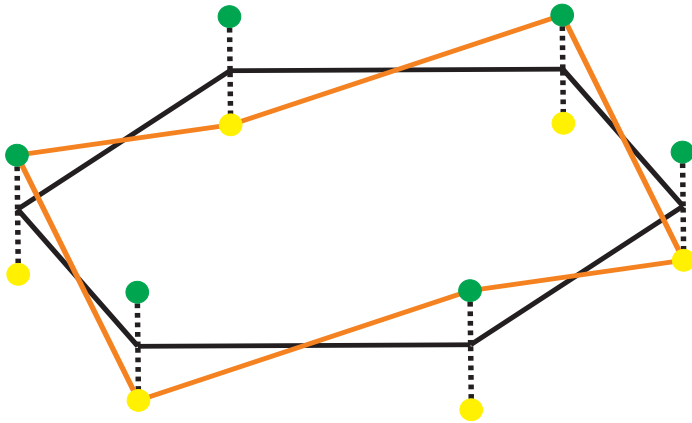
Combinatorial polyhedra

► polygon \longrightarrow 2-regular connected graph

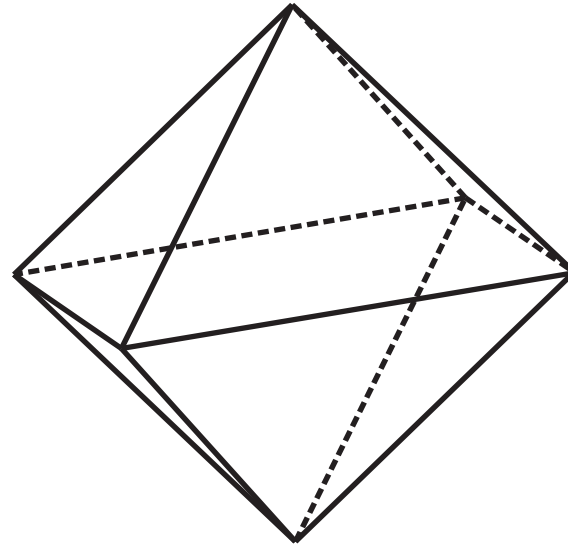
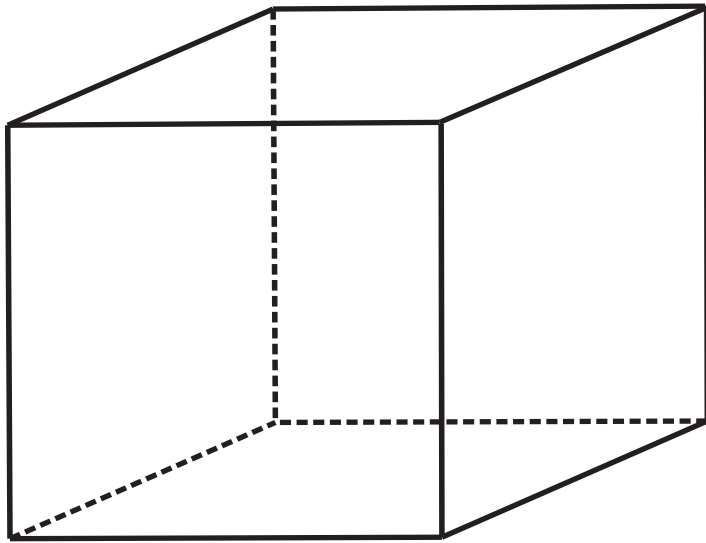


Combinatorial polyhedra

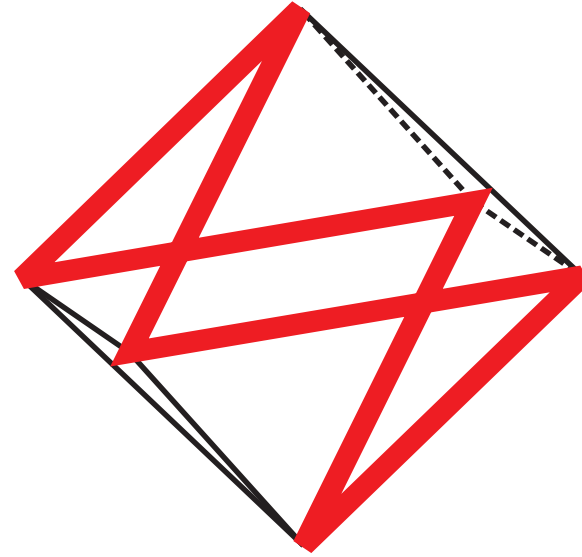
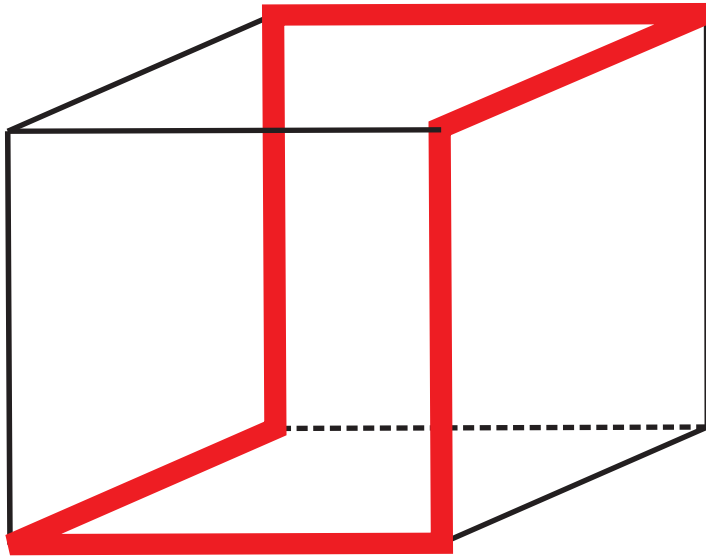
► polygon \longrightarrow 2-regular connected graph



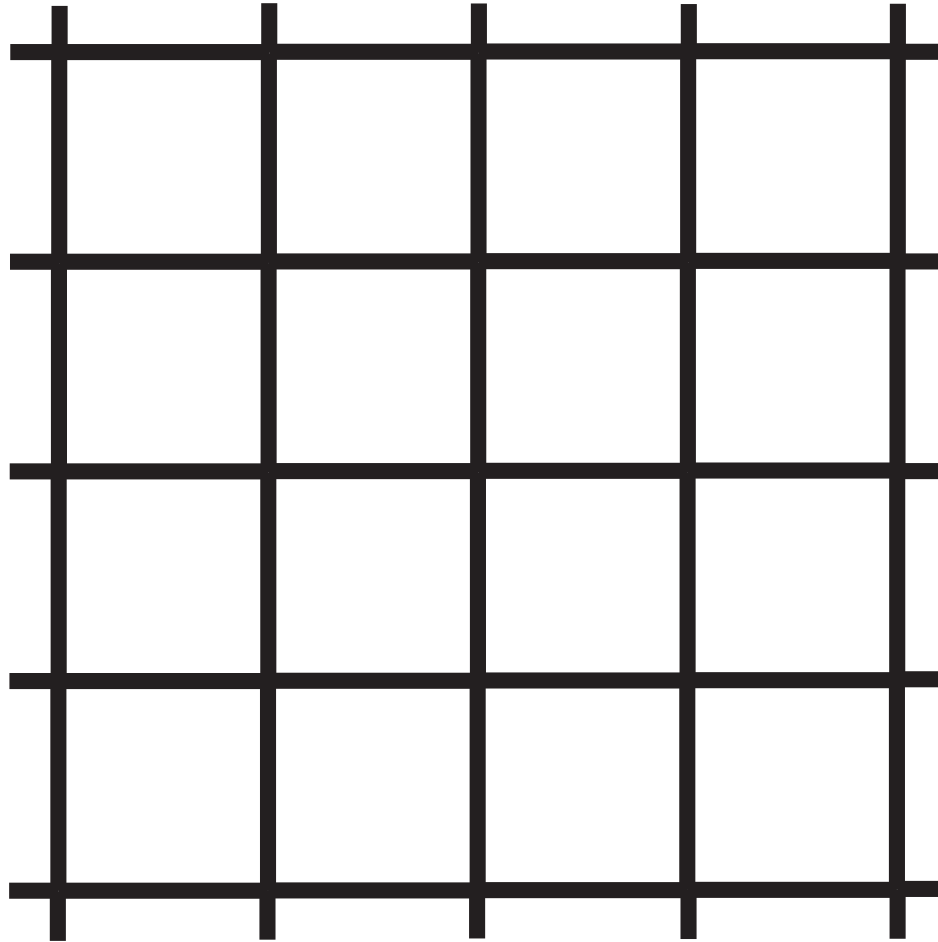
Combinatorial polyhedra



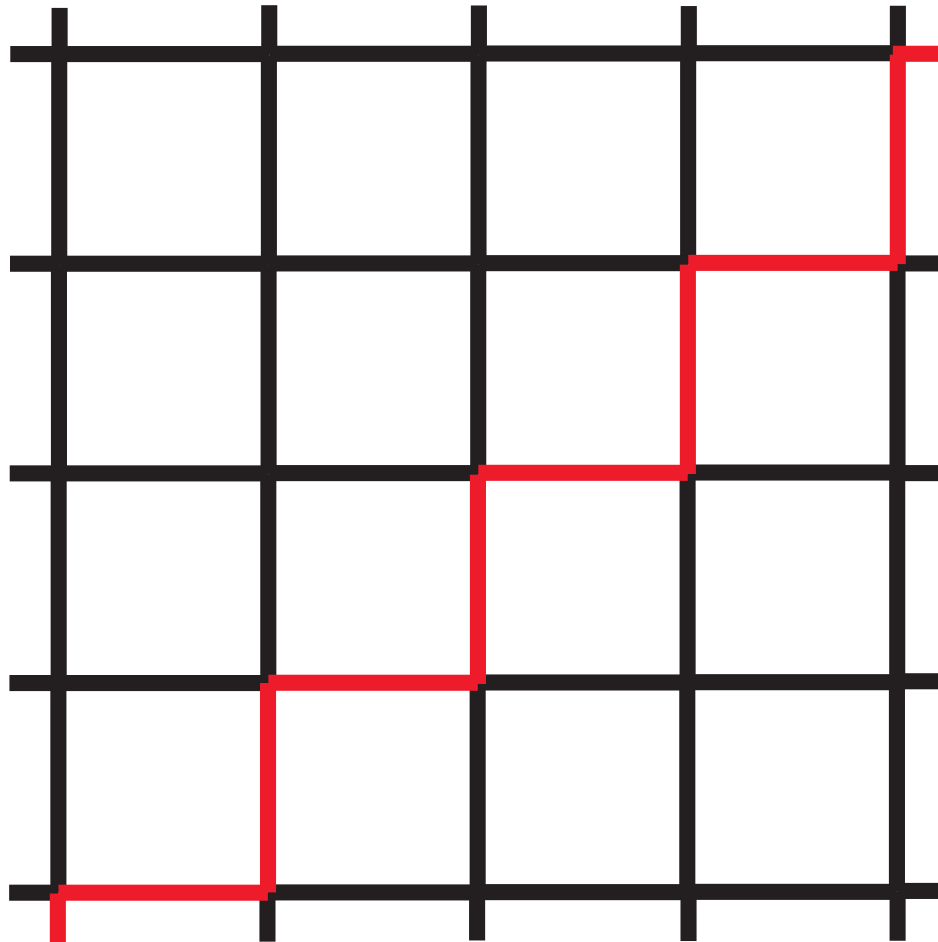
Combinatorial polyhedra



Planar polyhedra



Planar polyhedra



Regular polyhedra in the Euclidean space

Regular polyhedra in the Euclidean space

- 18 finite

Regular polyhedra in the Euclidean space

- 18 finite
- 6 infinite planar

Regular polyhedra in the Euclidean space

- 18 finite
- 6 infinite planar
- 24 infinite nonplanar

Regular polyhedra in the Euclidean space

- 18 finite
- 6 infinite planar
- 24 infinite nonplanar

- 48 in total

Polygons



► polygon \longrightarrow 2-regular connected graph

Polygons

- ▶ polygon \longrightarrow 2-regular connected graph
- ▶ regular \longrightarrow isometry group is dihedral on vertices

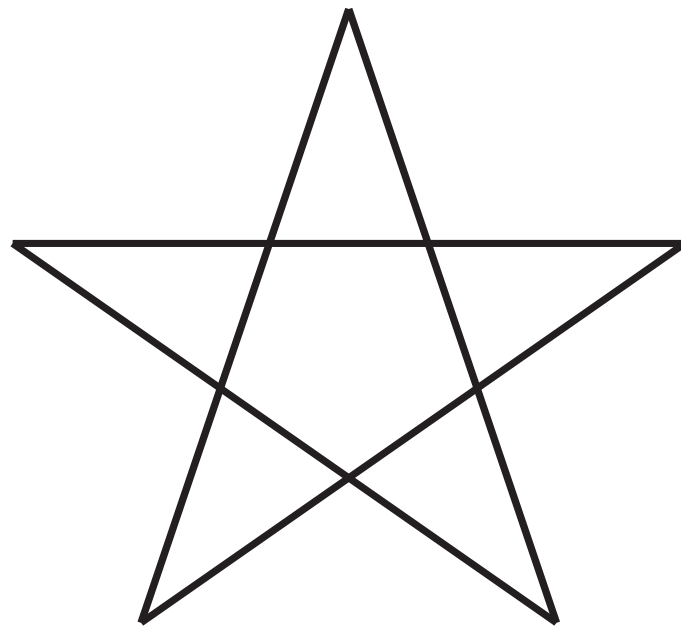
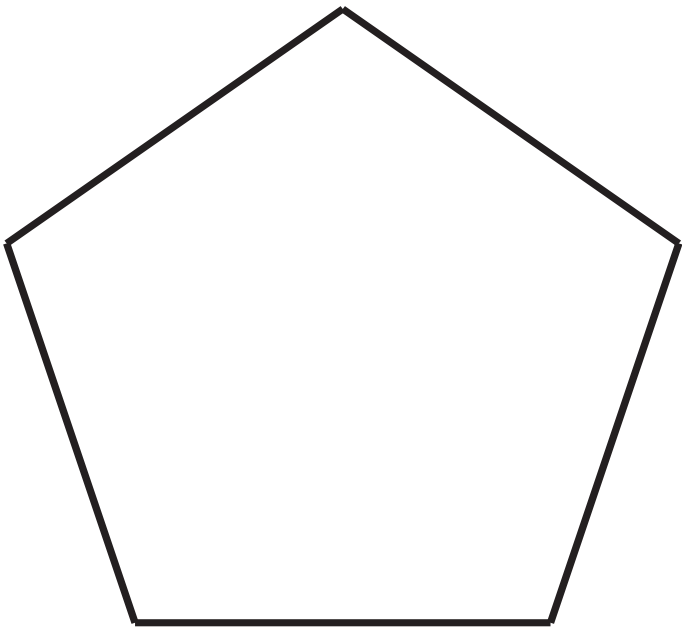
Polygons

A decorative graphic in the top left corner consisting of an orange triangle pointing right, followed by a thick black horizontal bar, and then a thinner grey horizontal bar.

Finite planar

Polygons

Finite planar



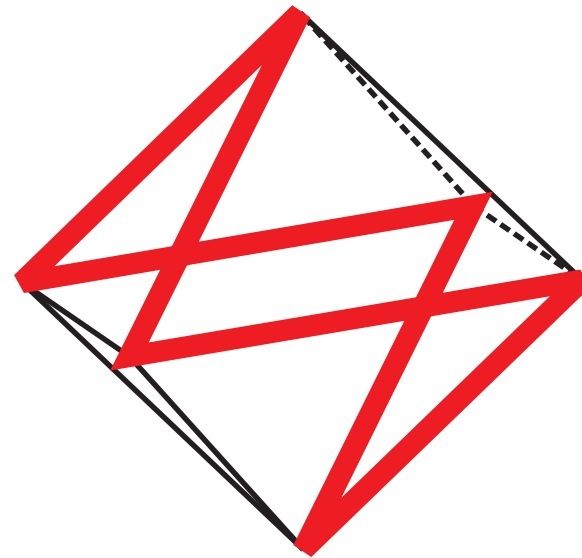
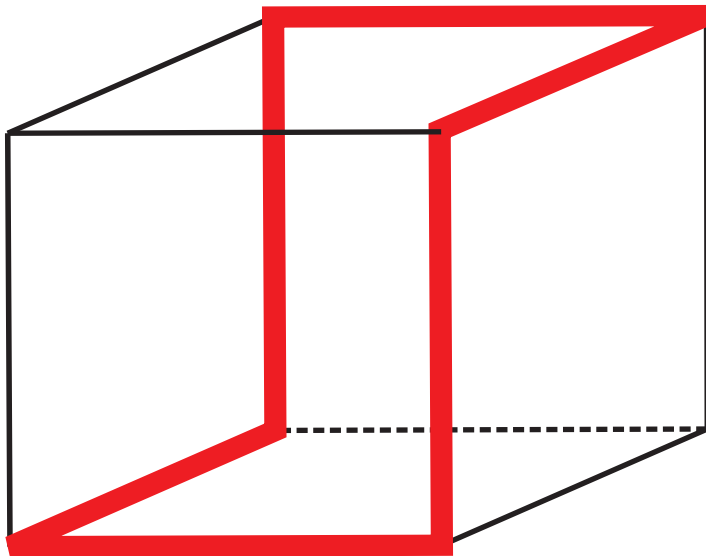
Polygons



Finite skew

Polygons

Finite skew



Polygons



Infinite planar

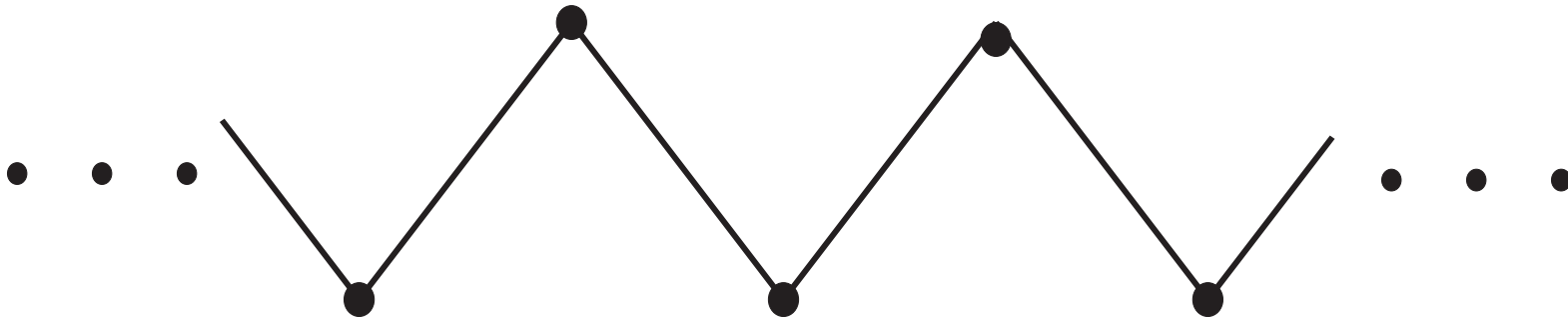
Polygons

Infinite planar



Polygons

Infinite planar



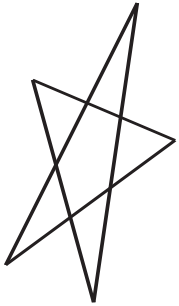
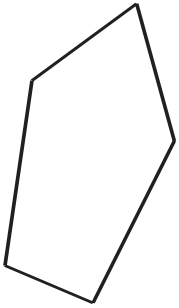
Polygons



Infinite helical

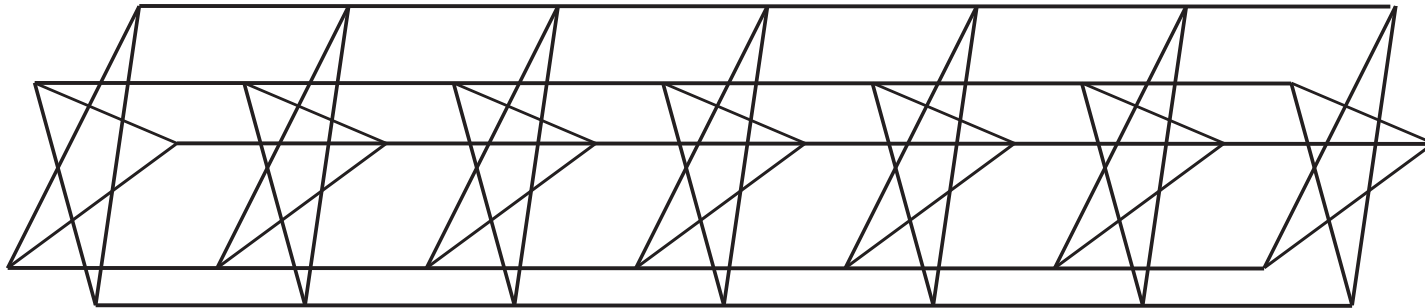
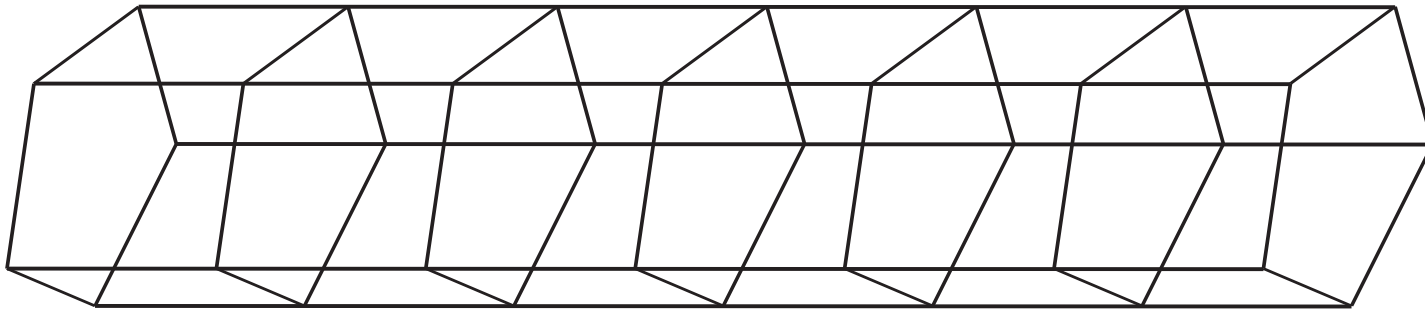
Polygons

Infinite helical



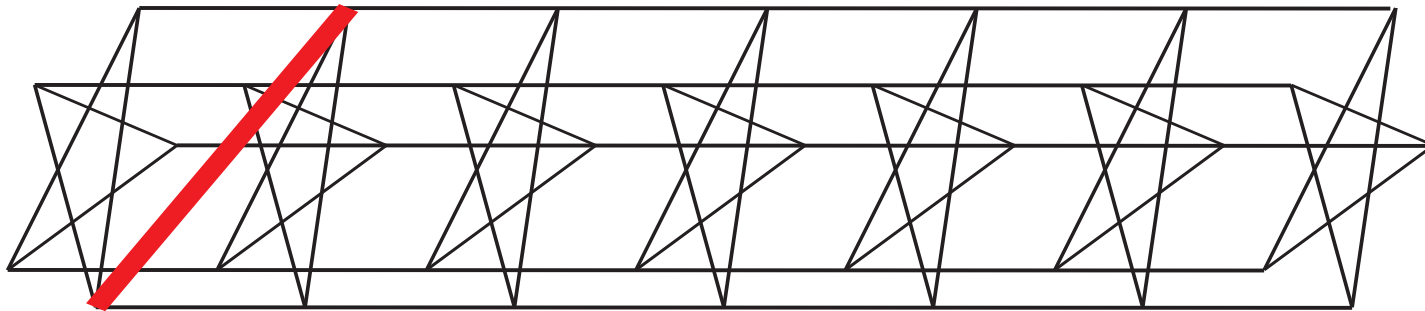
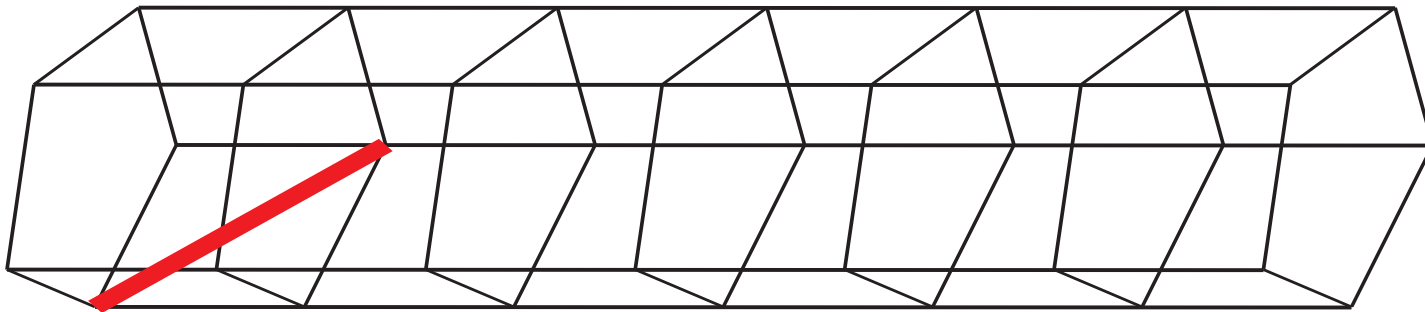
Polygons

Infinite helical



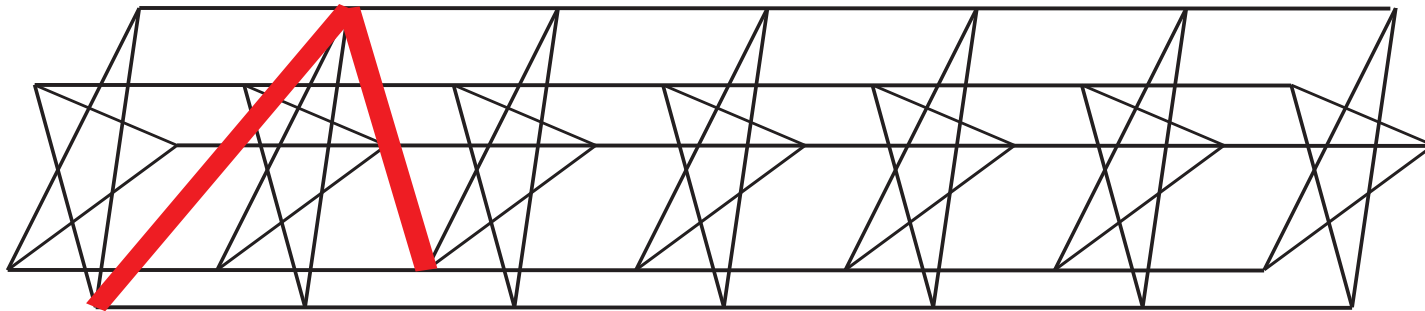
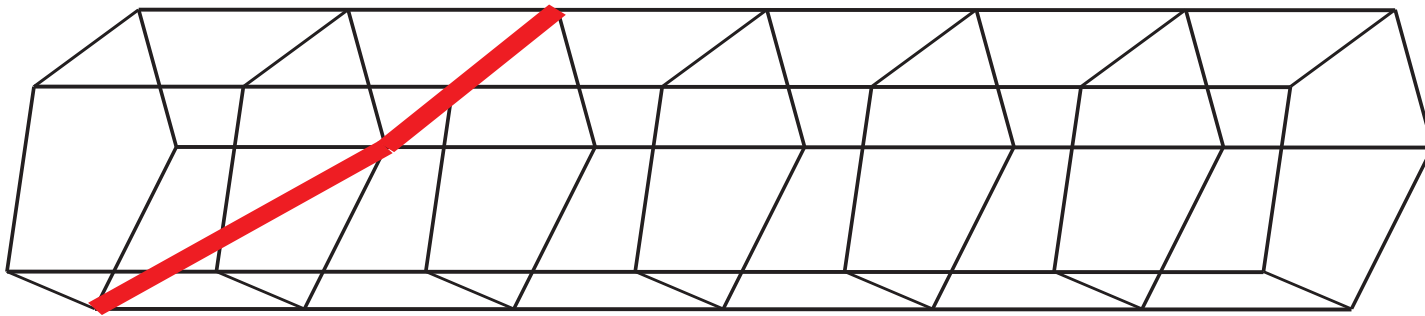
Polygons

Infinite helical



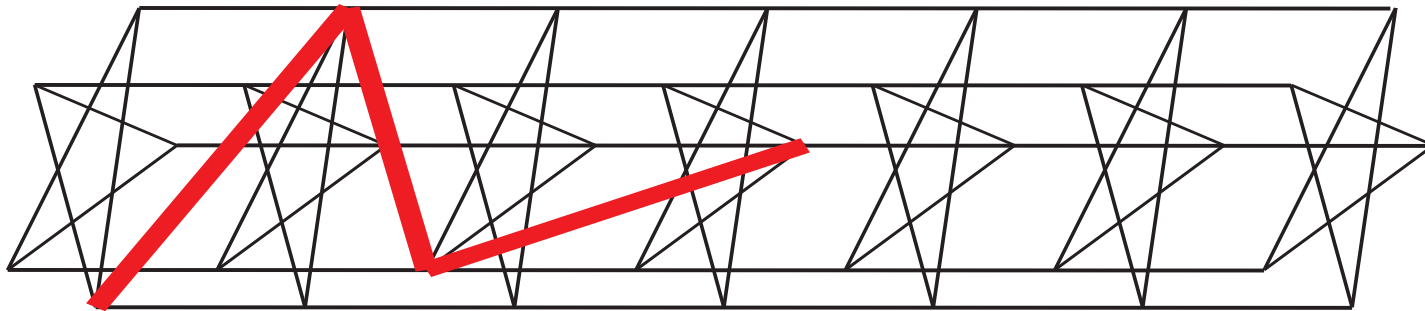
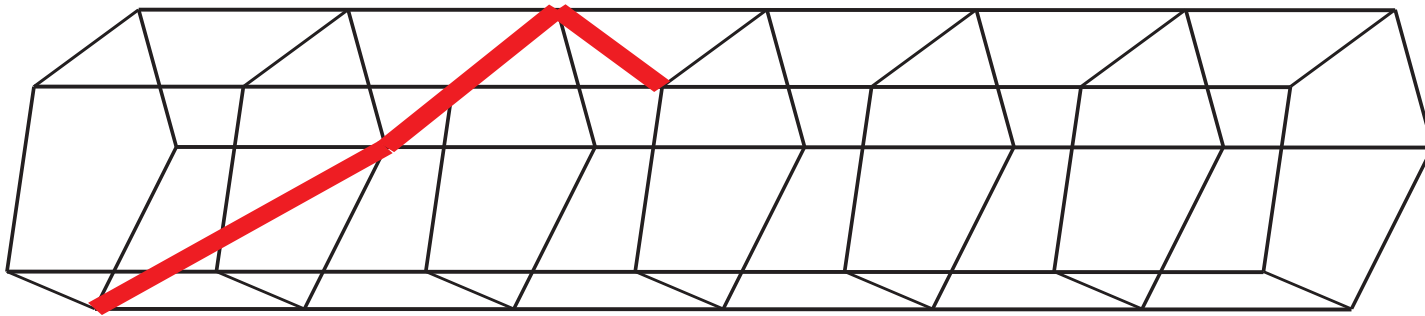
Polygons

Infinite helical



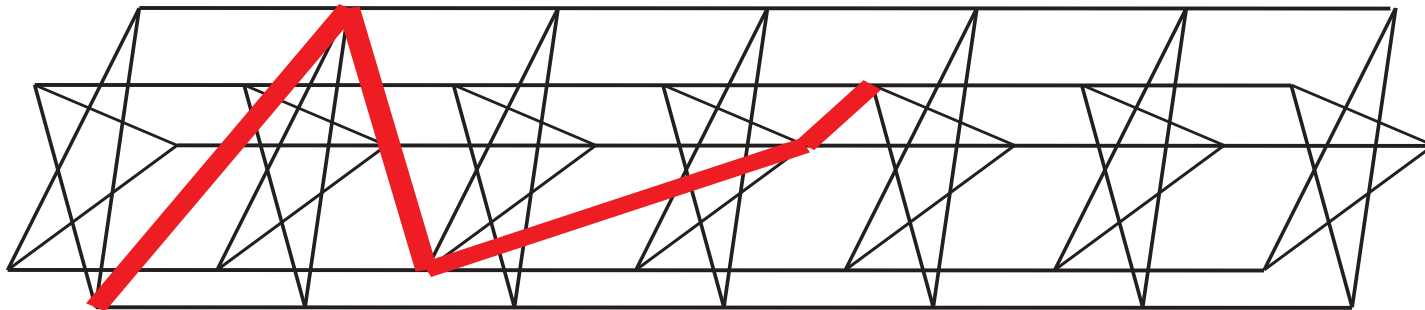
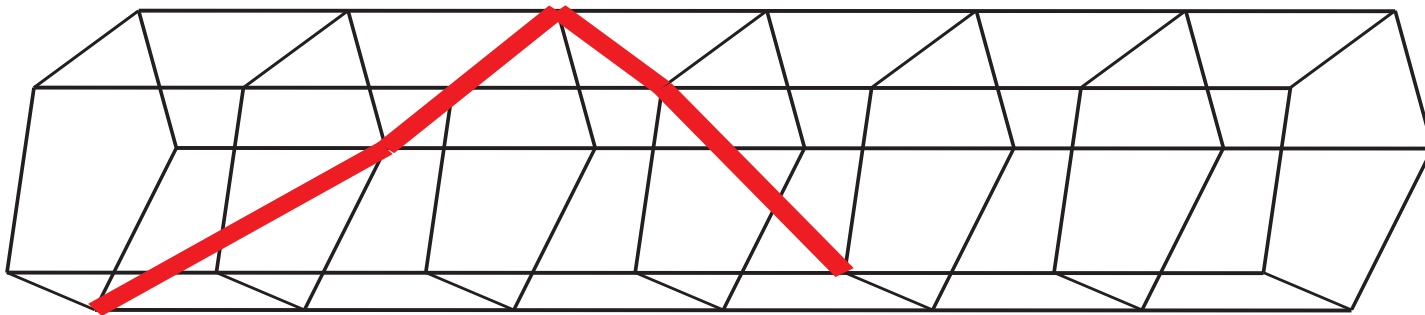
Polygons

Infinite helical



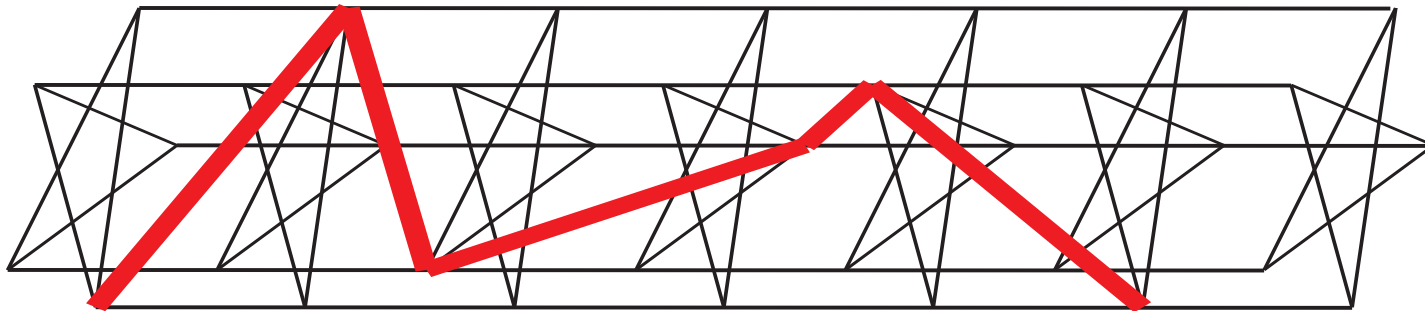
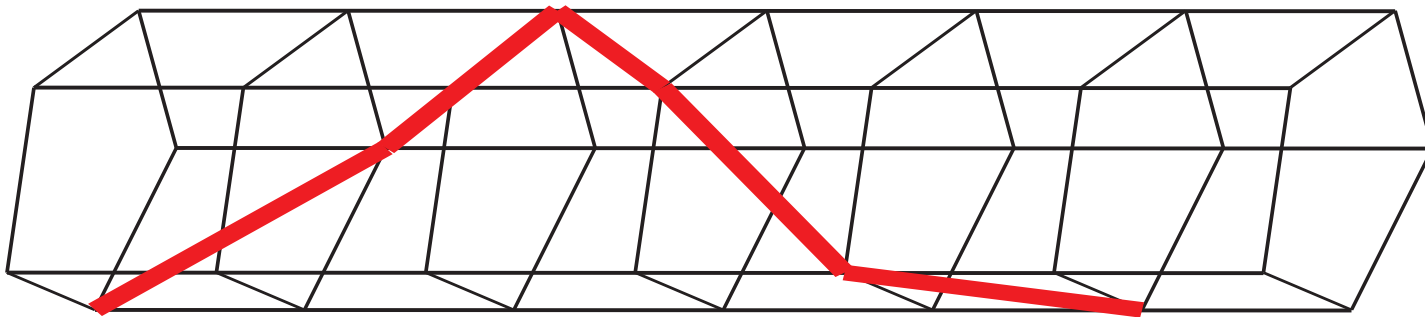
Polygons

Infinite helical



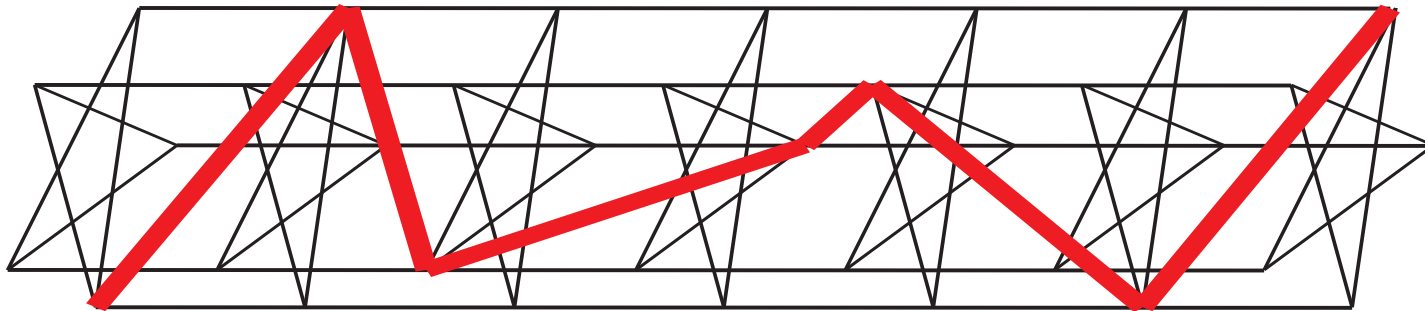
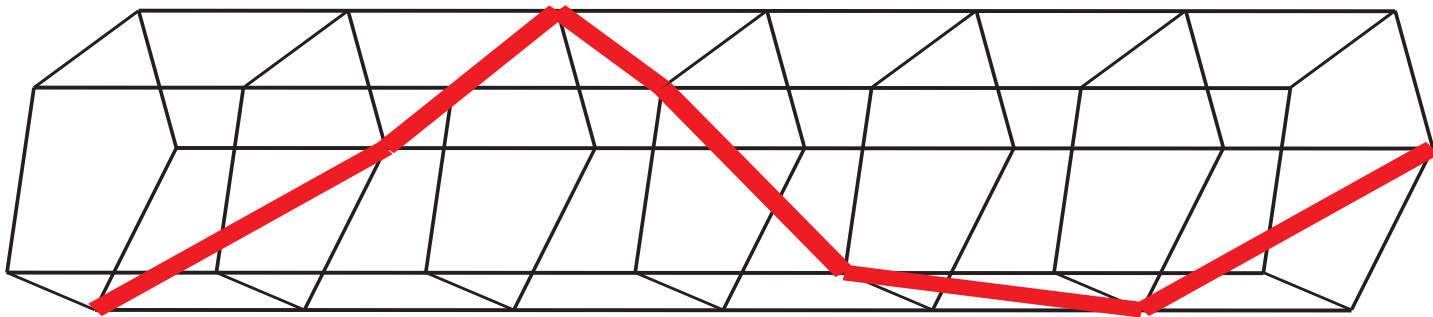
Polygons

Infinite helical



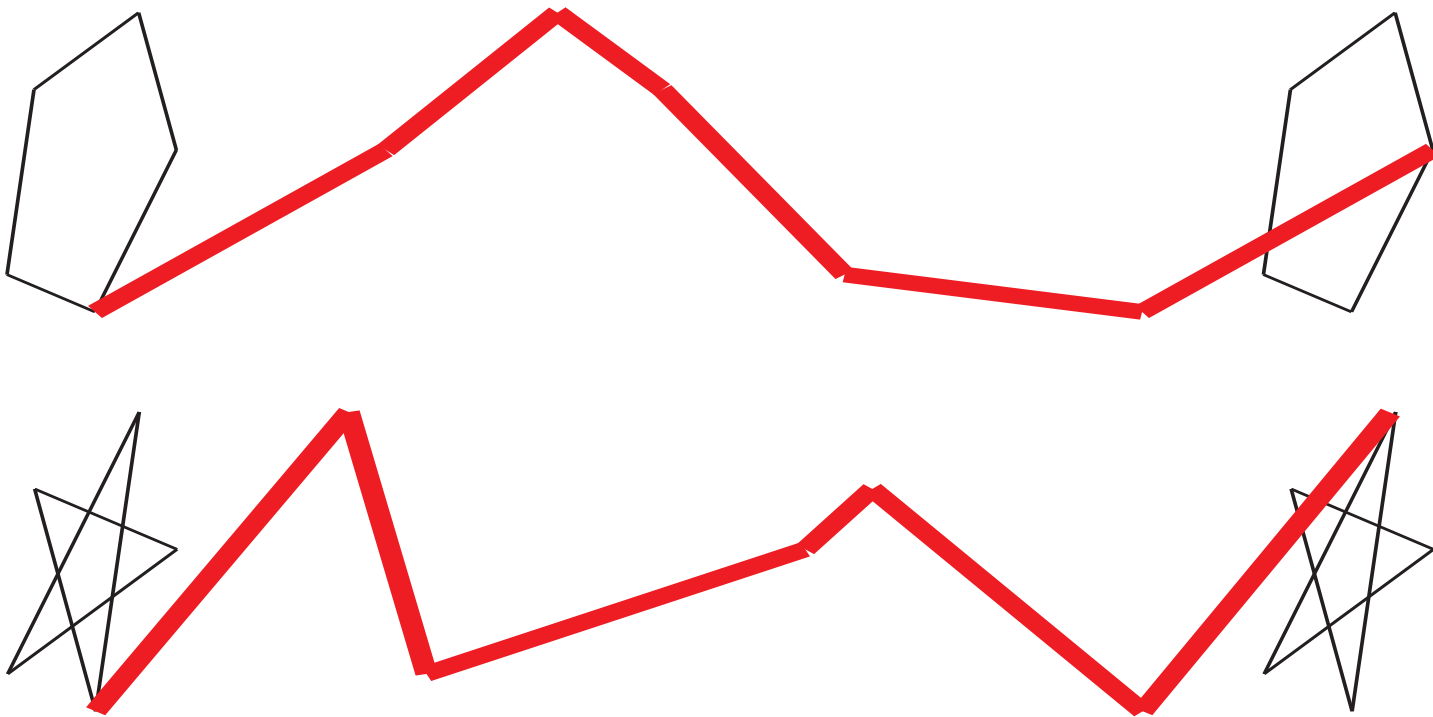
Polygons

Infinite helical



Polygons

Infinite helical



Regular polyhedra

Three horizontal lines of varying colors (yellow, black, grey) and thicknesses extending across the top of the slide.

Polyhedron

Regular polyhedra

A series of three horizontal bars: a thick yellow bar, a medium grey bar, and a thin black bar, all extending from the left edge of the slide.

Polyhedron

- ▶ Set of vertices, edges, polygons (faces)

Regular polyhedra

A decorative graphic consisting of three horizontal bars. The top bar is orange with a black border on its left and bottom edges. The middle bar is grey. The bottom bar is black.

Polyhedron

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected

Regular polyhedra



Polyhedron

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely two faces

Regular polyhedra

Polyhedron

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely two faces
- ▶ Regular \longrightarrow symmetry group transitive on flags

Regular polyhedra

Polyhedron

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely two faces
- ▶ Regular \longrightarrow symmetry group transitive on flags (*vertex, edge, face*)

Regular polyhedra



- 18 finite

Regular polyhedra

- 18 finite
 - 7 with convex faces
 - 2 with star faces
 - 9 with skew faces

Regular polyhedra

- 18 finite
 - 7 with convex faces
 - 2 with star faces
 - 9 with skew faces
- 6 planar

Regular polyhedra

- 18 finite
 - 7 with convex faces
 - 2 with star faces
 - 9 with skew faces
- 6 planar
 - 3 with convex faces
 - 3 with zigzag faces

Regular polyhedra

Three horizontal bars of varying lengths and colors (yellow, grey, black) extending from the left side of the slide.

- 24 nonplanar infinite

Regular polyhedra

- 24 nonplanar infinite
 - 3 with planar convex faces
 - 6 with zigzag faces
 - 6 with finite skew faces
 - 9 with helical faces

Regular polyhedra

A decorative graphic consisting of three horizontal bars. The top bar is orange with a black outline, the middle bar is grey with a black outline, and the bottom bar is solid black.

Polyhedron

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely two faces

Regular polyhedra



Polyhedron

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely two faces ←

Regular polyhedra

Polyhedron

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely two faces ←
↑

Regular polyhedra

Polyhedron

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely two faces ←
↑ ↑

Regular polyhedra

Polyhedron

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely two faces ←
↑ ↑ ↑

Regular polyhedra

Polyhedron

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely two faces ←
↑ ↑ ↑ ↑

Regular polyhedra

Polyhedron

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely two faces ←
↑ ↑ ↑ ↑ ↑

Regular polyhedra

Polyhedron

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely two faces ←
↑ ↑ ↑ ↑ ↑ ↑

Regular polyhedra

Polyhedron

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely two faces ←
↑ ↑ ↑ ↑ ↑ ↑ ↑

Regular polyhedra

Polyhedron

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely two faces ←
↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

Polygonal complexes

Polygonal complex

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely k faces

Polygonal complexes

Polygonal complex

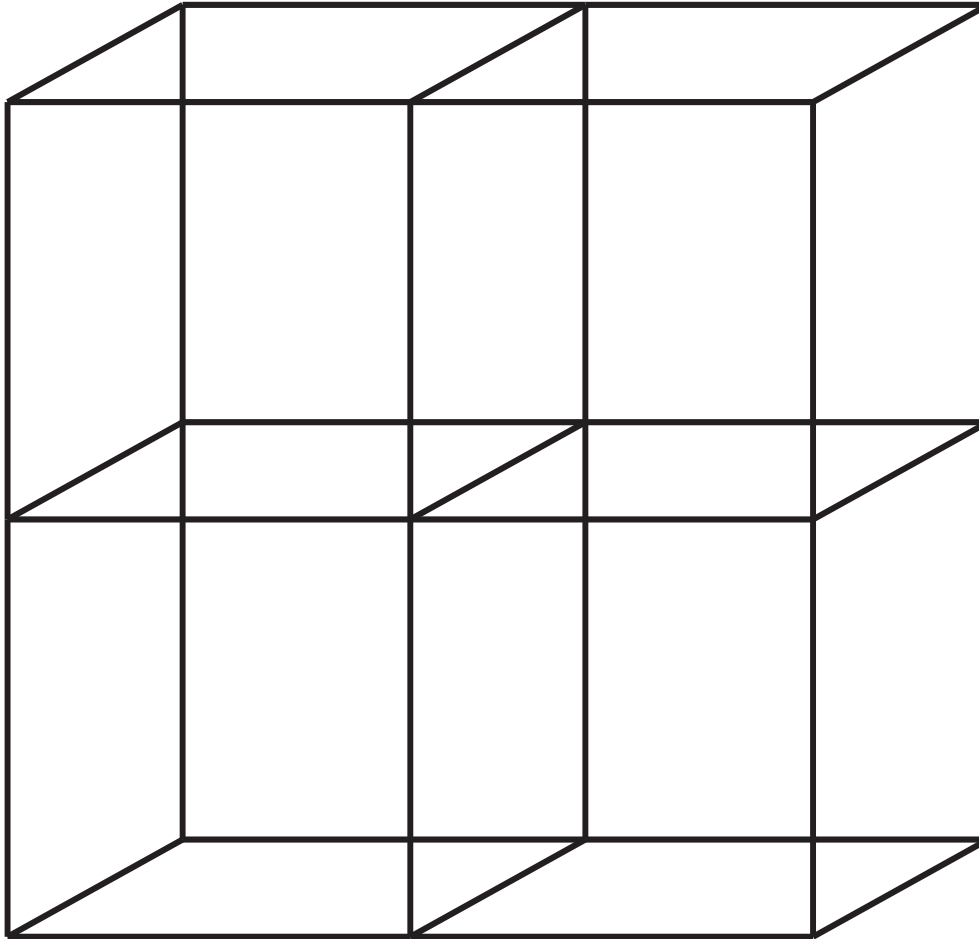
- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely k faces
($k = 2 \longrightarrow$ polyhedron)

Polygonal complexes

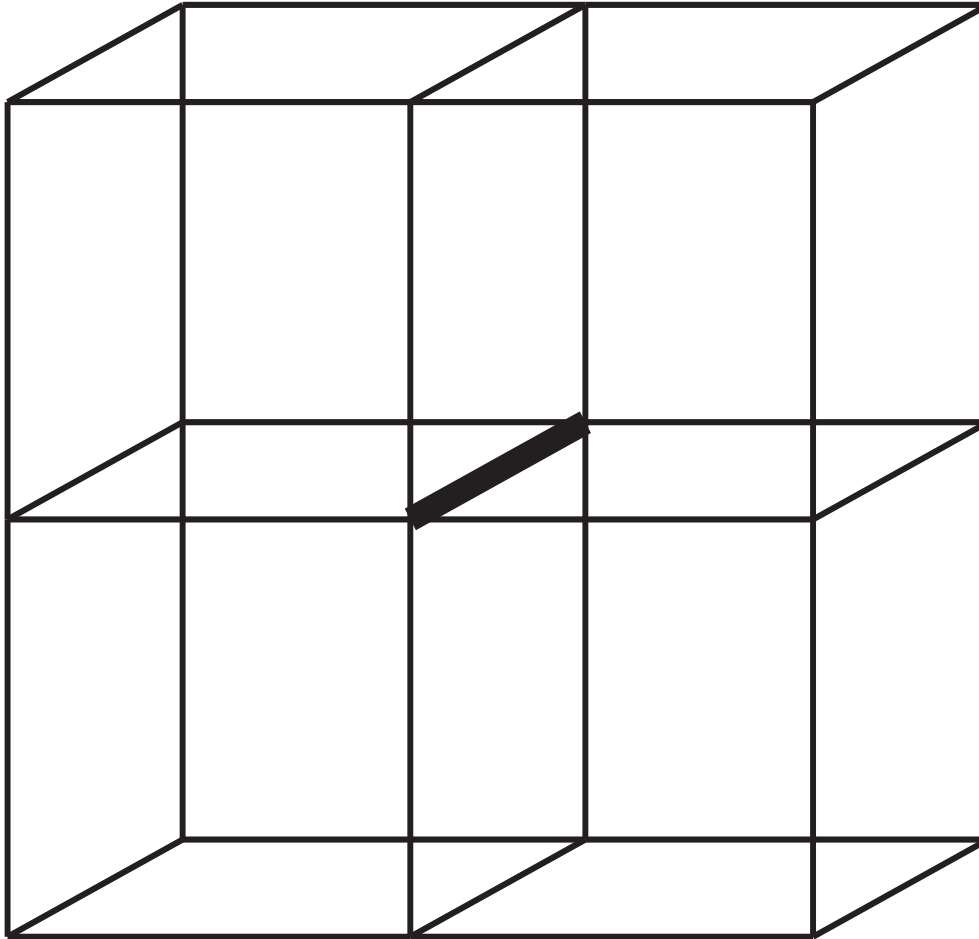
Polygonal complex

- ▶ Set of vertices, edges, polygons (faces)
- ▶ Discrete, connected
- ▶ Every edge belongs to precisely k faces
($k = 2 \longrightarrow$ polyhedron)
- ▶ Regular \longrightarrow symmetry group transitive on flags

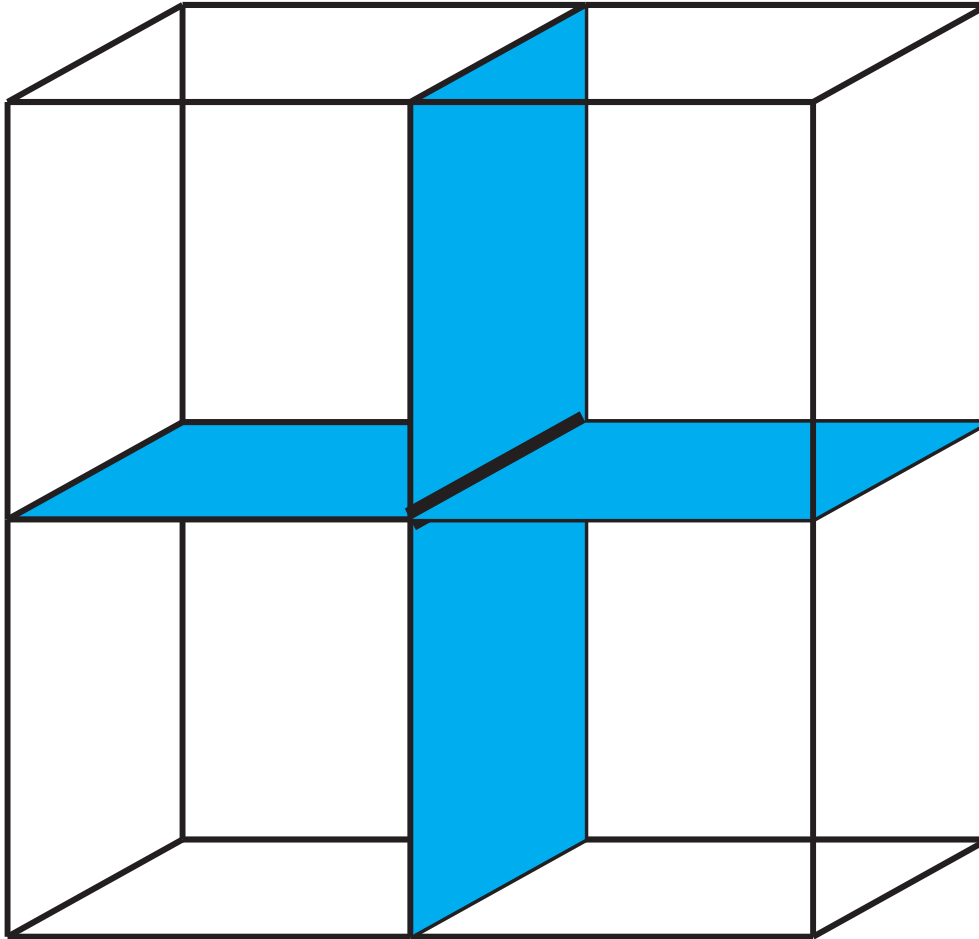
Polygonal complexes



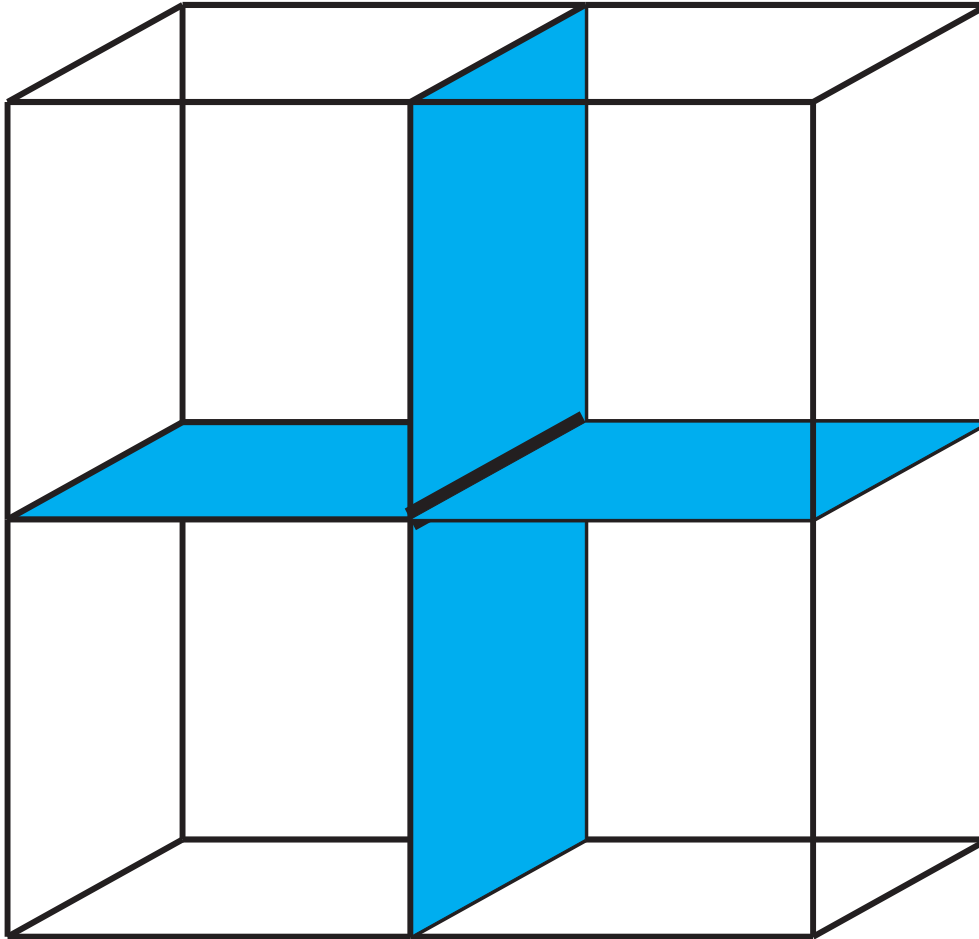
Polygonal complexes



Polygonal complexes

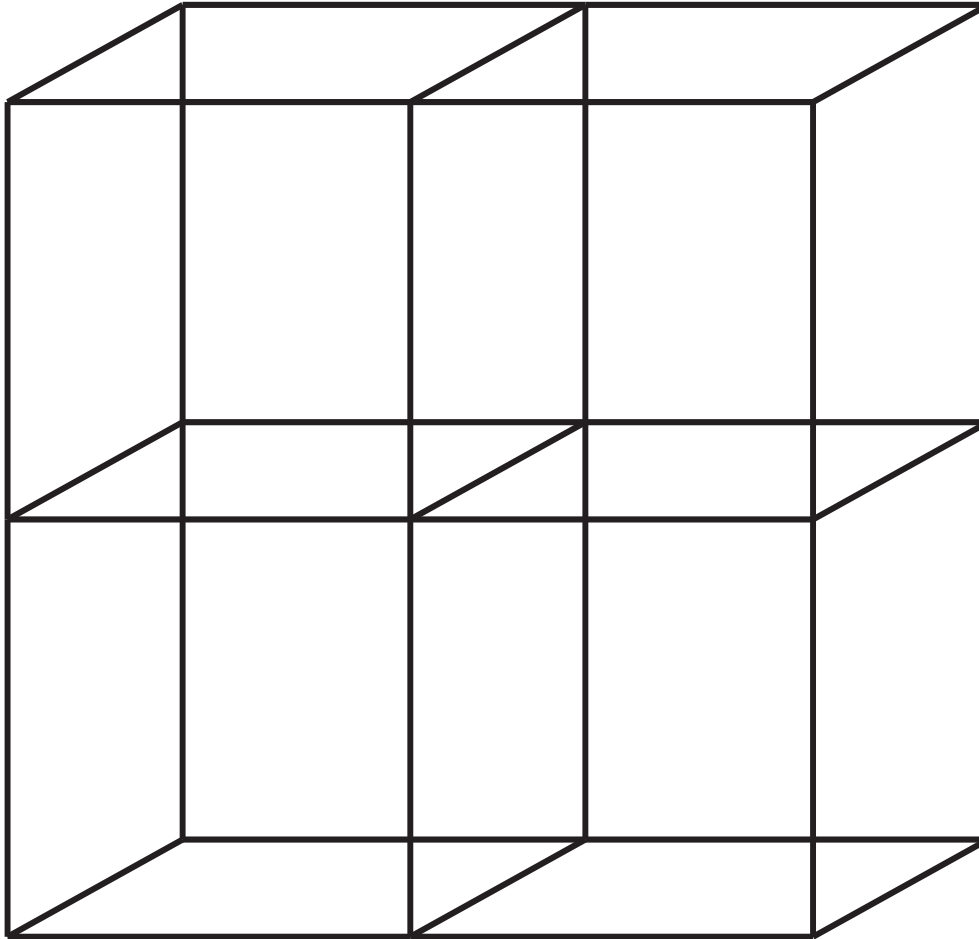


Polygonal complexes



$$k = 4$$

Polygonal complexes



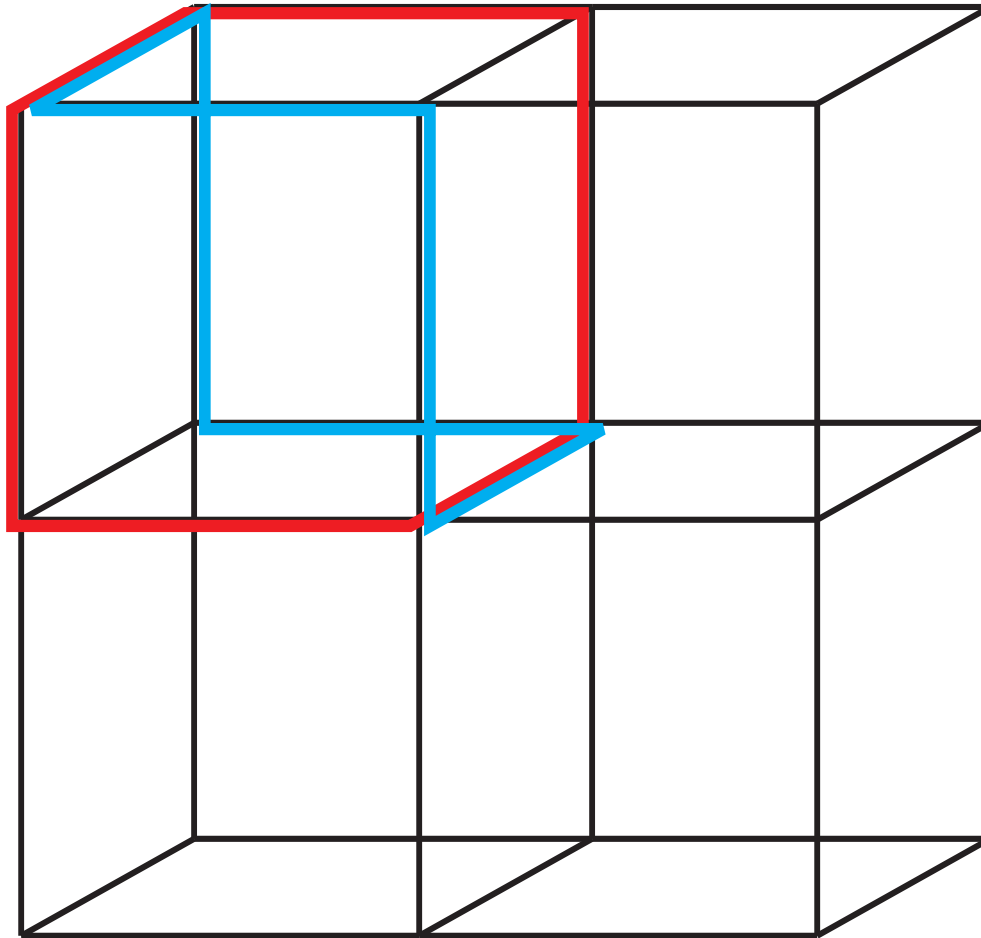
Polygonal complexes



Polygonal complexes



Polygonal complexes



$$k = 8$$

Polygonal complexes

A decorative graphic consisting of three horizontal bars. The top bar is black and has a yellow-to-orange gradient background that tapers to the right. The middle bar is gray, and the bottom bar is black.

Theorem

All finite regular polygonal complexes in space are polyhedra.

Polygonal complexes

Theorem

All finite regular polygonal complexes in space are polyhedra.

Theorem

There are no regular polygonal complexes in space with affinely irreducible group of symmetries, except for polyhedra.

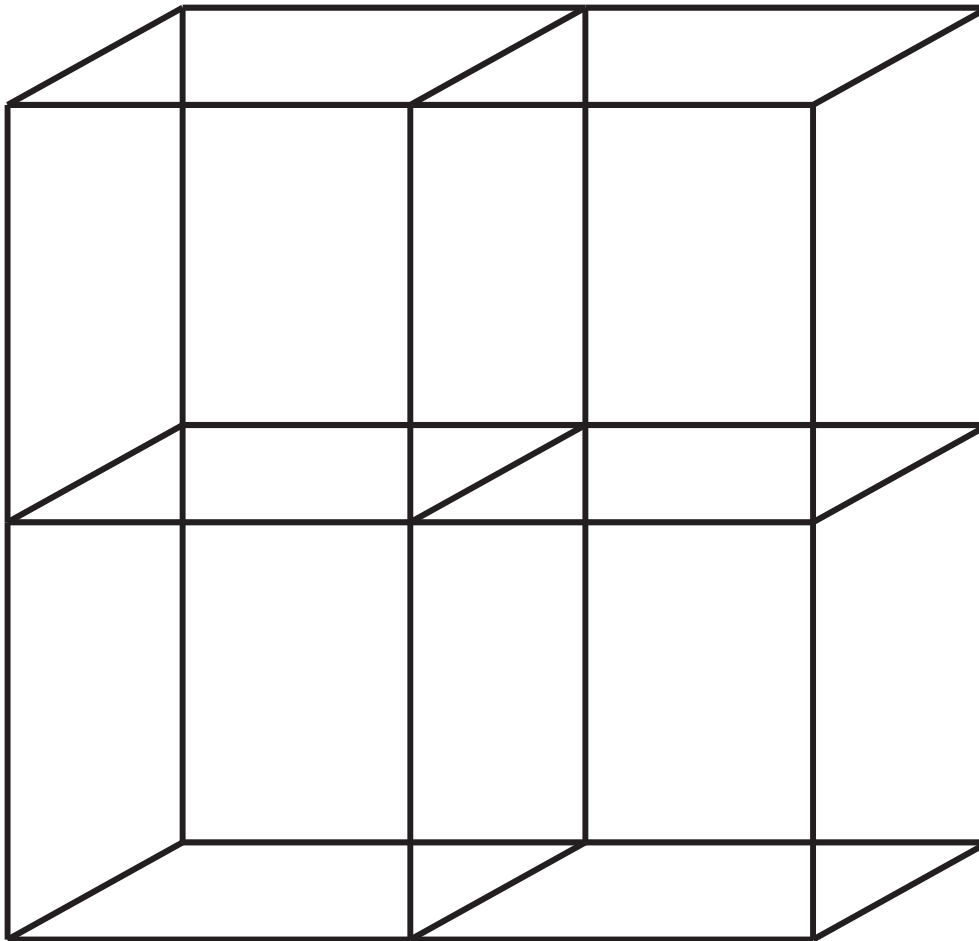
flags



► Flag \longrightarrow triple of incident vertex, edge and face

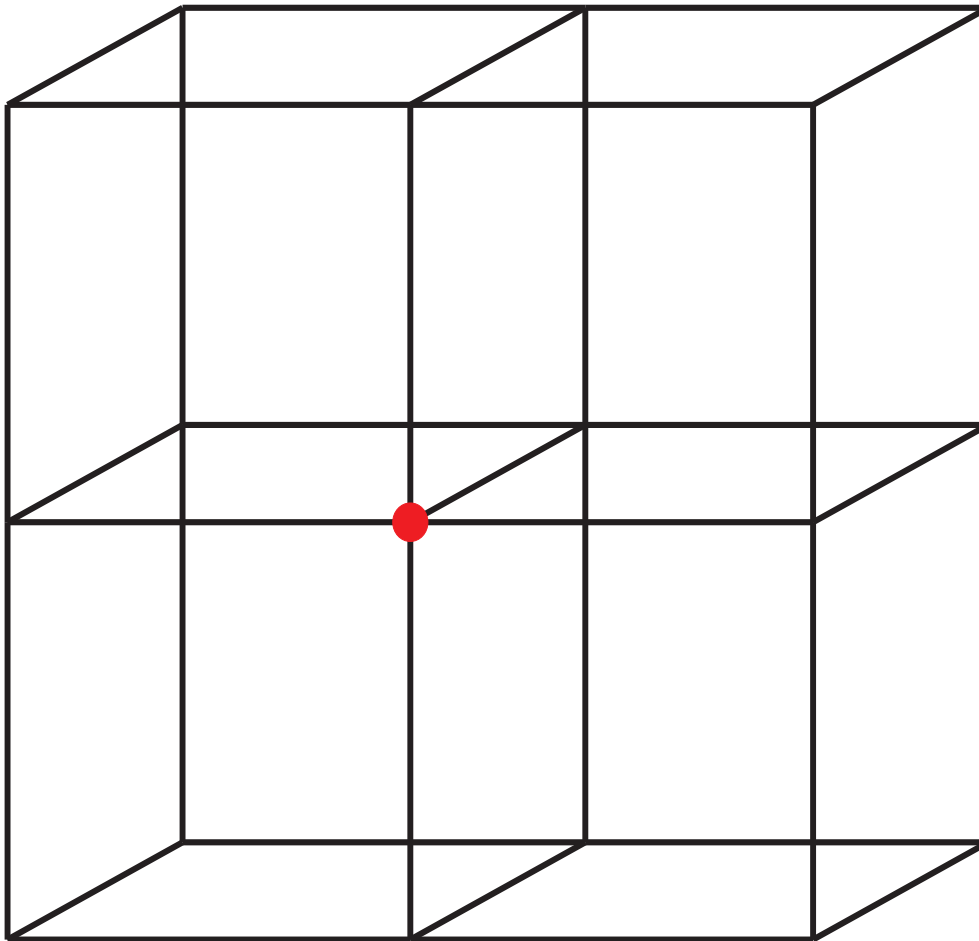
flags

► Flag \longrightarrow triple of incident vertex, edge and face



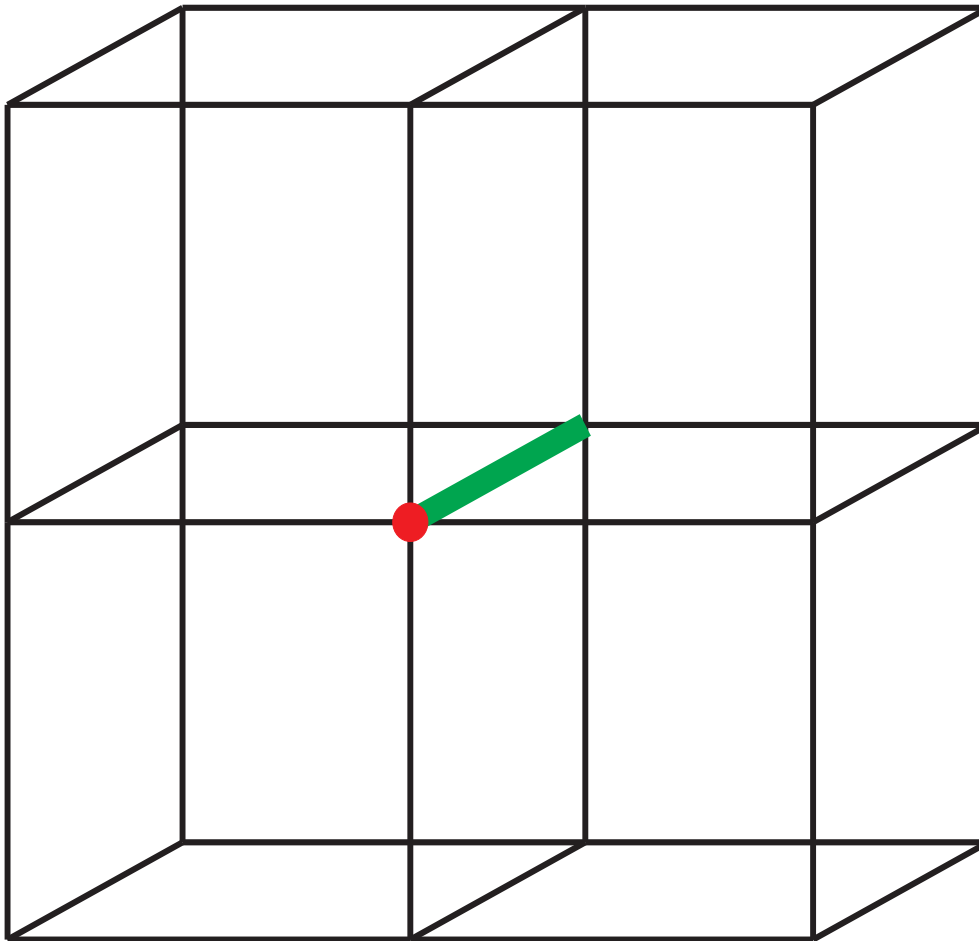
flags

► Flag \longrightarrow triple of incident vertex, edge and face



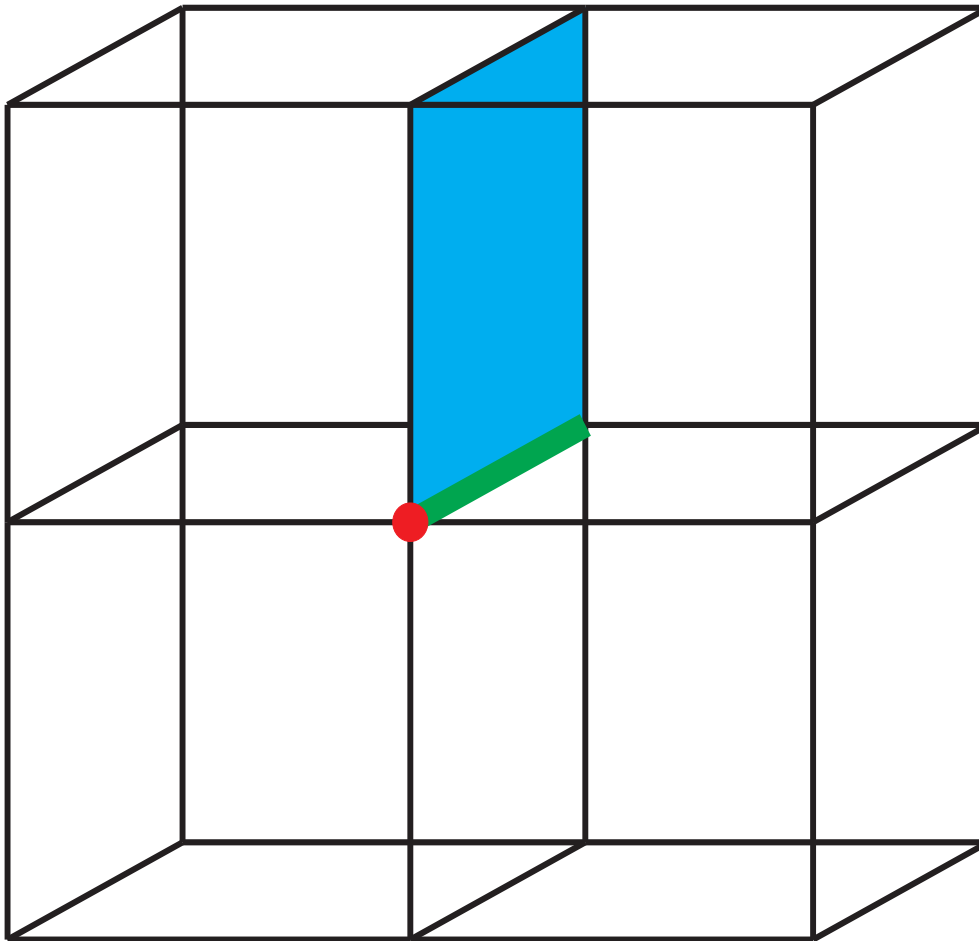
flags

► Flag \longrightarrow triple of incident vertex, edge and face



flags

► Flag \longrightarrow triple of incident vertex, edge and face



Flag stabilizer

Three horizontal lines of varying colors (yellow, black, grey) and lengths extending from the left edge of the slide.

Planar faces

Flag stabilizer

Three horizontal bars of varying lengths and colors (yellow, black, grey) extending from the left side of the slide.

Planar faces

- Trivial stabilizer

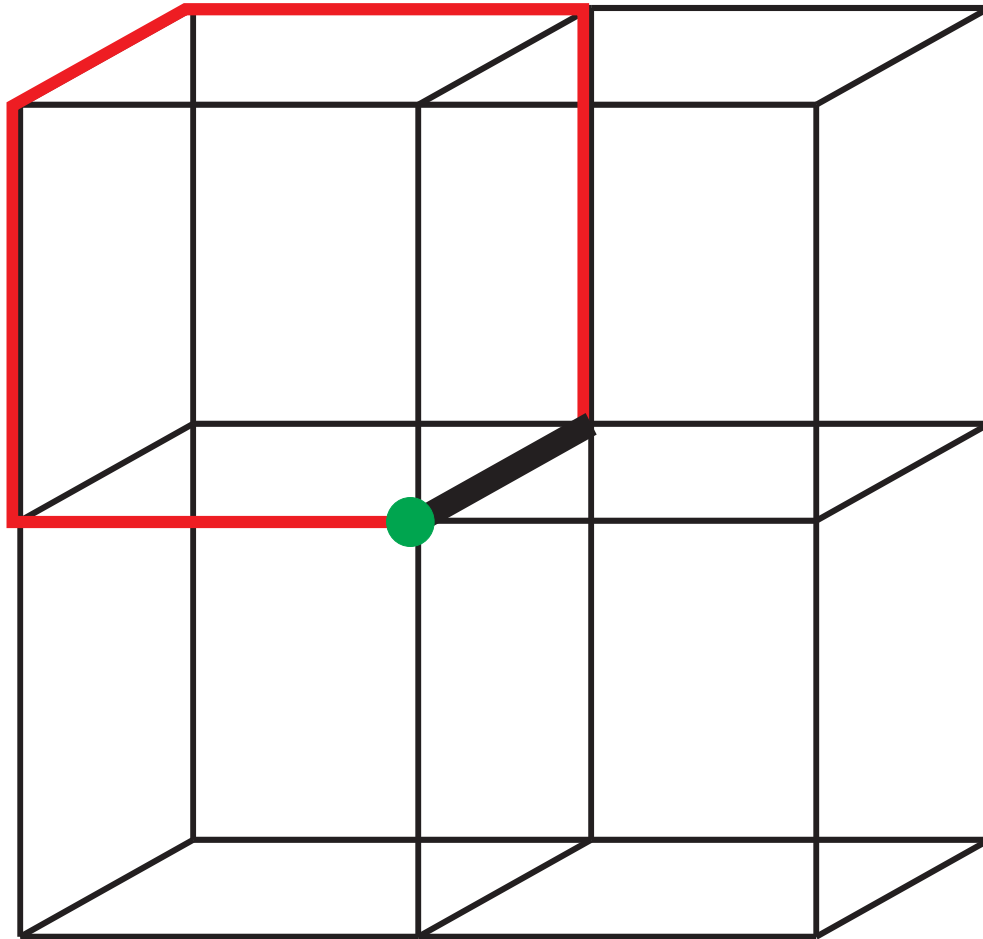
Flag stabilizer

A decorative graphic consisting of three horizontal bars. The top bar is yellow with a black outline and a black shadow. The middle bar is gray with a black outline and a black shadow. The bottom bar is black with a black outline and a black shadow.

Planar faces

- Trivial stabilizer
- Plane reflection

Flag stabilizer



Flag stabilizer

A decorative graphic consisting of three horizontal bars. The top bar is yellow with a black outline and a black shadow. The middle bar is gray with a black outline and a black shadow. The bottom bar is black with a black outline and a black shadow.

Planar faces

- Trivial stabilizer
- Plane relection

Flag stabilizer

A decorative graphic consisting of three horizontal bars. The top bar is yellow with a black outline and a black shadow. The middle bar is a solid black line. The bottom bar is a solid grey line.

Planar faces

- Trivial stabilizer
- Plane relection

Non planar faces

Flag stabilizer

A decorative graphic consisting of three horizontal bars. The top bar is yellow with a black outline and a black shadow. The middle bar is a solid black line. The bottom bar is a solid grey line.

Planar faces

- Trivial stabilizer
- Plane relection

Non planar faces

- Trivial stabilizer

Flag stabilizer

Three horizontal lines of varying colors (yellow, black, grey) and lengths extending from the left edge of the slide.

Theorem

Flag stabilizer

Three horizontal bars of varying lengths and colors (yellow, black, grey) extending from the left side of the slide.

Theorem

Every regular polygonal complex with nontrivial flag stabilizer is the 2-skeleton of a regular 4-polytope in space.

Flag stabilizer



Theorem

Every regular polygonal complex with nontrivial flag stabilizer is the 2-skeleton of a regular 4-polytope in space.

There are 4 instances

Group structure

► $G_0 \longrightarrow$ stabilizes edge and face,

Group structure

- ▶ $G_0 \longrightarrow$ stabilizes edge and face,
(2 elements)

Group structure

- ▶ $G_0 \longrightarrow$ stabilizes edge and face,
(2 elements)
- ▶ $G_1 \longrightarrow$ stabilizes vertex and face,

Group structure

- ▶ $G_0 \longrightarrow$ stabilizes edge and face,
(2 elements)
- ▶ $G_1 \longrightarrow$ stabilizes vertex and face,
(2 elements)

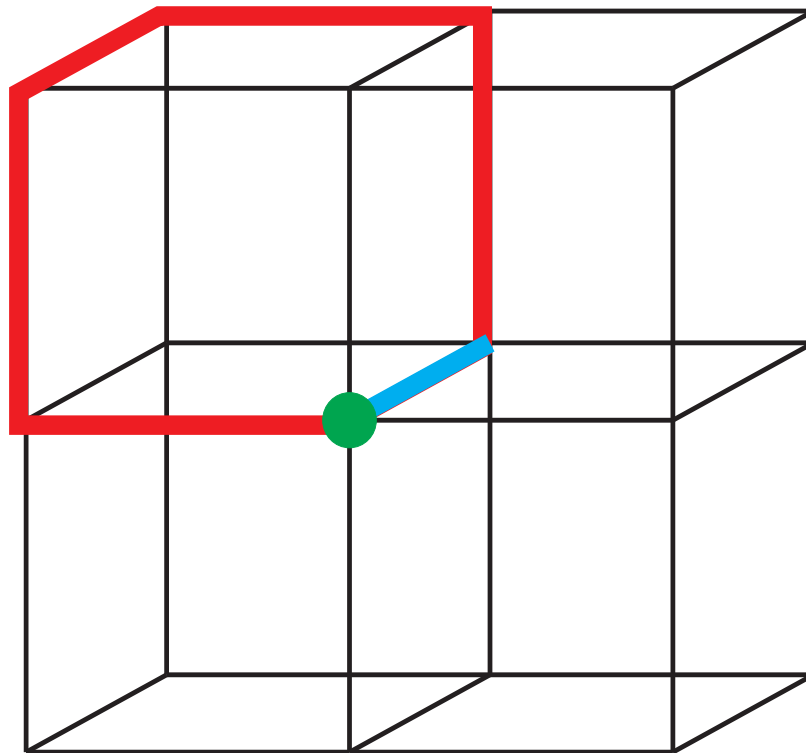
Group structure

- ▶ $G_0 \longrightarrow$ stabilizes edge and face,
(2 elements)
- ▶ $G_1 \longrightarrow$ stabilizes vertex and face,
(2 elements)
- ▶ $G_2 \longrightarrow$ stabilizes vertex and edge,

Group structure

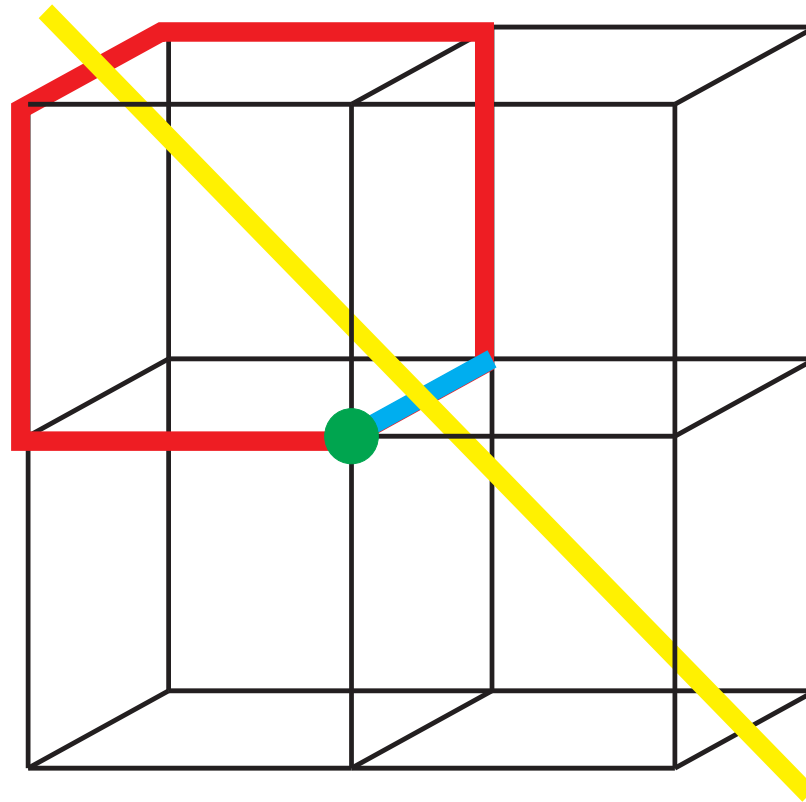
- ▶ $G_0 \longrightarrow$ stabilizes edge and face,
(2 elements)
- ▶ $G_1 \longrightarrow$ stabilizes vertex and face,
(2 elements)
- ▶ $G_2 \longrightarrow$ stabilizes vertex and edge,
(cyclic or dihedral)

Group structure



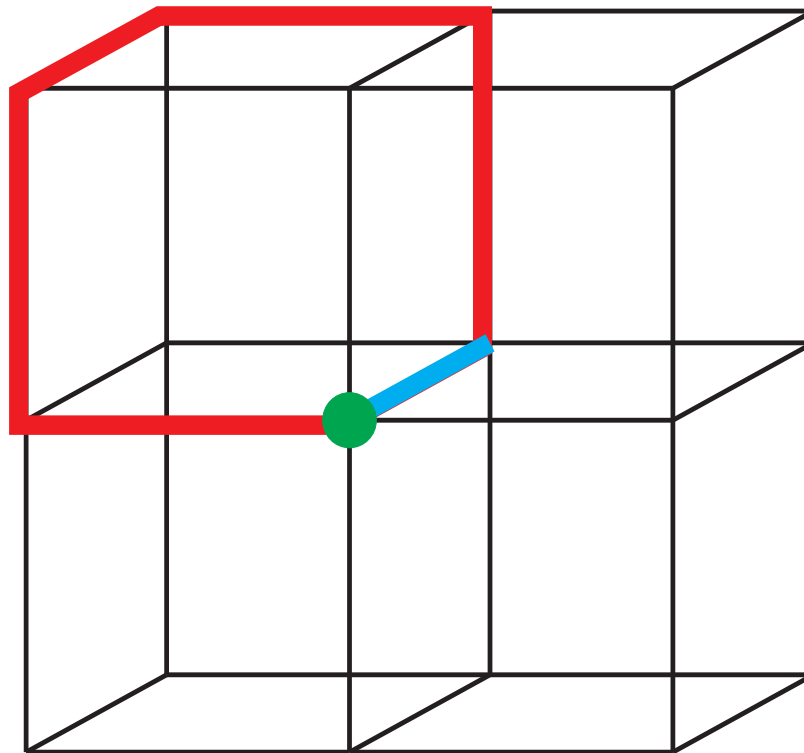
G_0

Group structure



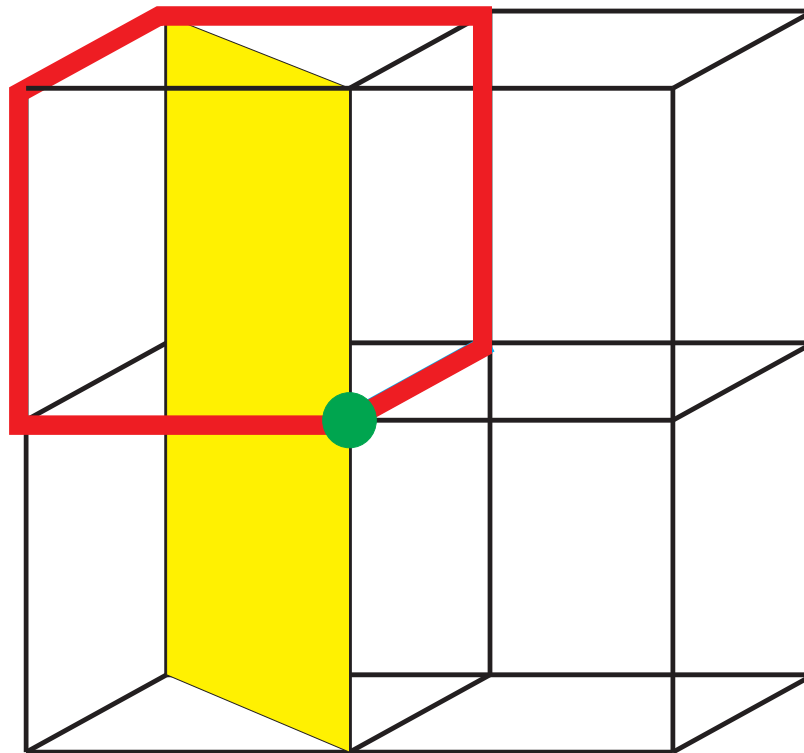
G_0

Group structure



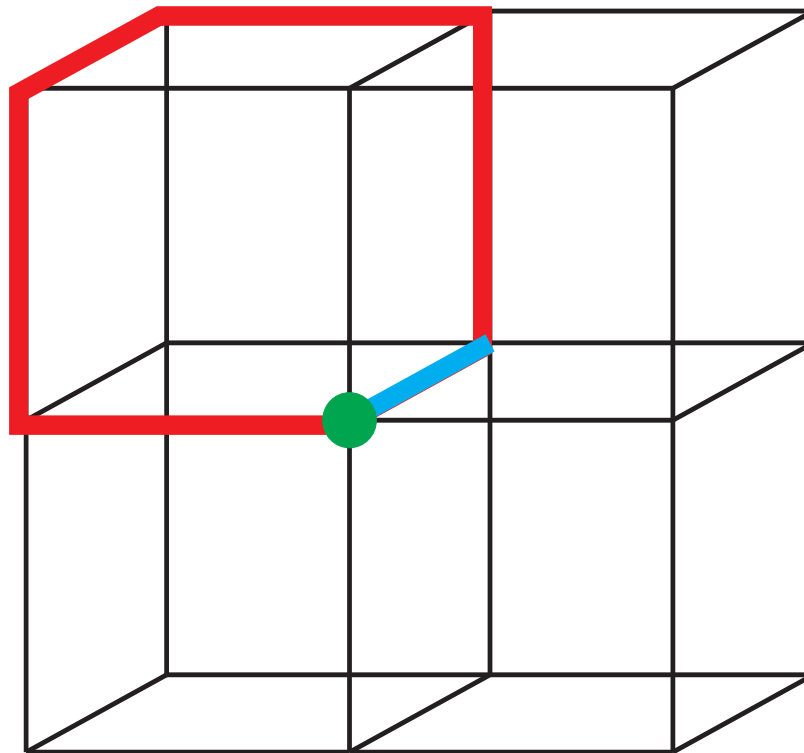
G_1

Group structure



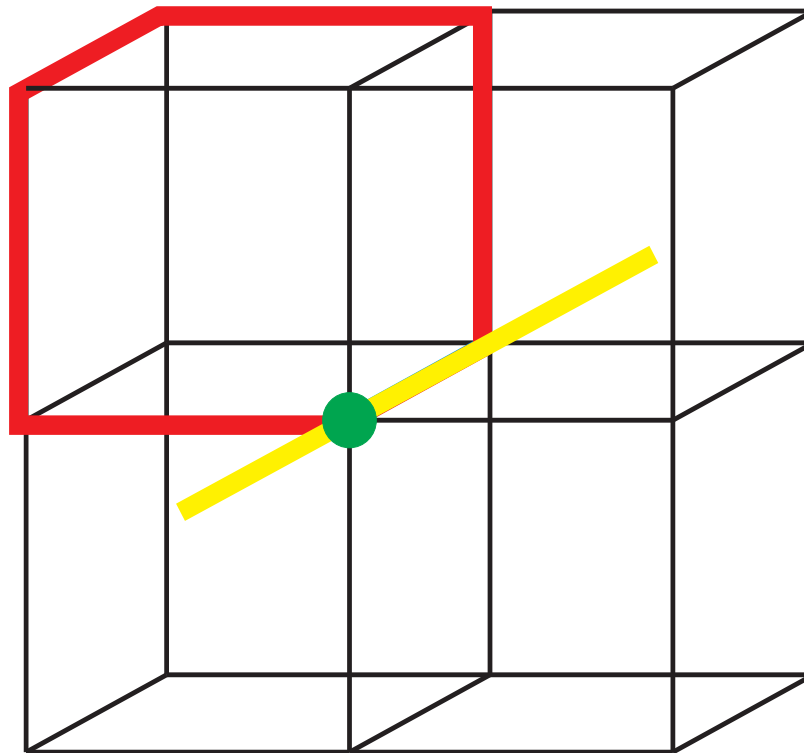
G_1

Group structure



G_2

Group structure



G_2

Group structure

Three horizontal lines of varying colors (yellow, black, grey) and lengths are positioned below the title.

- ▶ Symmetry group

Group structure

- Symmetry group \longrightarrow crystallographic group generated by G_0 , G_1 and G_2

Group structure

- ▶ Symmetry group \longrightarrow crystallographic group generated by G_0 , G_1 and G_2
- ▶ Point group

Group structure

- ▶ Symmetry group \longrightarrow crystallographic group generated by G_0 , G_1 and G_2
- ▶ Point group \longrightarrow Symmetry group modulo translations

Group structure

- ▶ Symmetry group \longrightarrow crystallographic group generated by G_0 , G_1 and G_2
- ▶ Point group \longrightarrow Symmetry group modulo translations

Theorem

The point group is a subgroup of the octahedral group

Procedure

- Choose appropriate G_0 , G_1 and G_2 in the point group

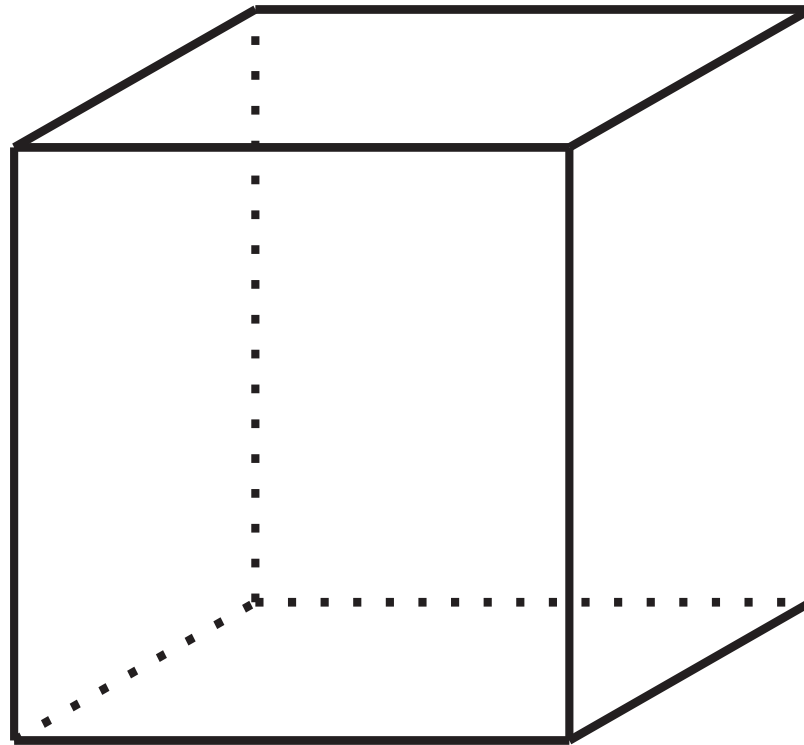
Procedure

- Choose appropriate G_0 , G_1 and G_2 in the point group
- Choose the translation vector for G_0

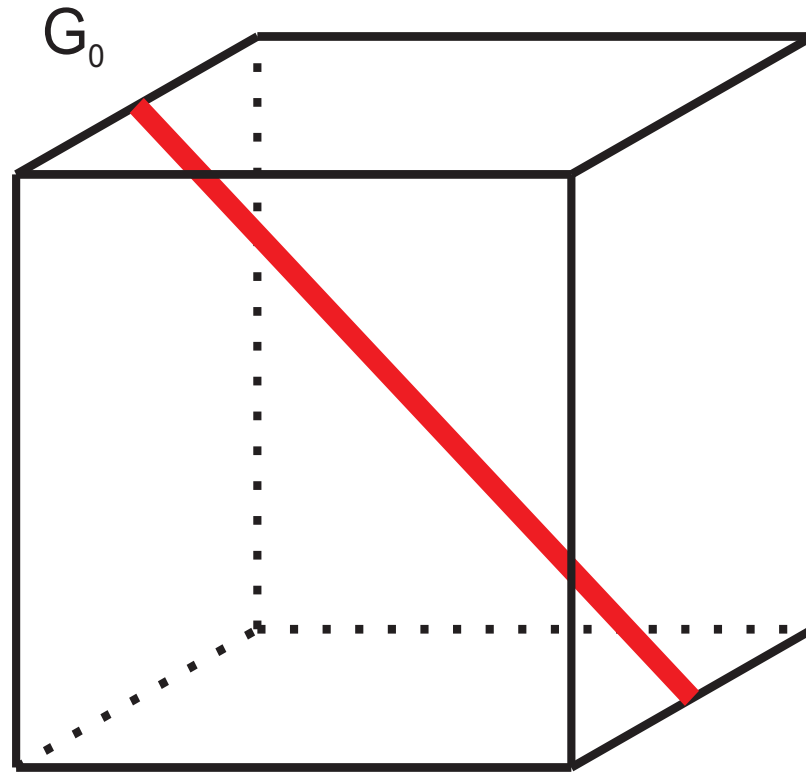
Procedure

- Choose appropriate G_0 , G_1 and G_2 in the point group
- Choose the translation vector for G_0
- Do the necessary computations!!

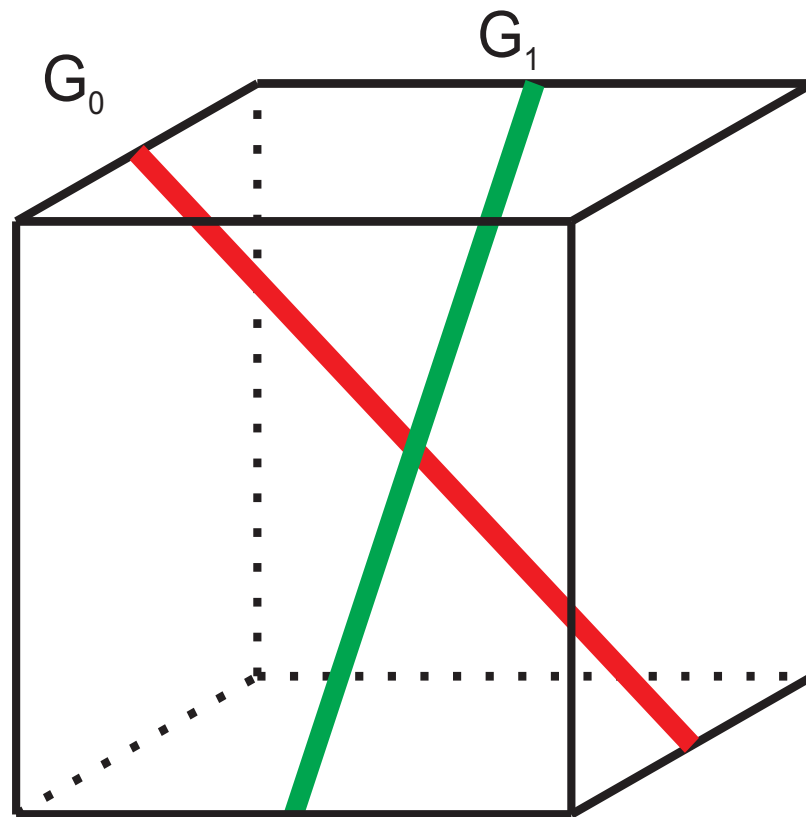
Procedure



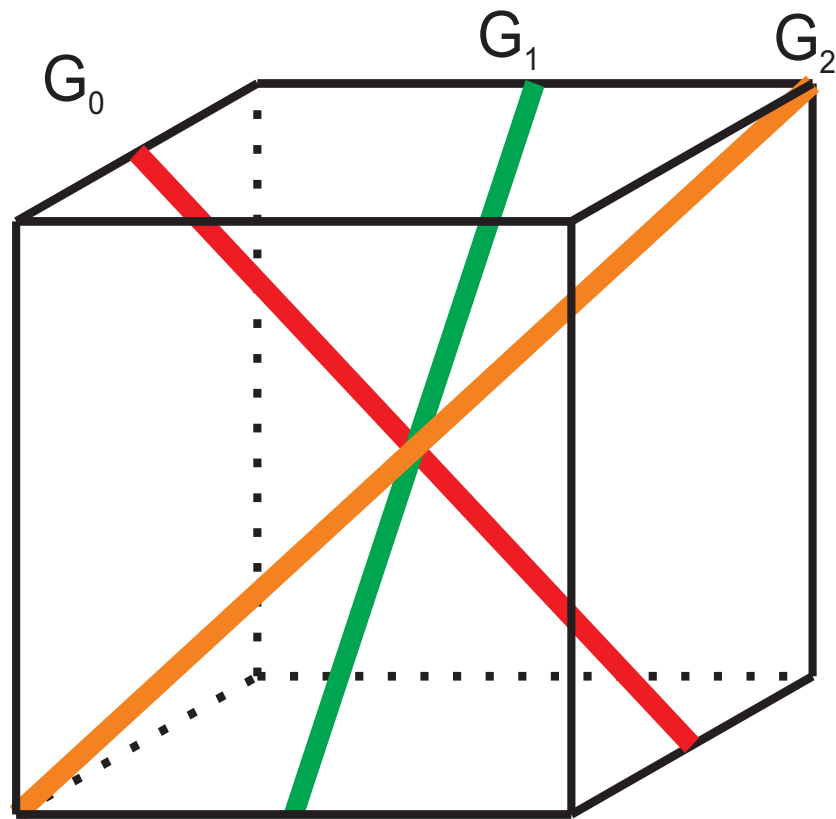
Procedure



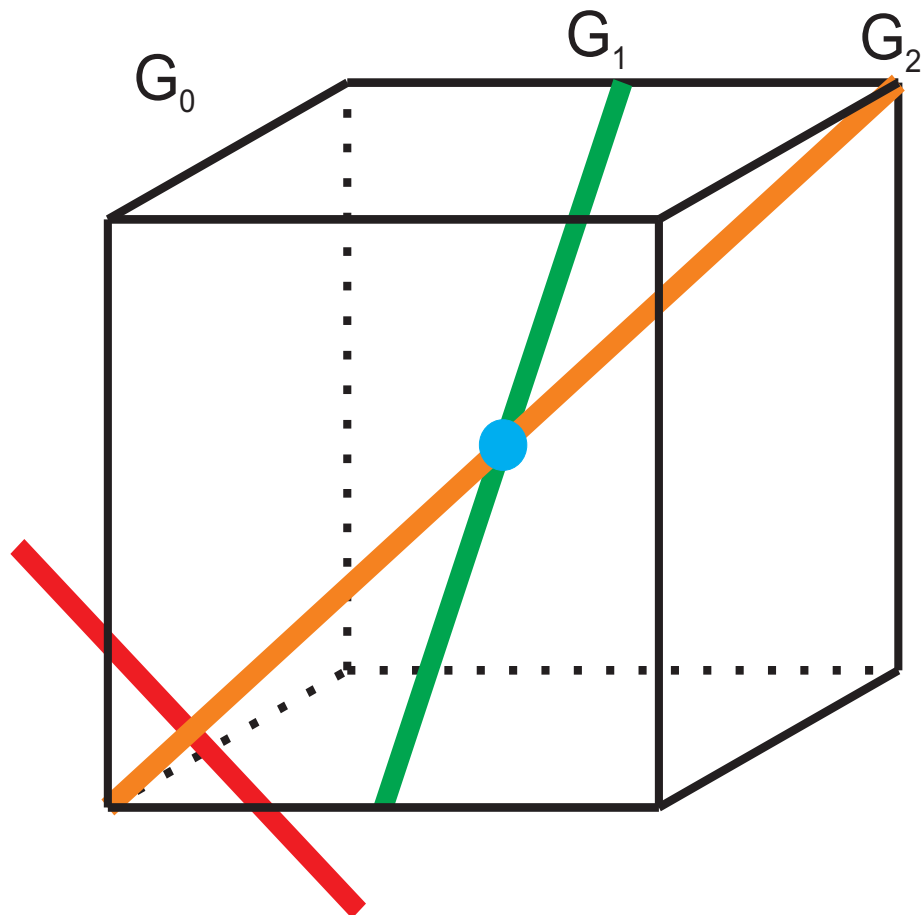
Procedure



Procedure



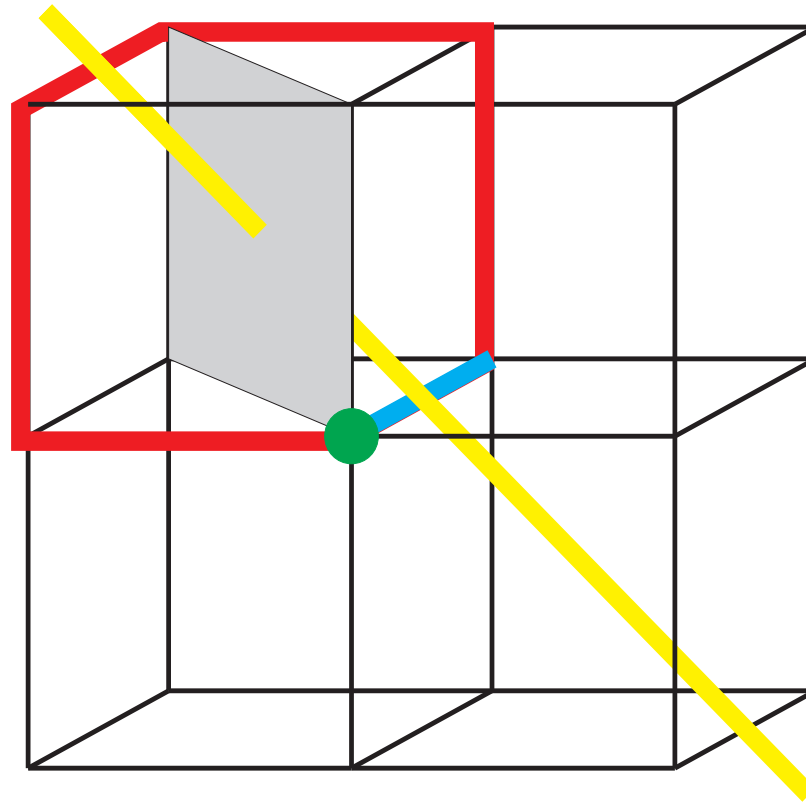
Procedure



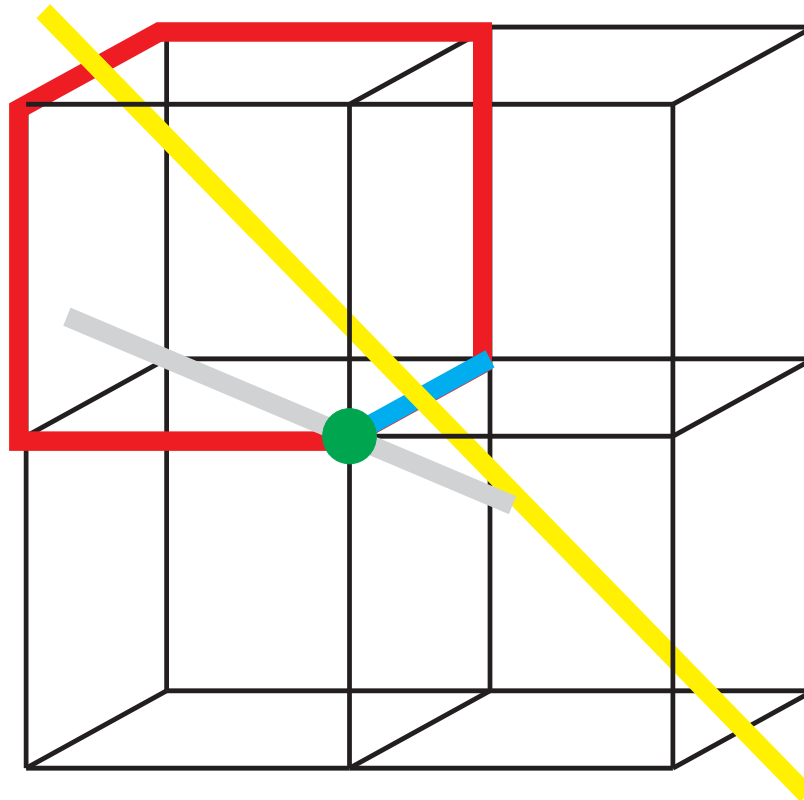
Shortcuts



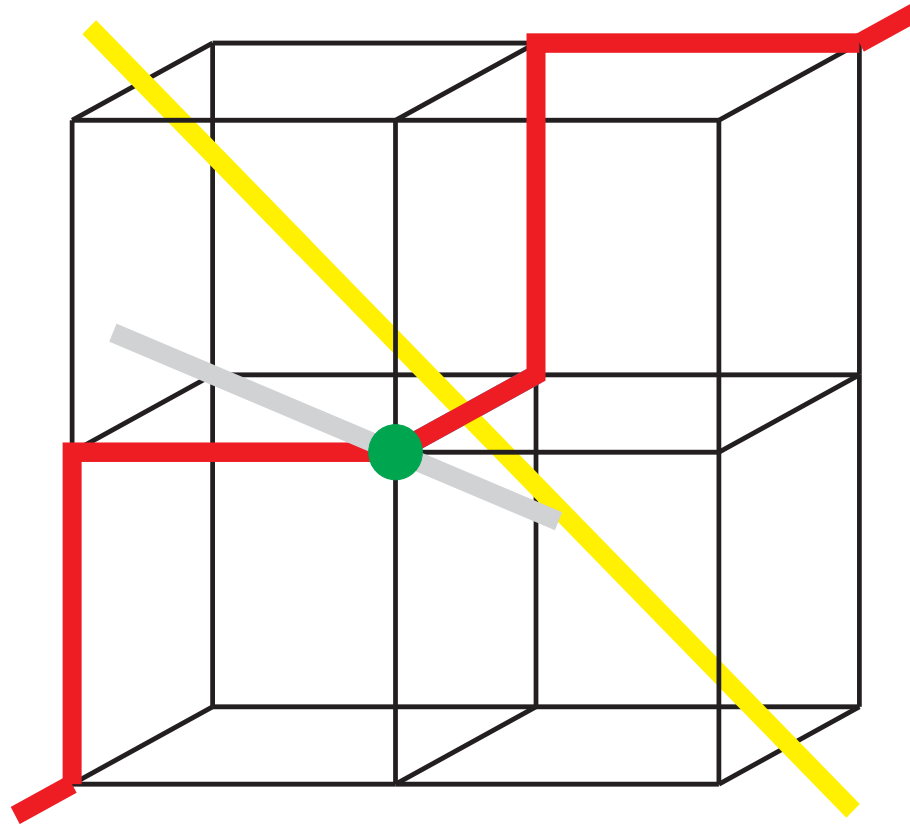
Shortcuts



Shortcuts



Shortcuts



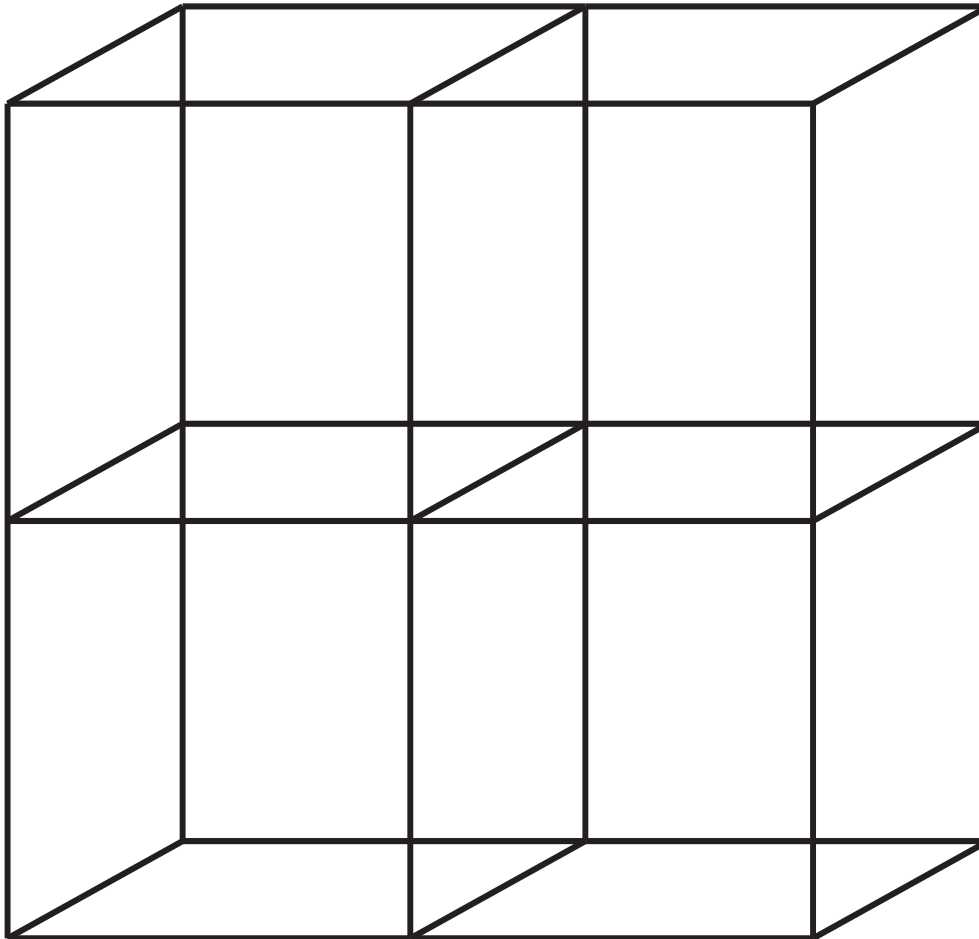
Lattices

A decorative header element consisting of a yellow wedge on the left, followed by a black horizontal line, a grey horizontal line, and another black horizontal line.

Cubical lattice

Lattices

Cubical lattice



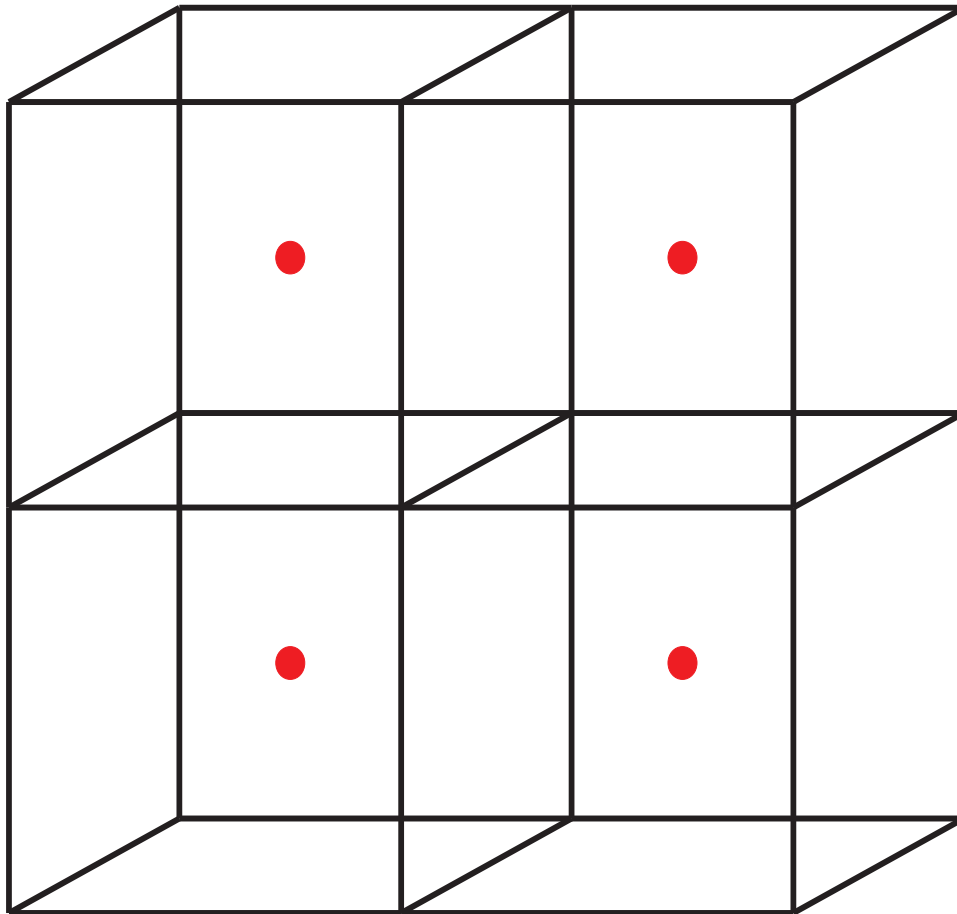
Lattices

A yellow wedge shape points from the left edge of the slide towards the title 'Lattices'. Below the title, there are three horizontal lines: a thin grey line, a thick black line, and another thin grey line.

Body centred cubic lattice

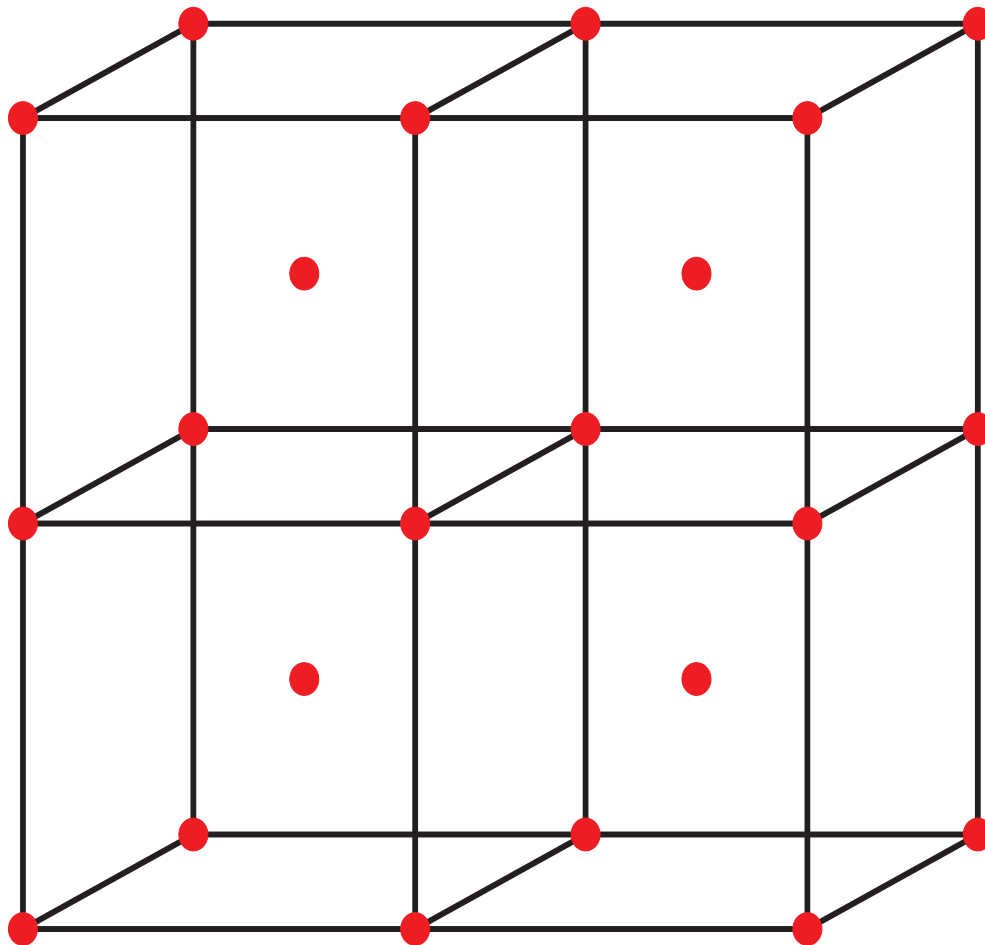
Lattices

Body centred cubic lattice



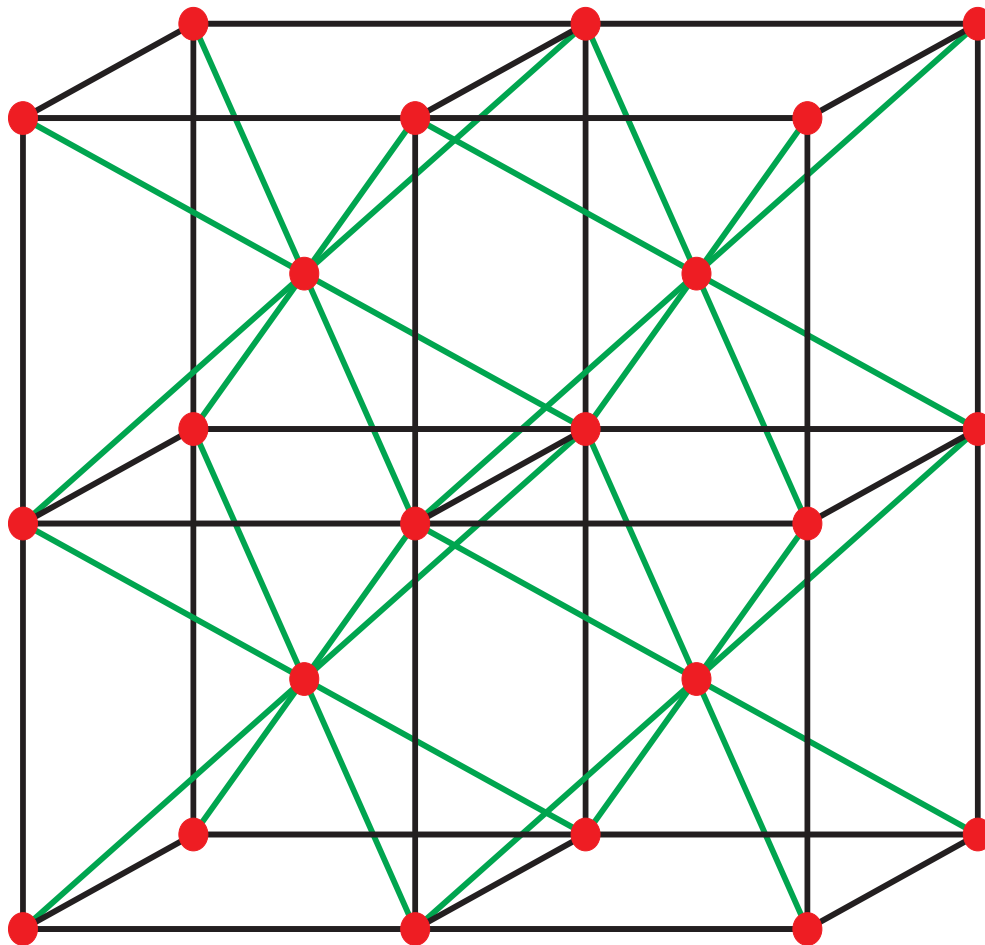
Lattices

Body centred cubic lattice



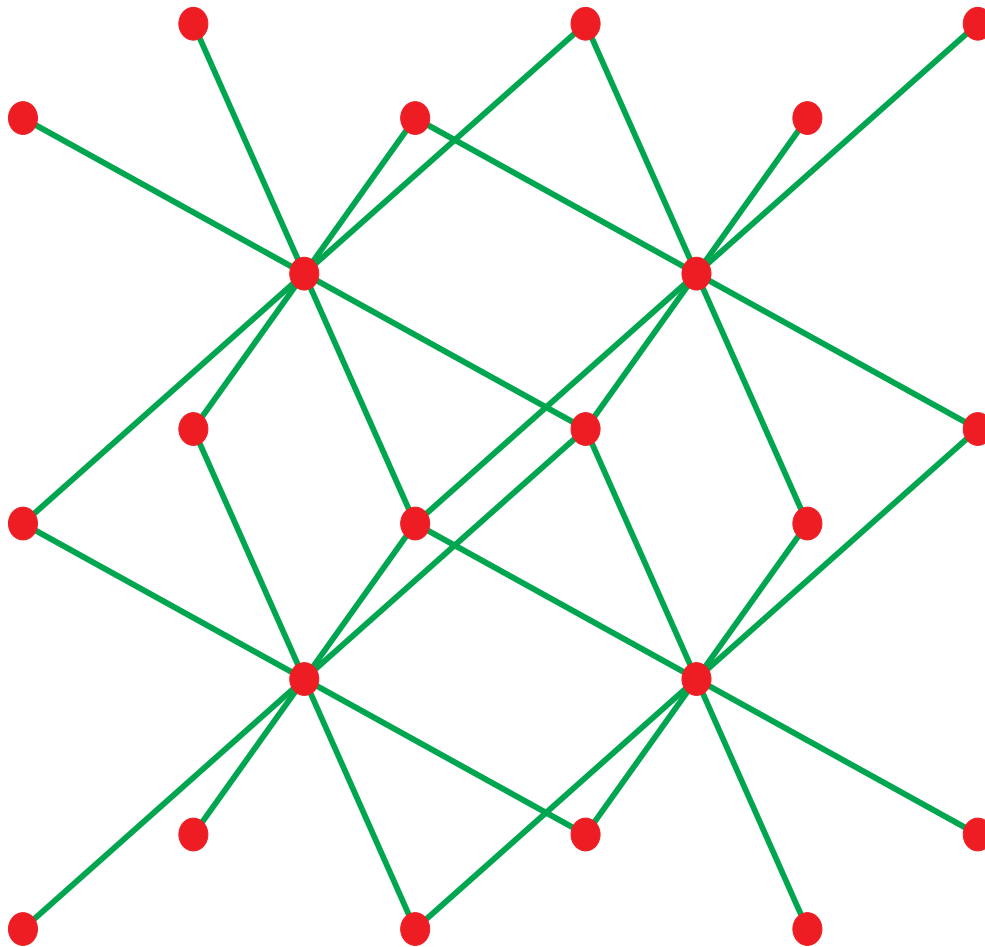
Lattices

Body centred cubic lattice



Lattices

Body centred cubic lattice



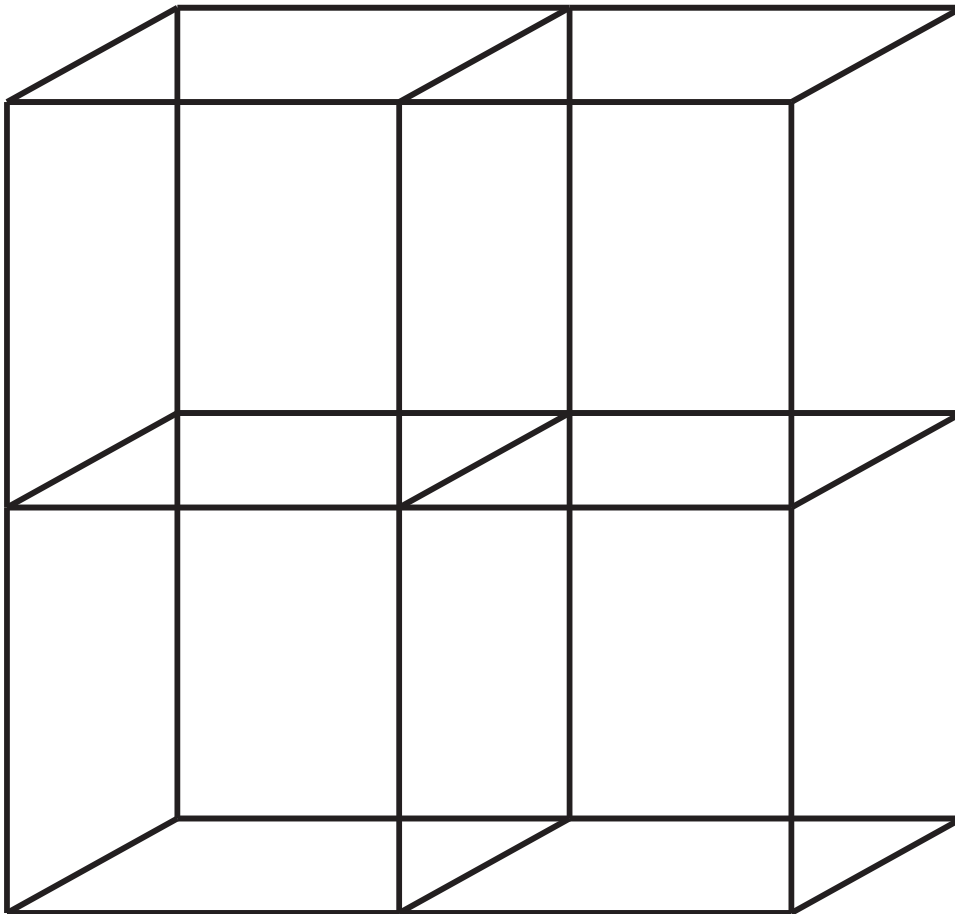
Lattices

A decorative header element consisting of a yellow wedge on the left, followed by a black horizontal line, a grey horizontal line, and another black horizontal line.

Face centred cubic lattice

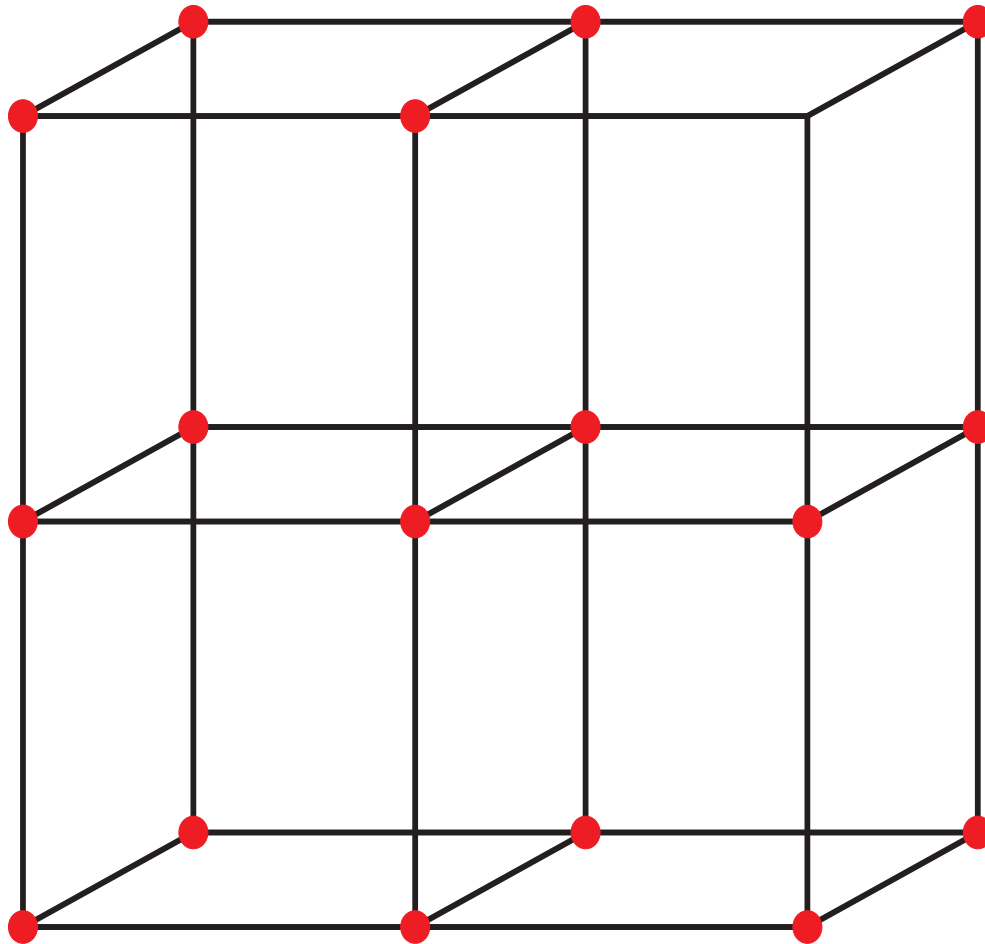
Lattices

Face centred cubic lattice



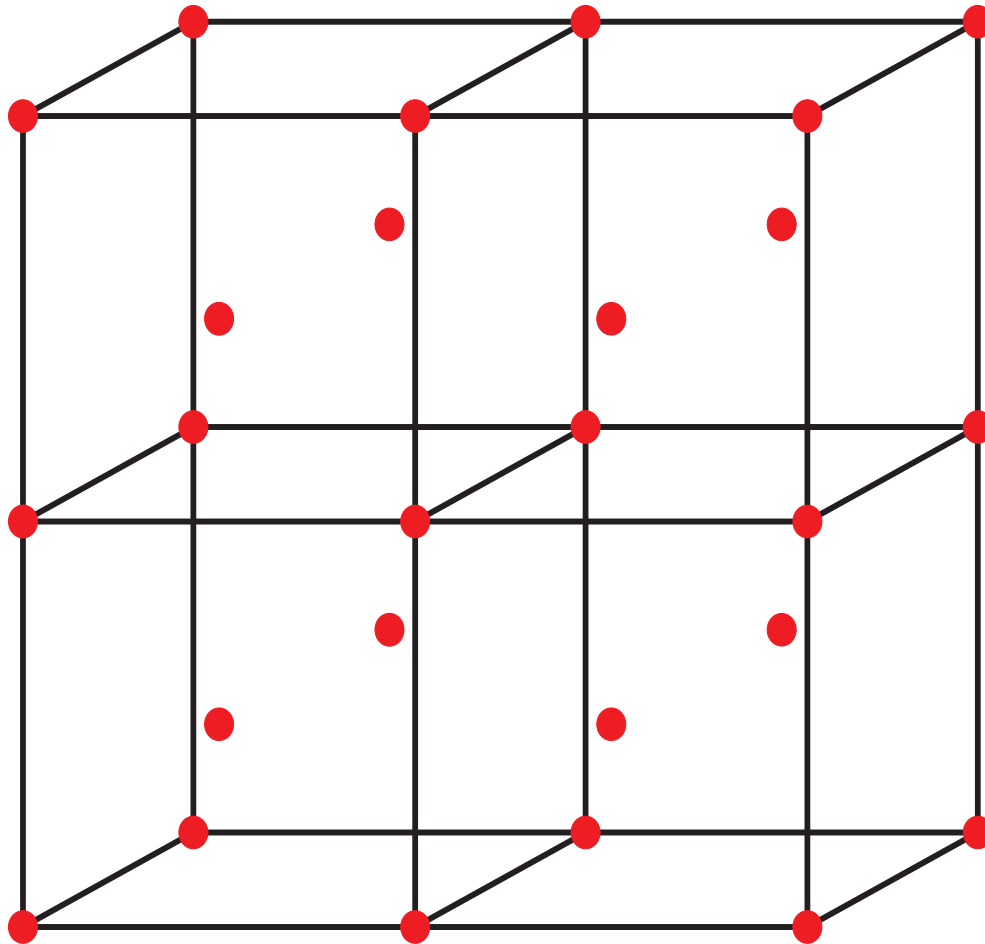
Lattices

Face centred cubic lattice



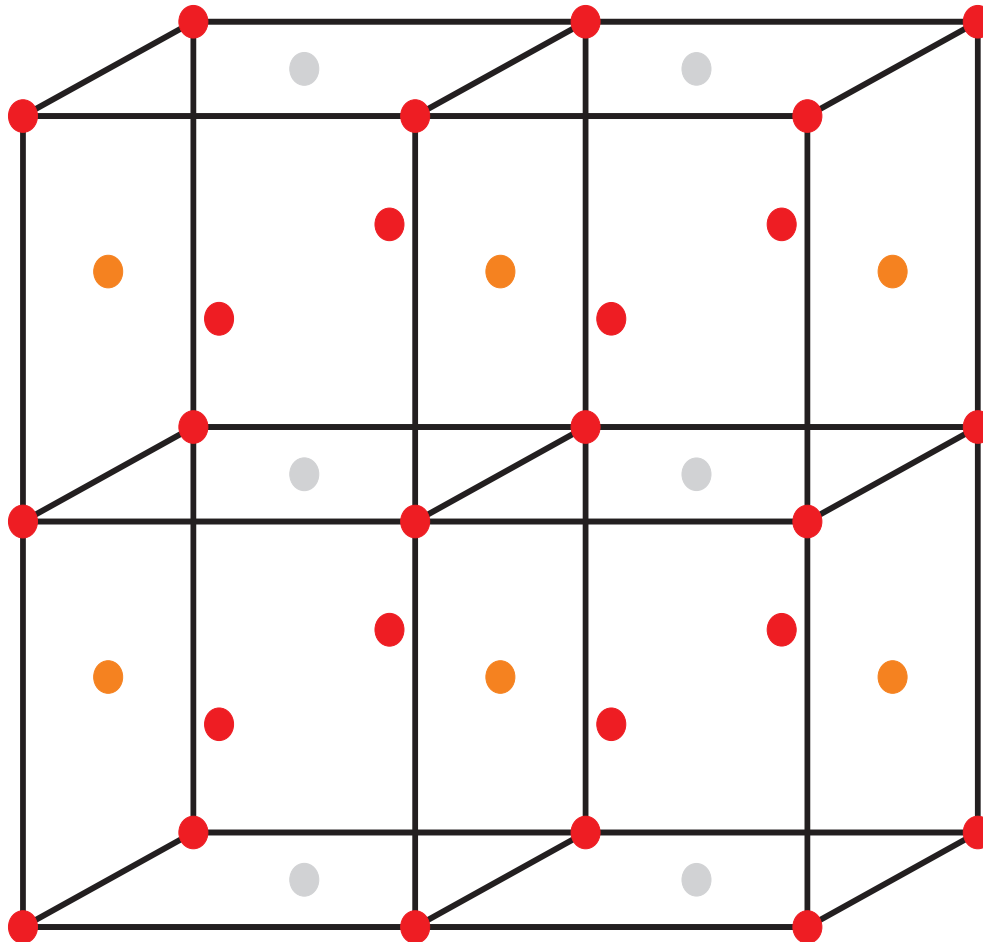
Lattices

Face centred cubic lattice



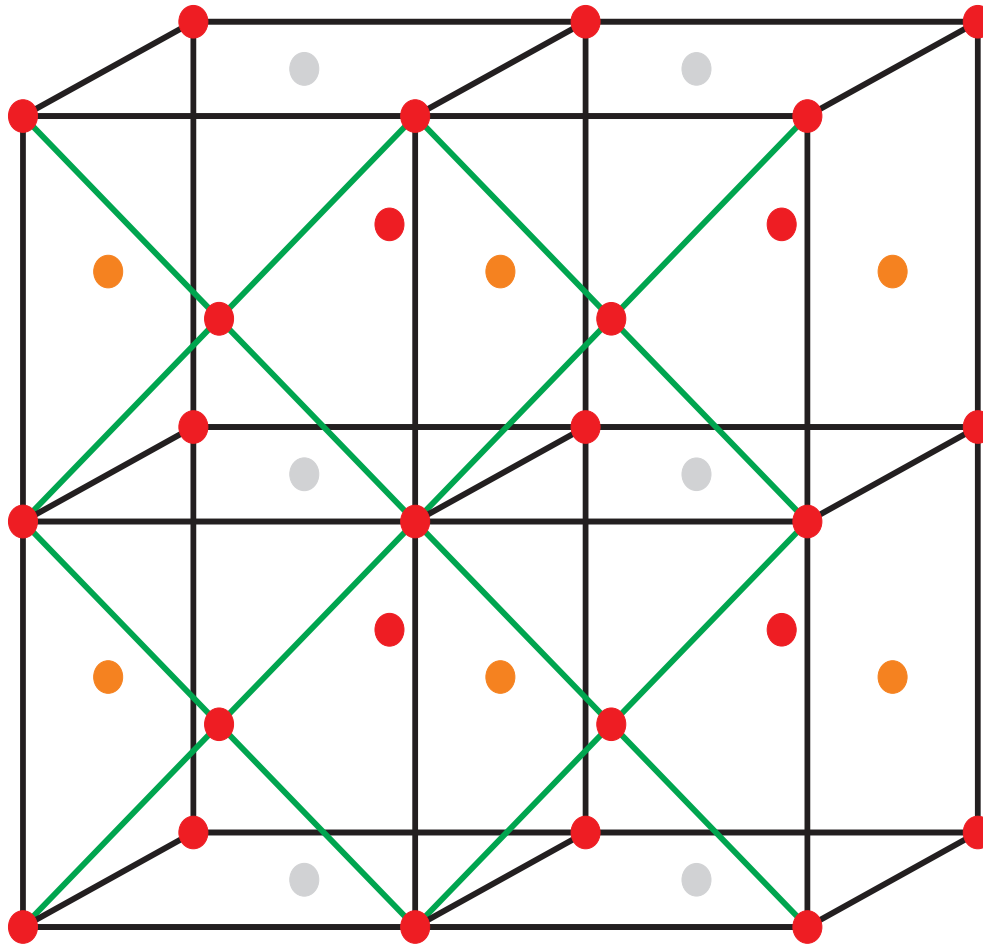
Lattices

Face centred cubic lattice



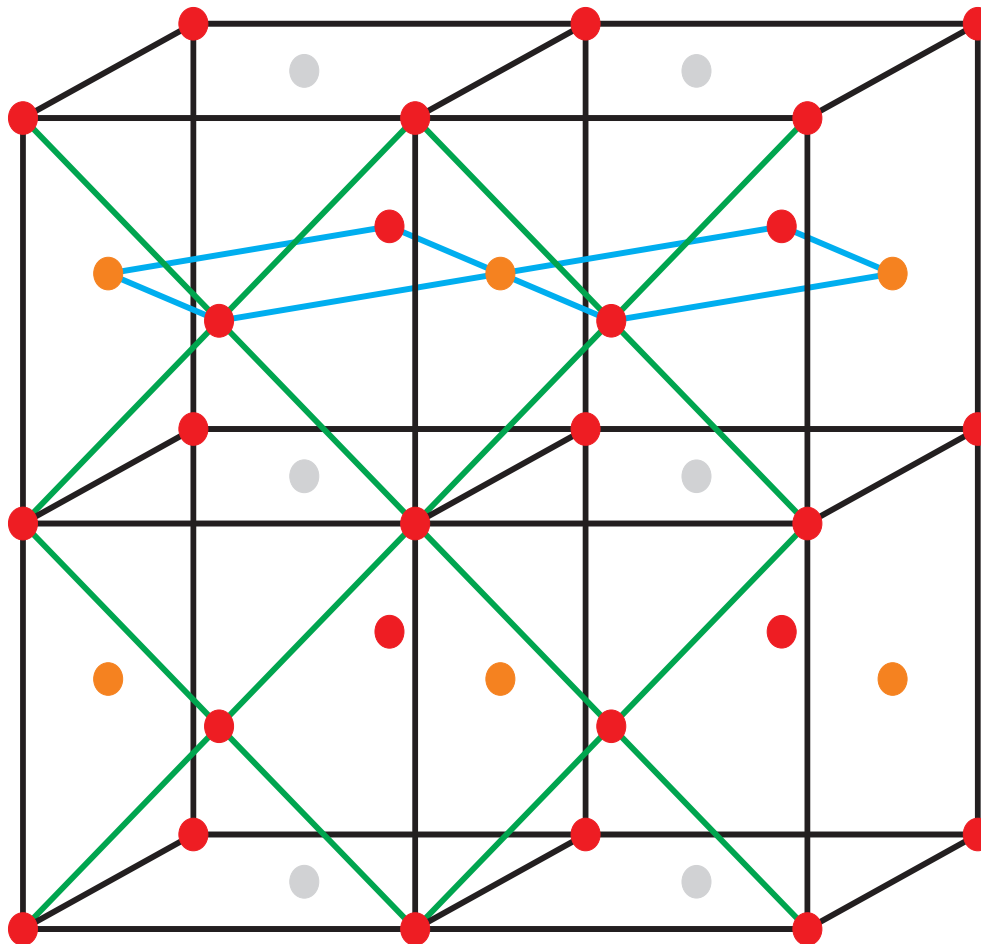
Lattices

Face centred cubic lattice



Lattices

Face centred cubic lattice



Enumeration



- ▶ 25 regular polygonal complexes

Enumeration

- ▶ 25 regular polygonal complexes
- 4 with non-trivial flag-stabilizers

Enumeration

- ▶ 25 regular polygonal complexes
 - 4 with non-trivial flag-stabilizers
 - 21 with trivial flag-stabilizers

Enumeration

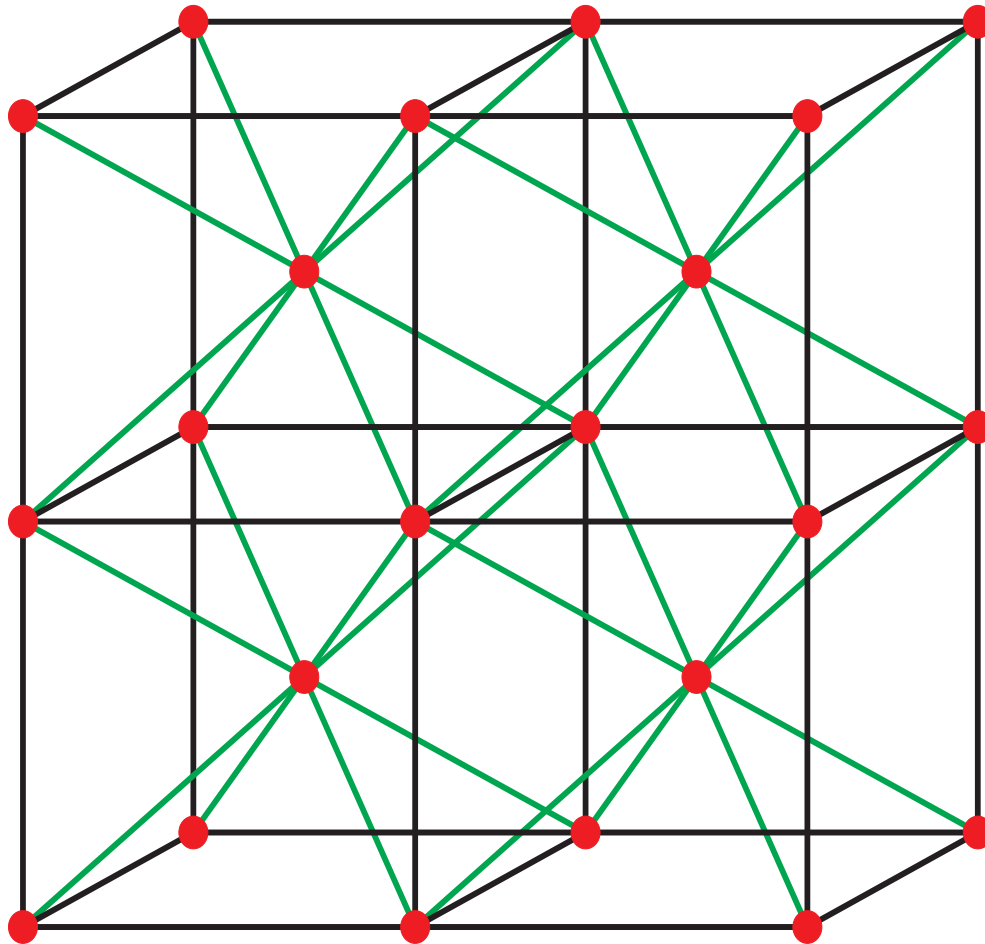
- 3 with finite planar faces

Enumeration

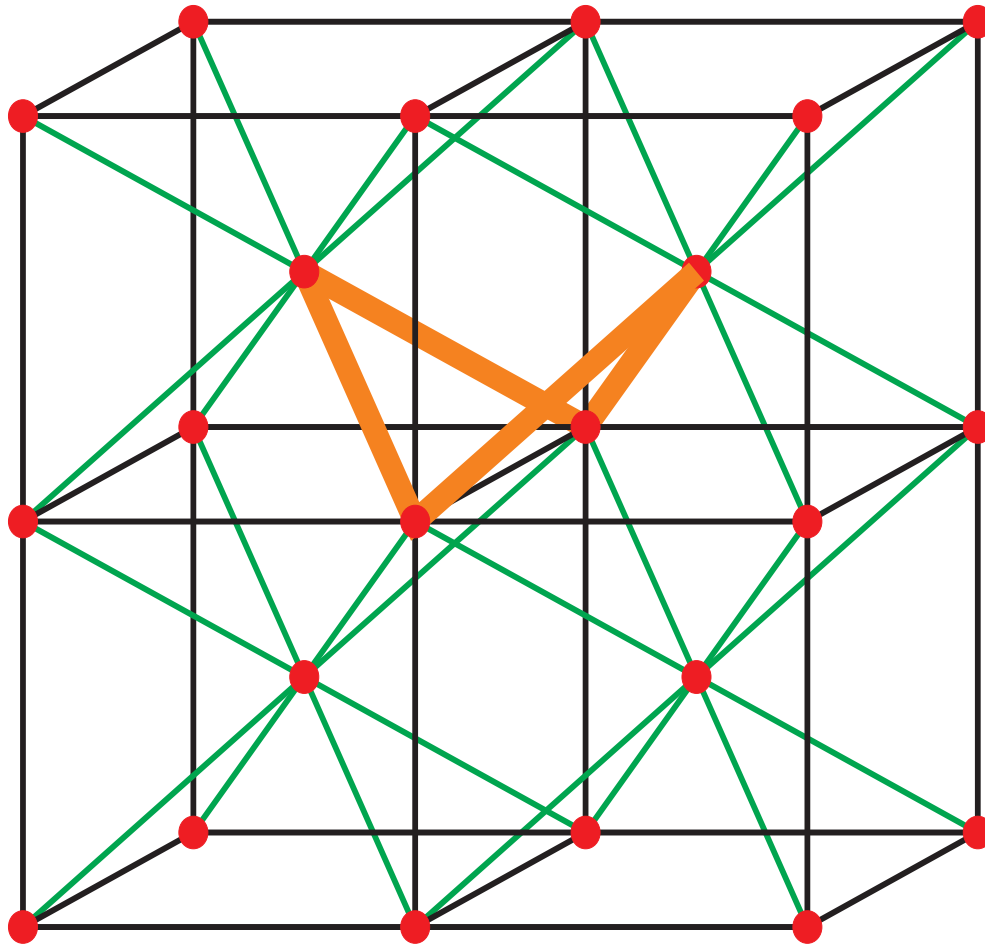


- 3 with finite planar faces
- 8 with finite skew faces

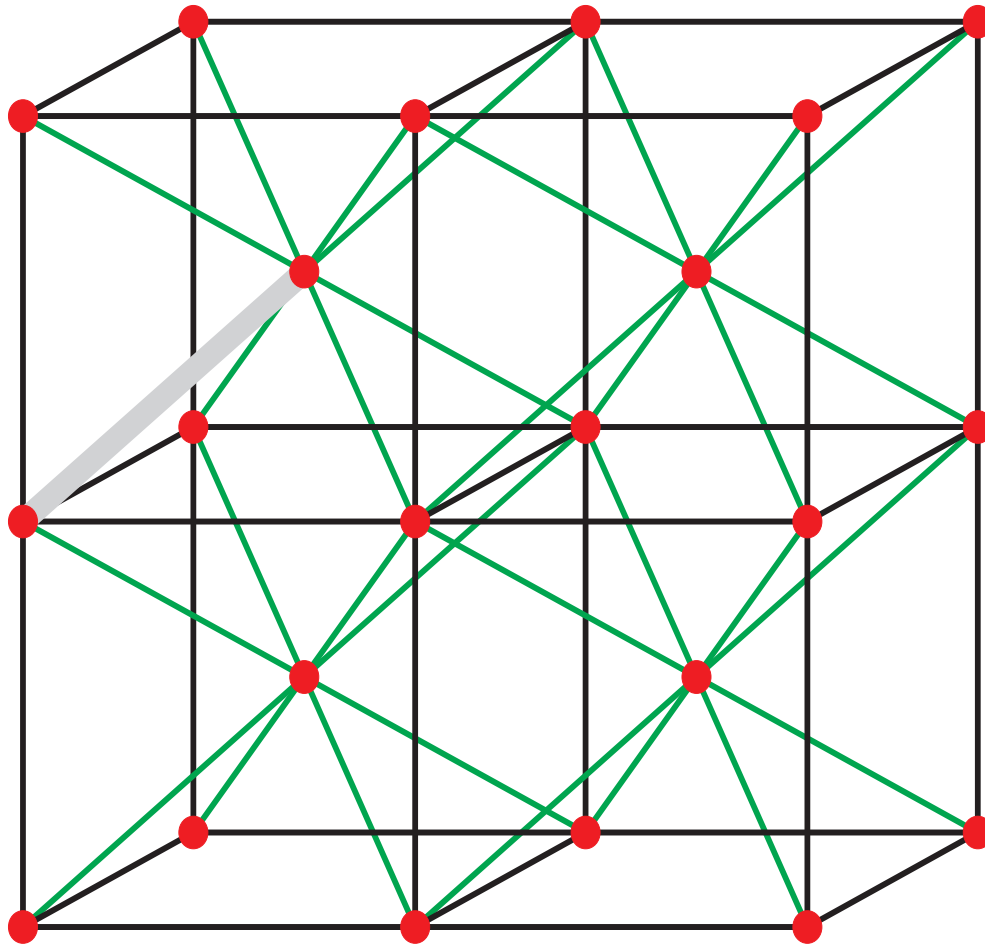
Enumeration



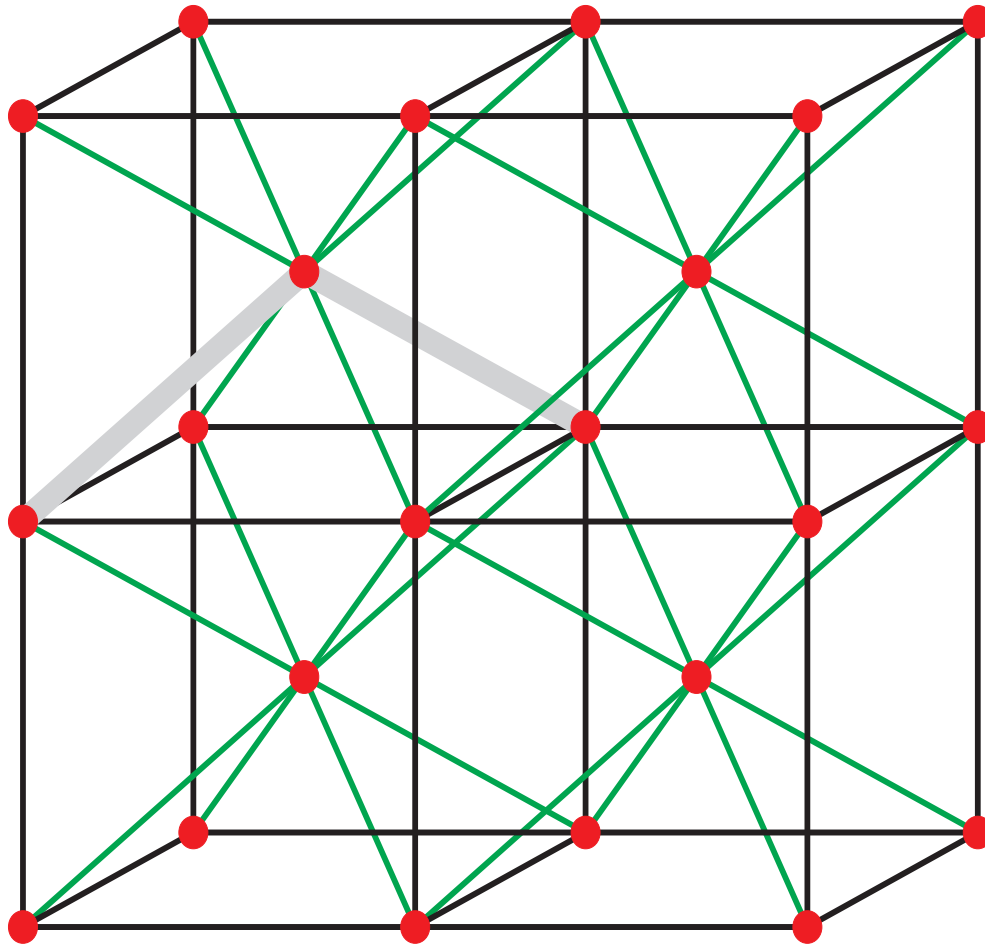
Enumeration



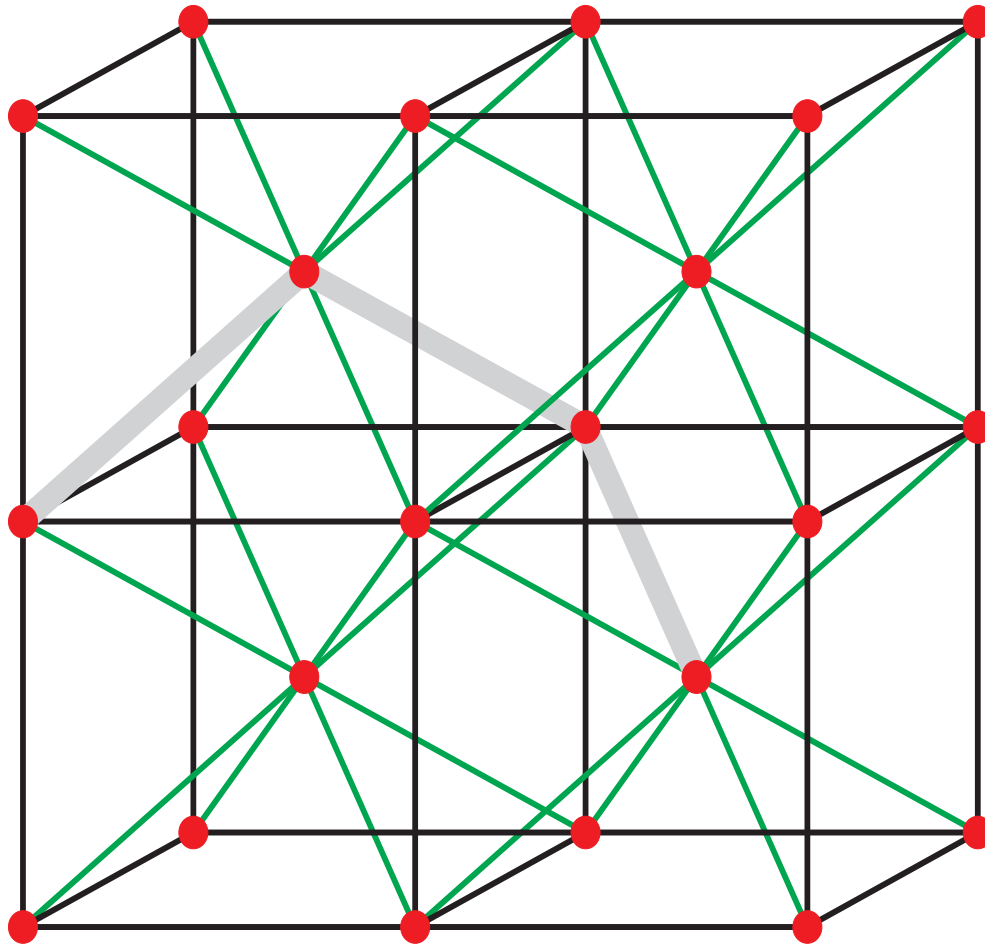
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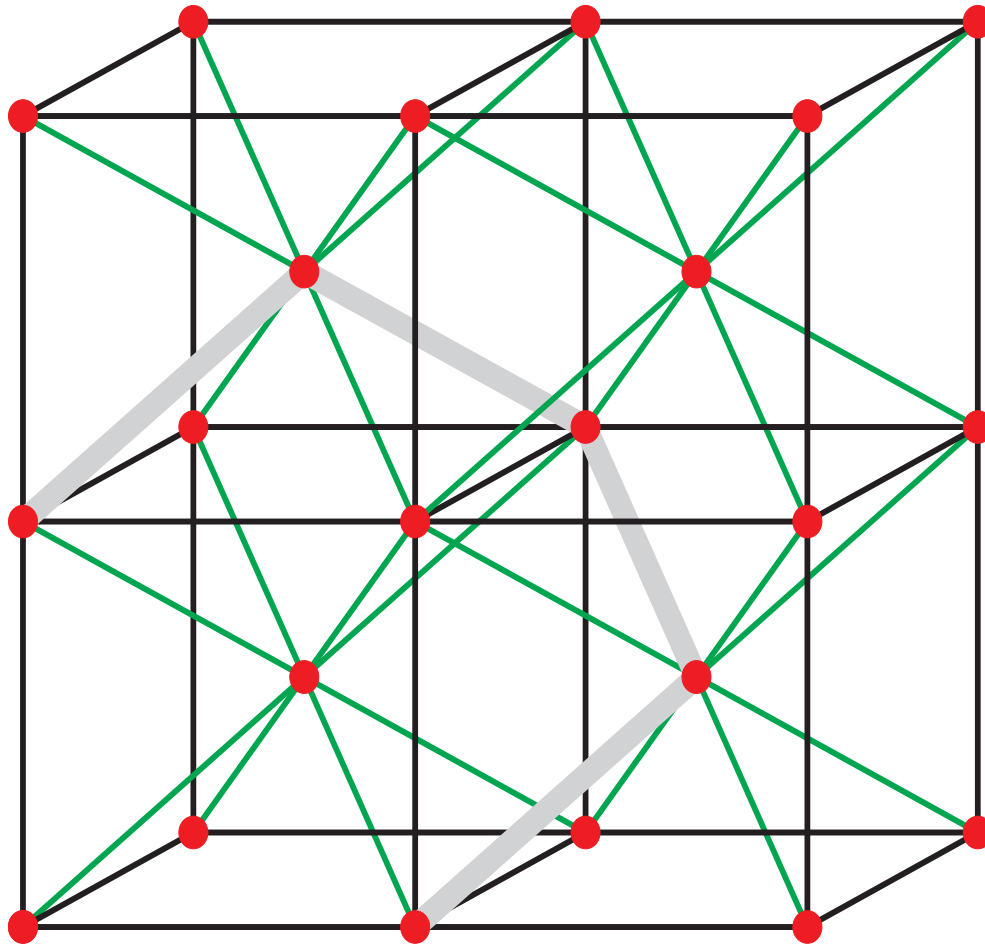
Enumeration



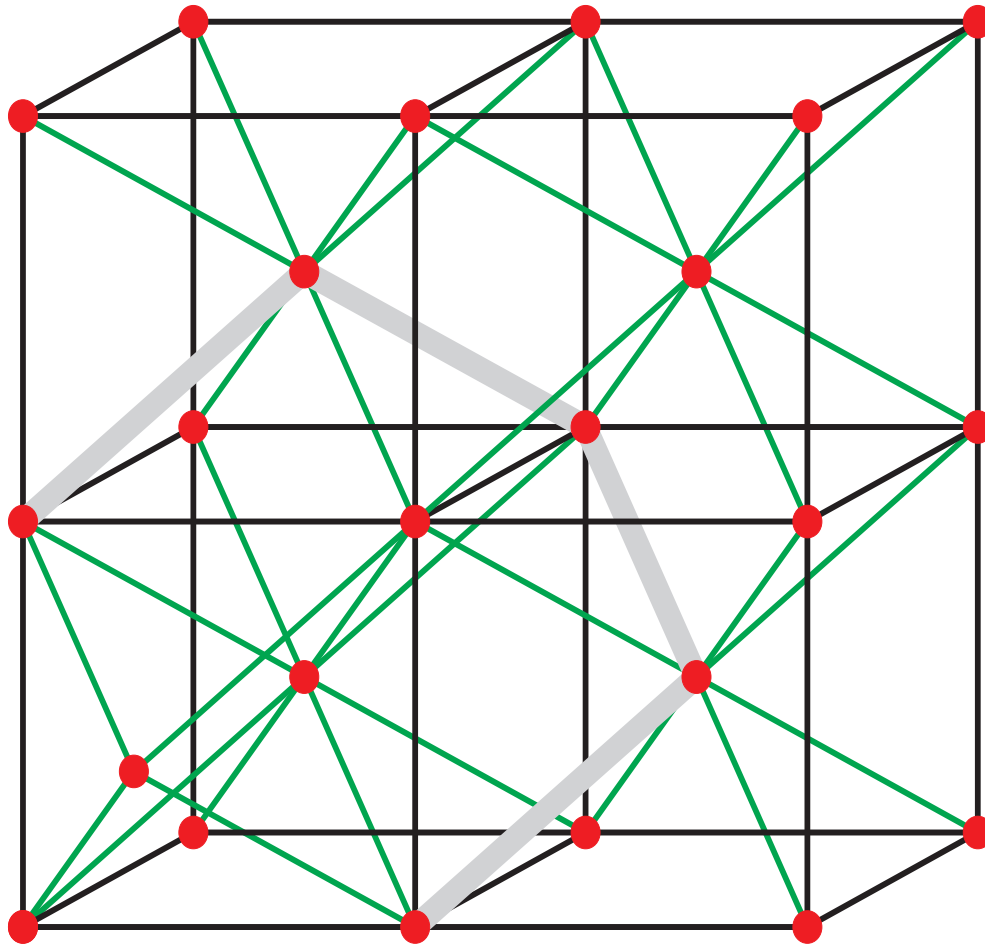
Enumeration



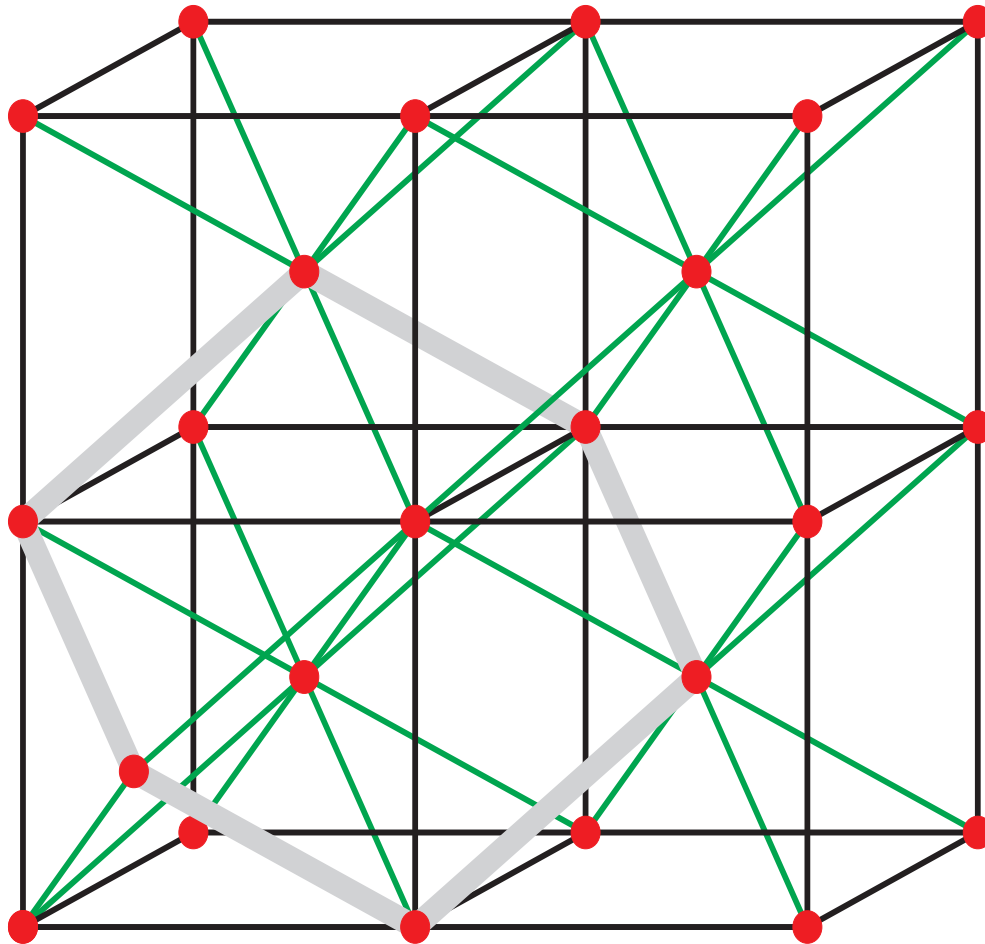
Enumeration



Enumeration



Enumeration



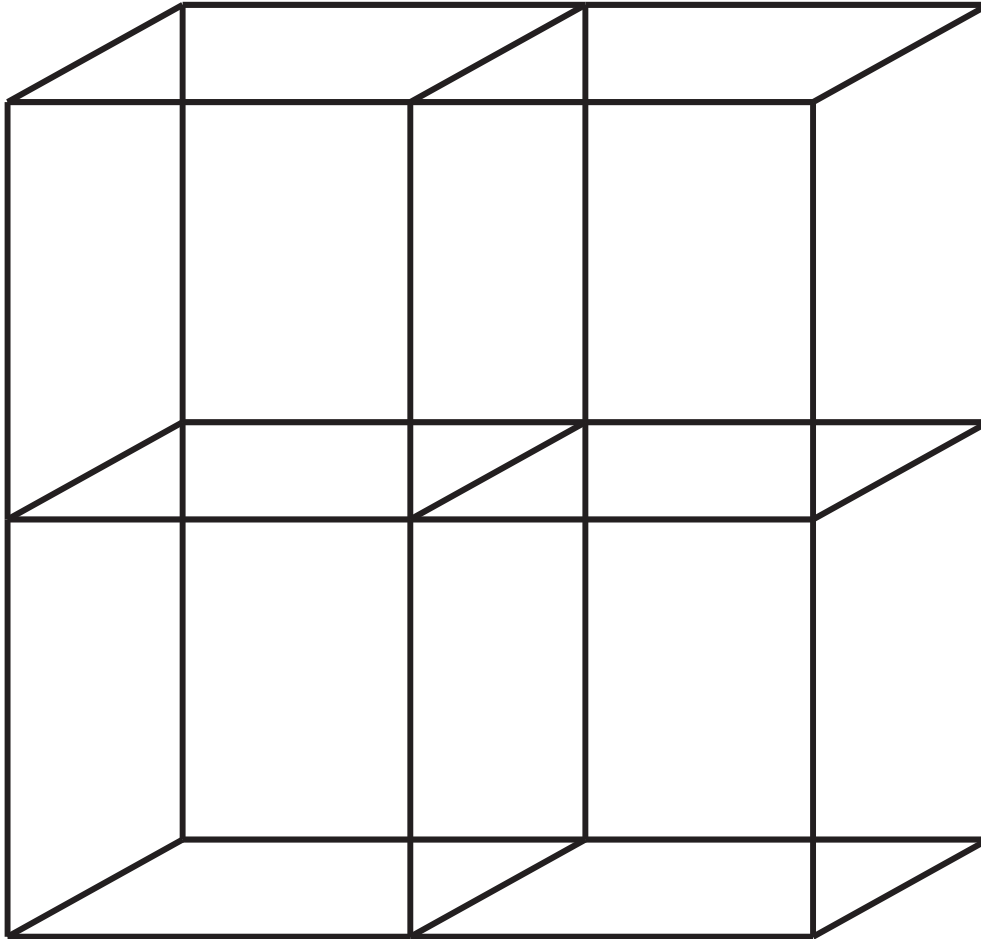
Enumeration

- 3 with finite planar faces
- 8 with finite skew faces

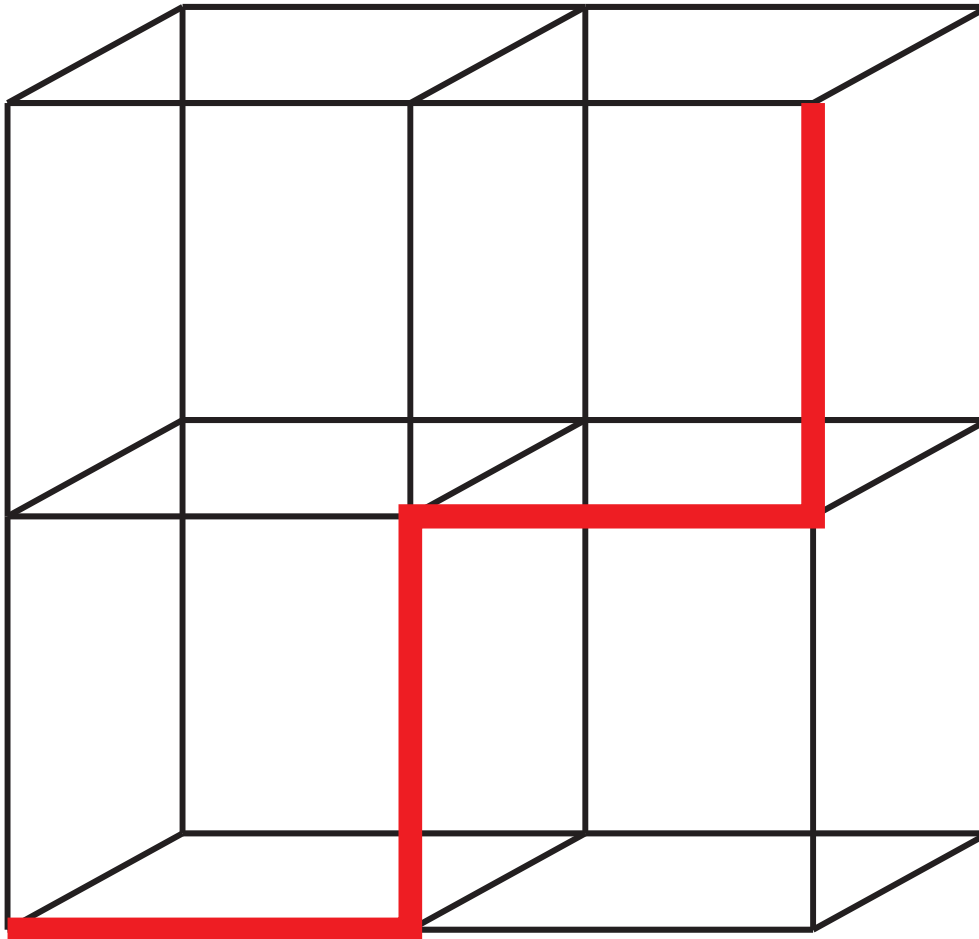
Enumeration

- 3 with finite planar faces
- 8 with finite skew faces
- 5 with zigzag faces

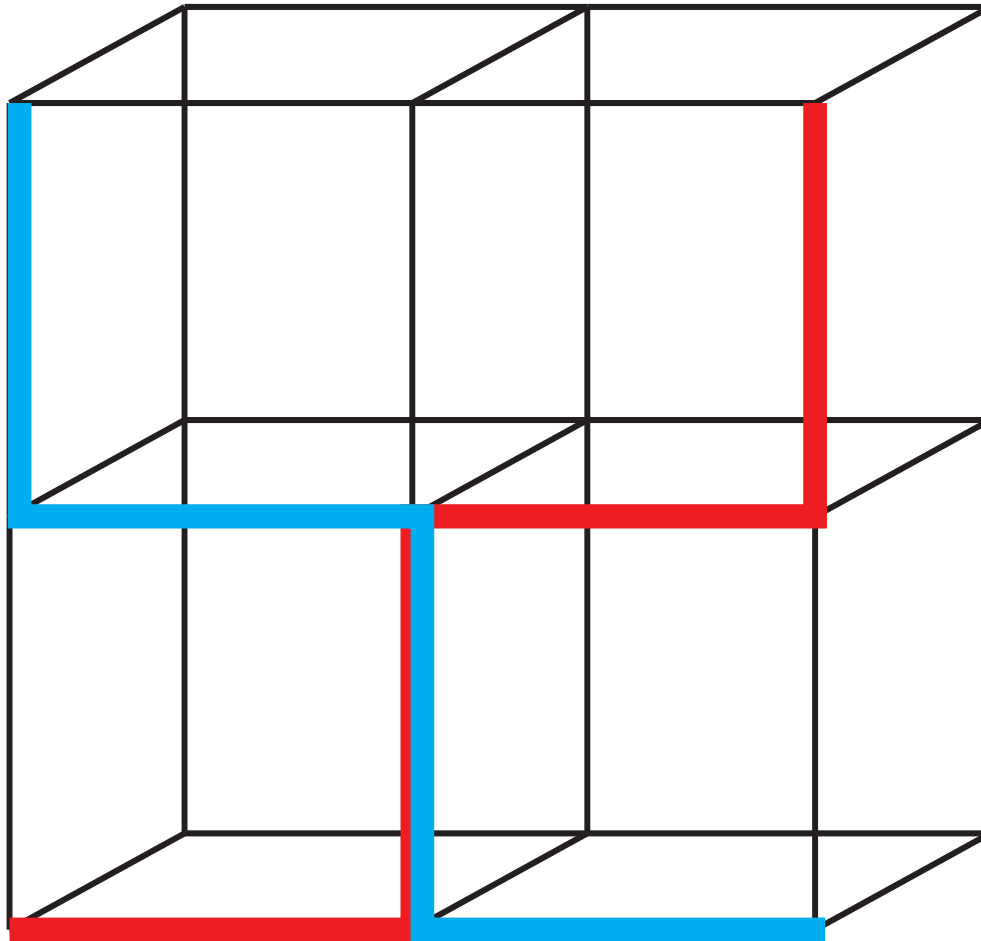
Enumeration



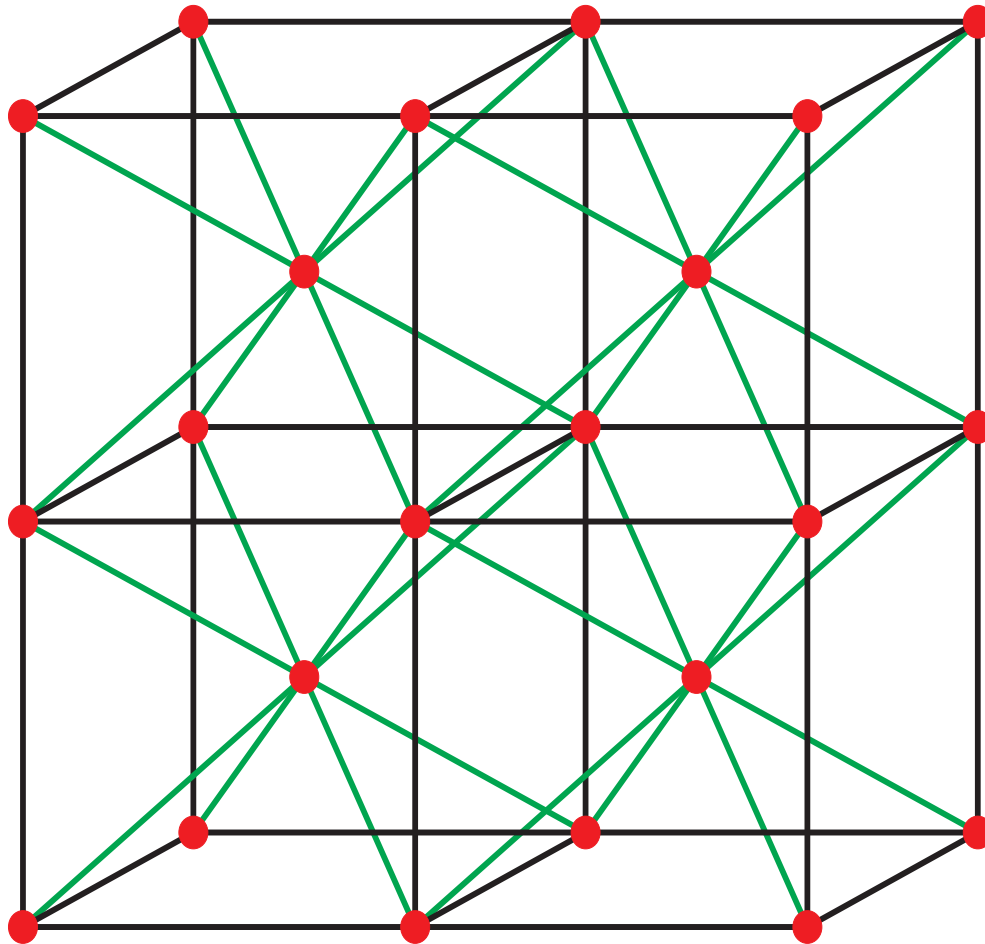
Enumeration



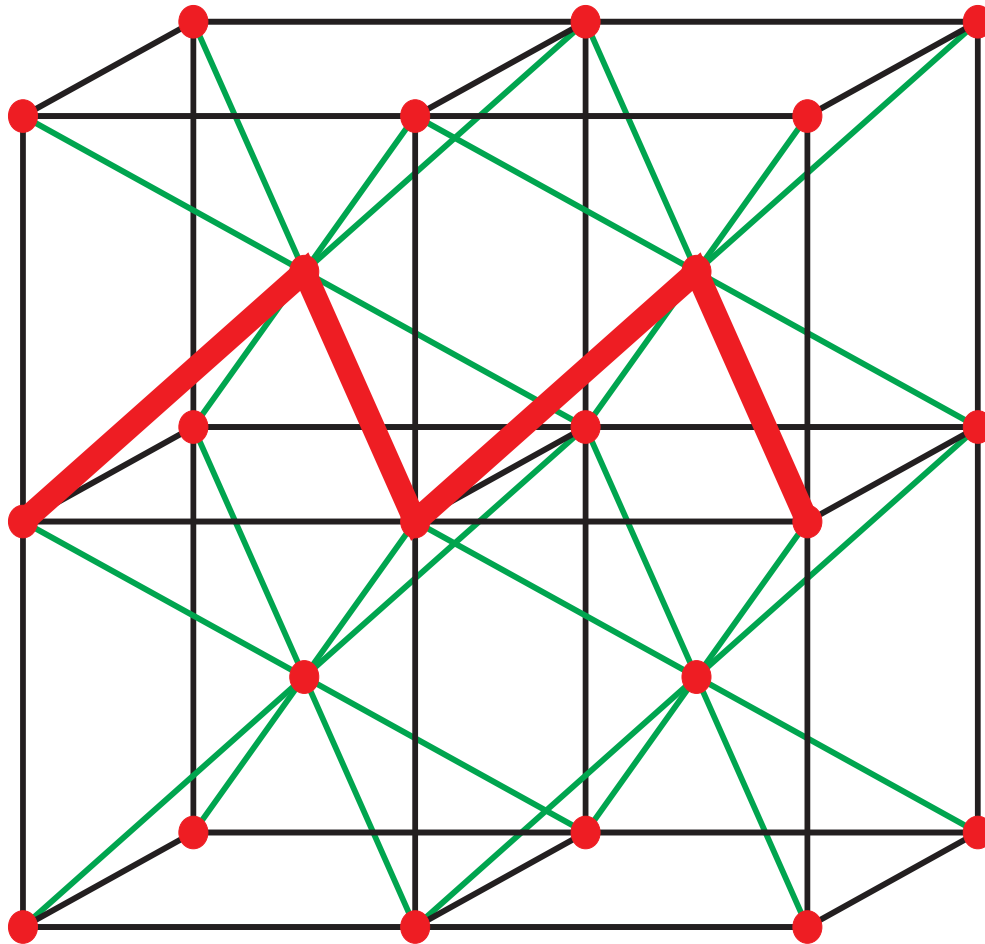
Enumeration



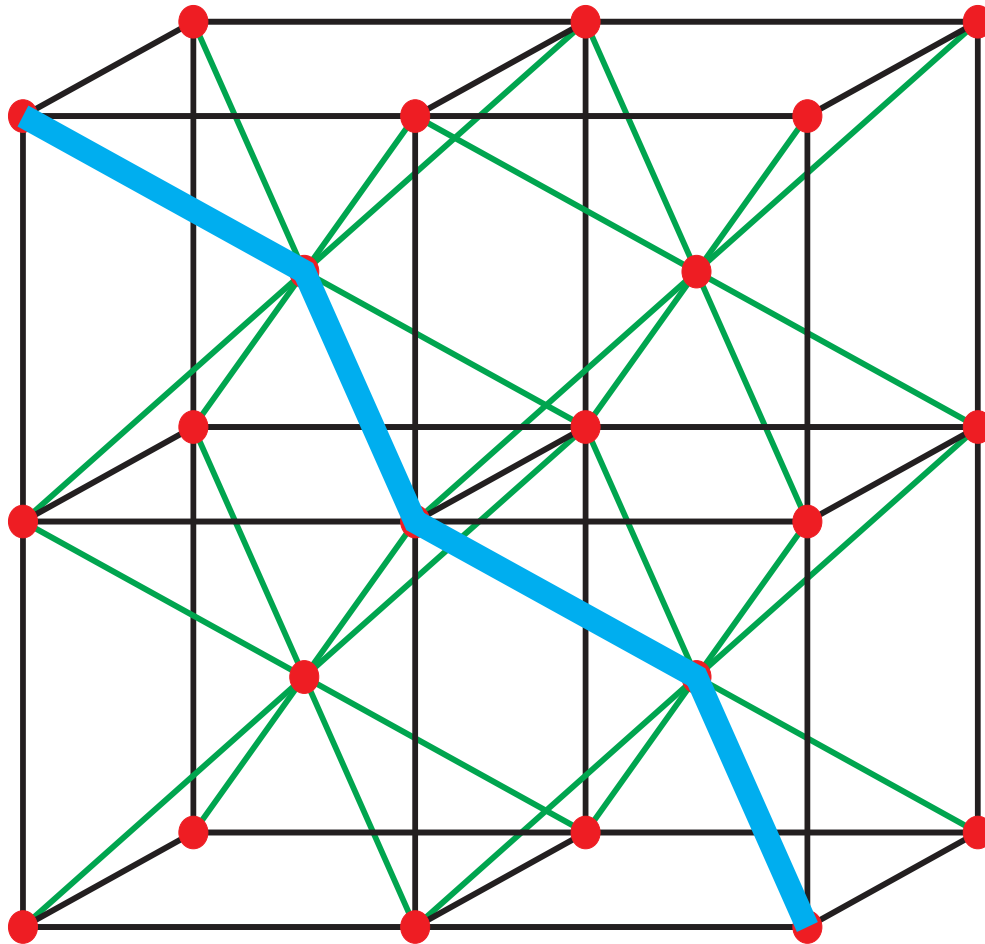
Enumeration



Enumeration



Enumeration



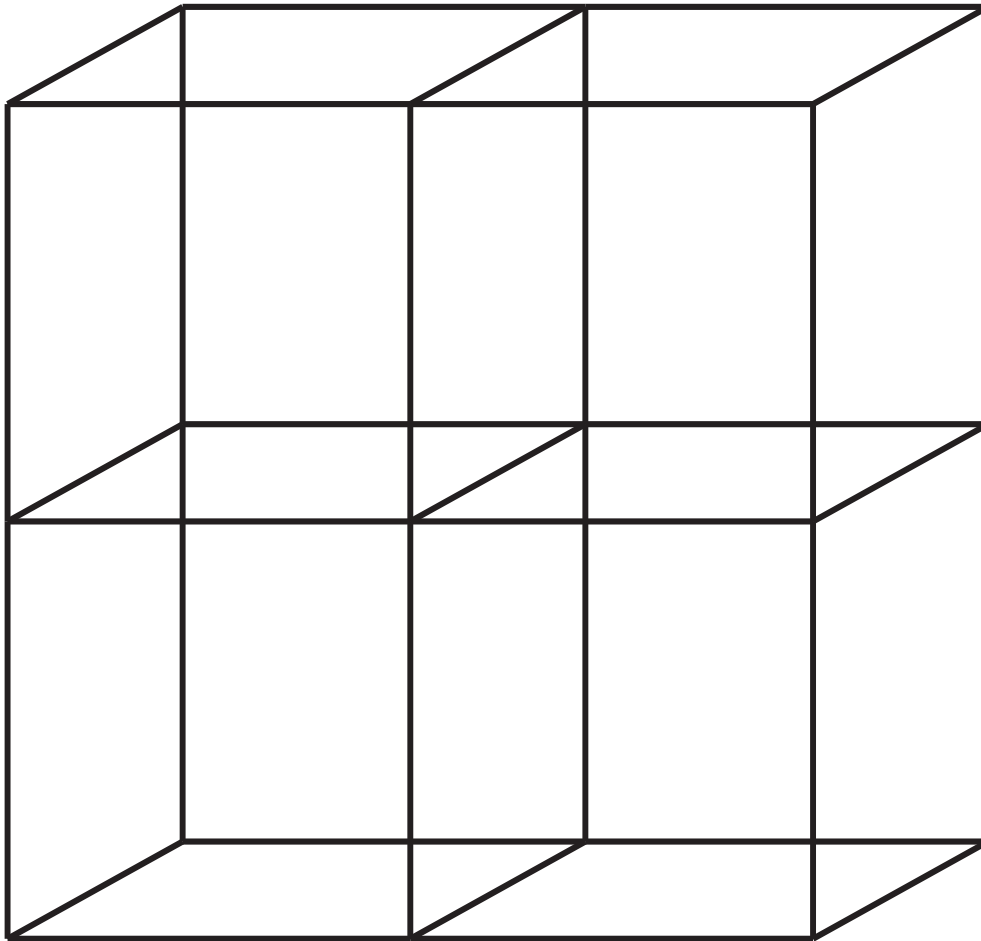
Enumeration

- 3 with finite planar faces
- 8 with finite skew faces
- 5 with zigzag faces

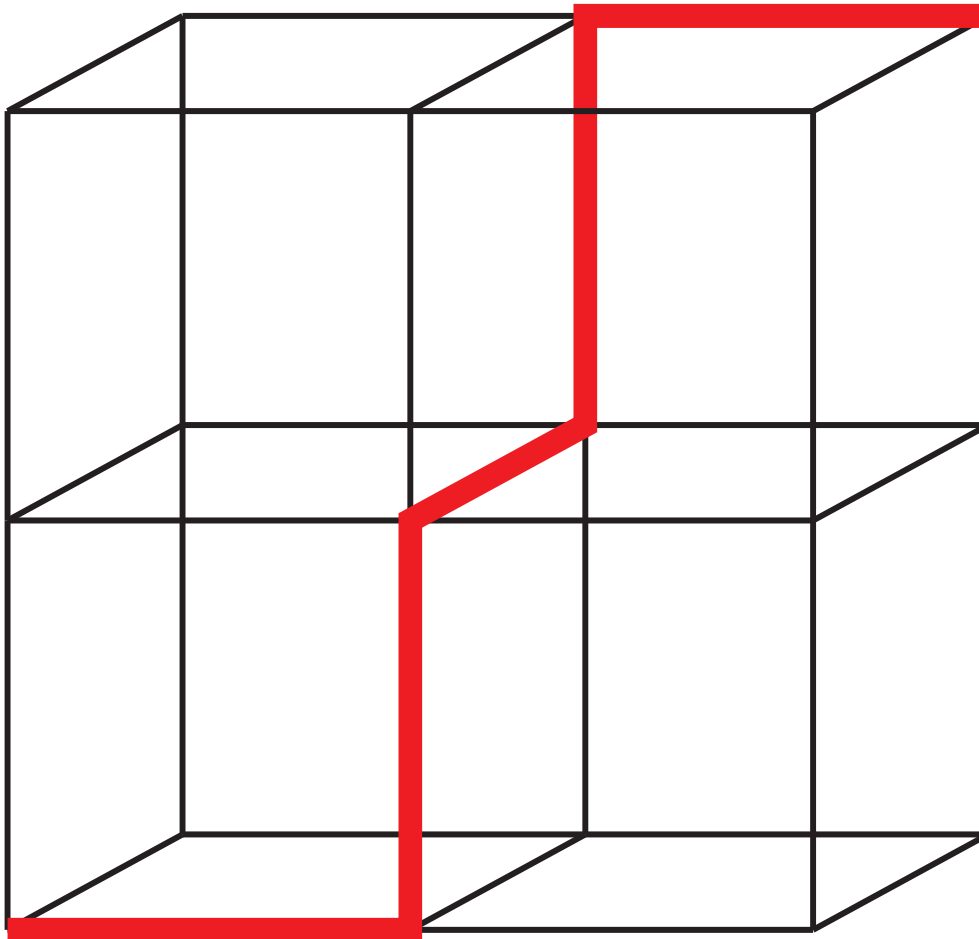
Enumeration

- 3 with finite planar faces
- 8 with finite skew faces
- 5 with zigzag faces
- 9 with helical faces

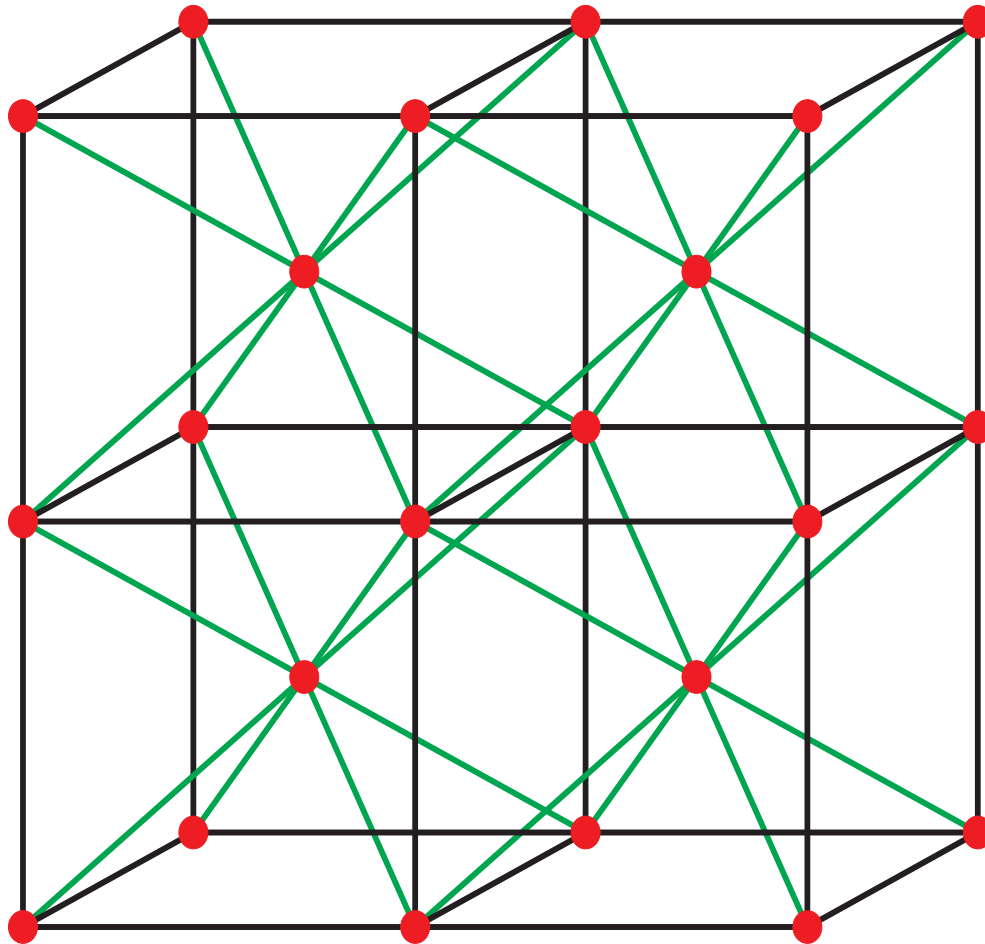
Enumeration



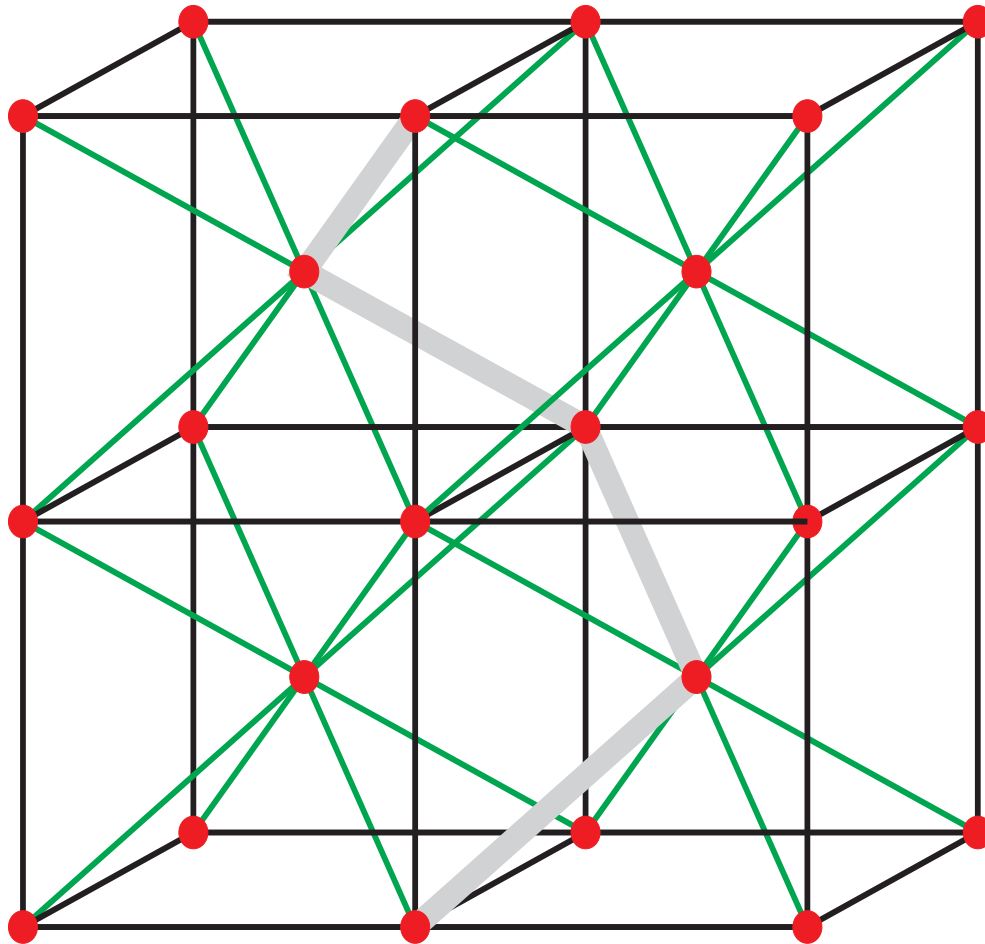
Enumeration



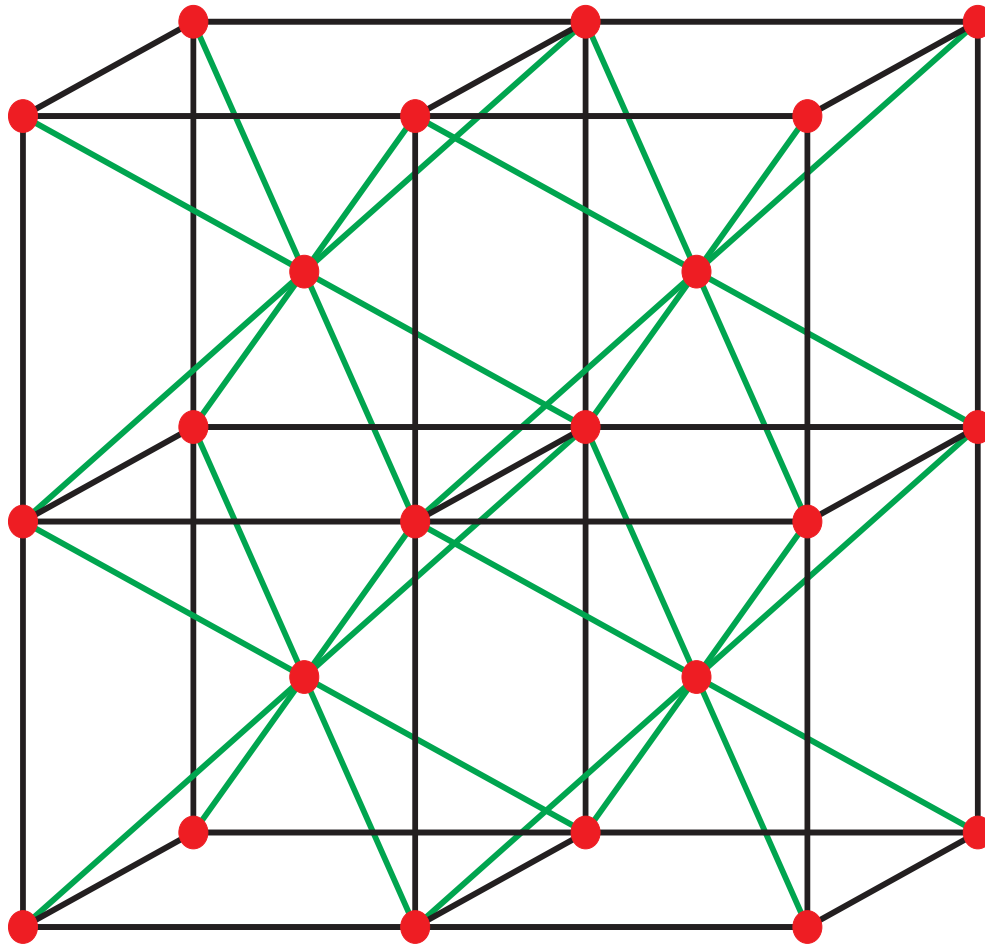
Enumeration



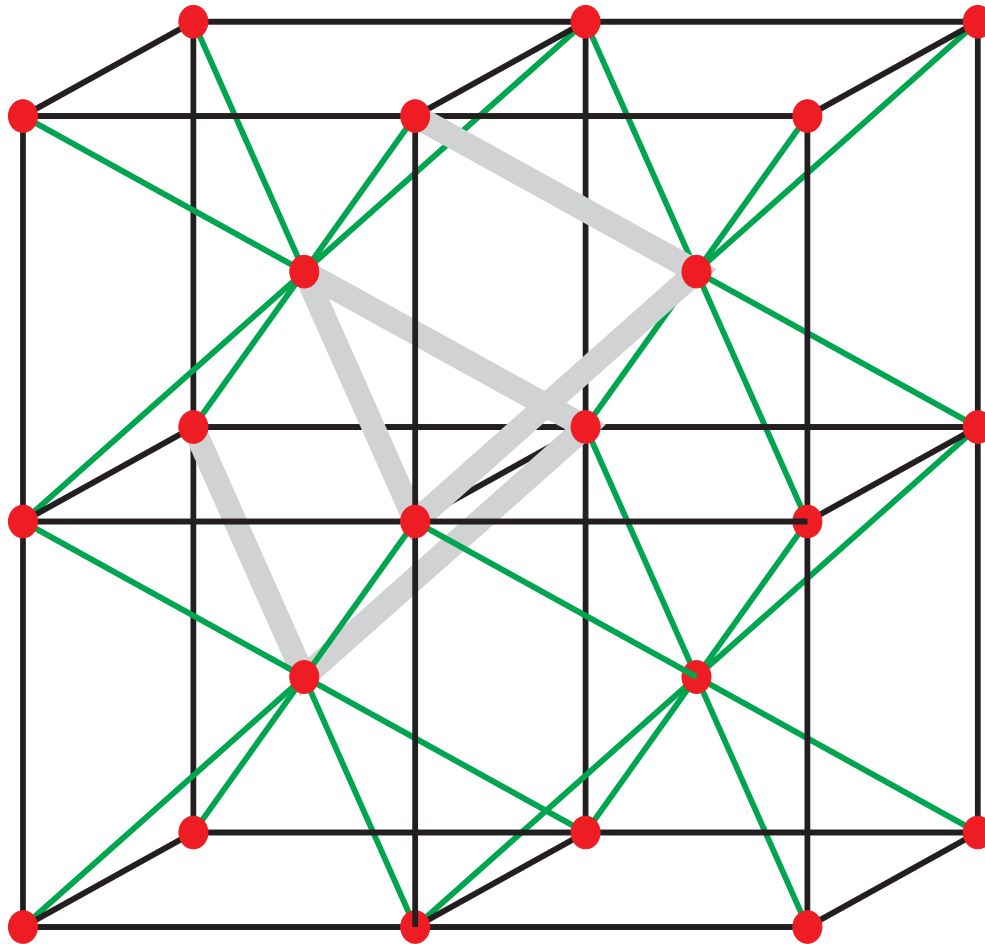
Enumeration



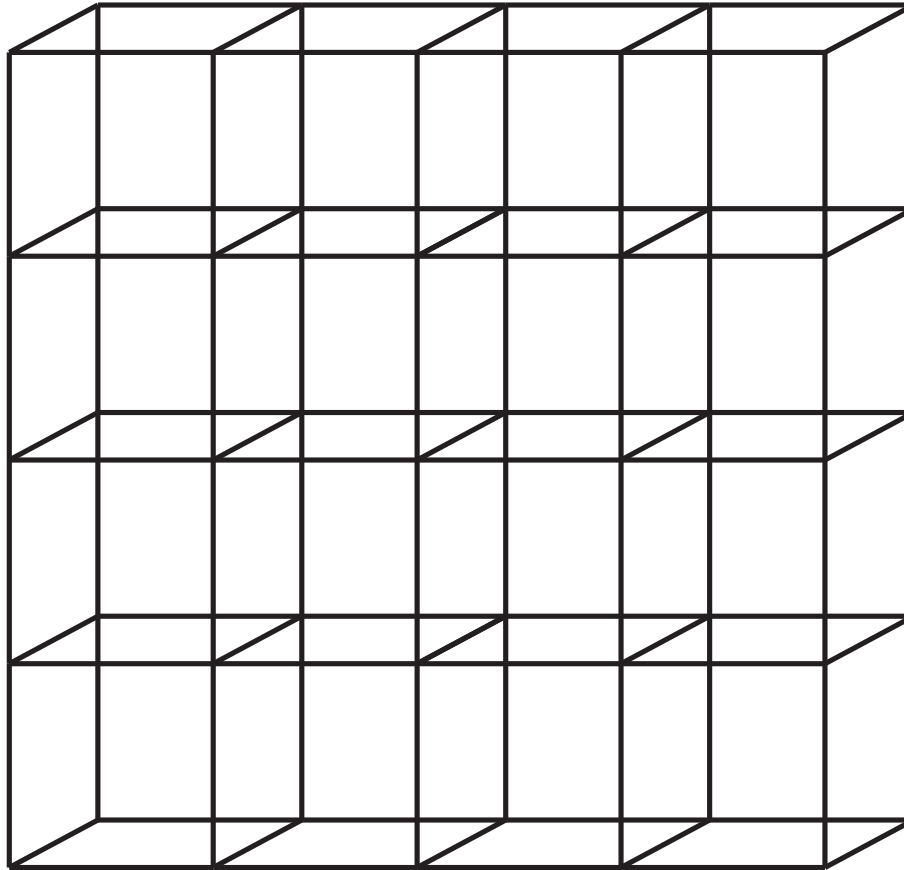
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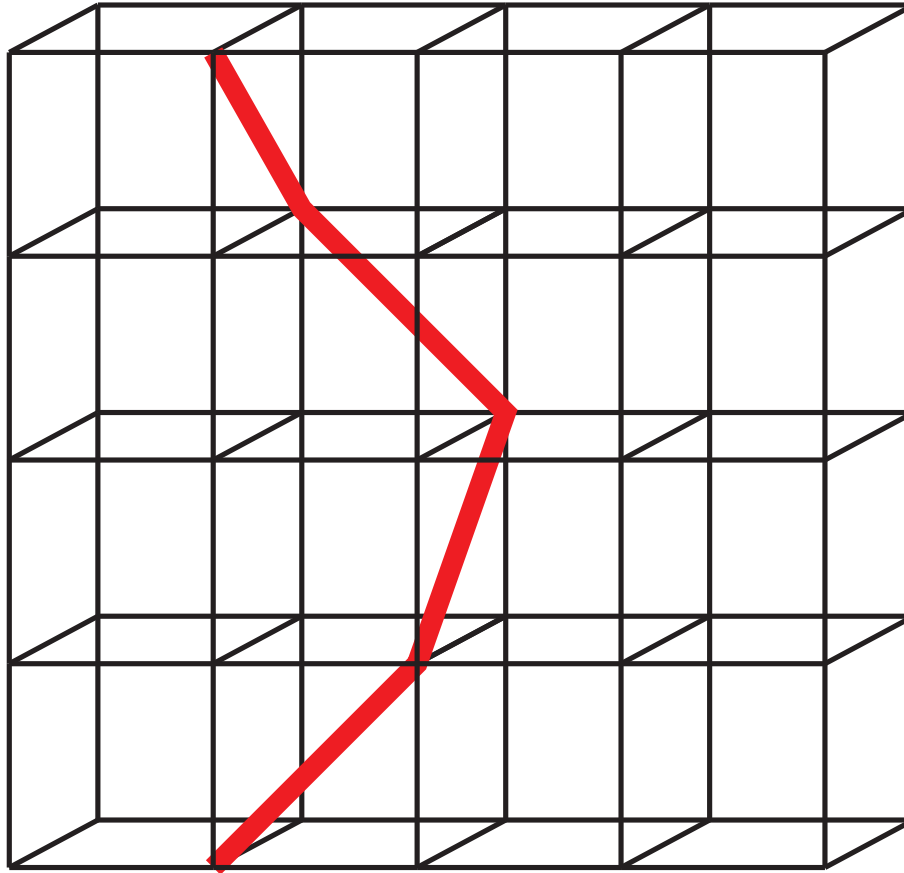
Enumeration



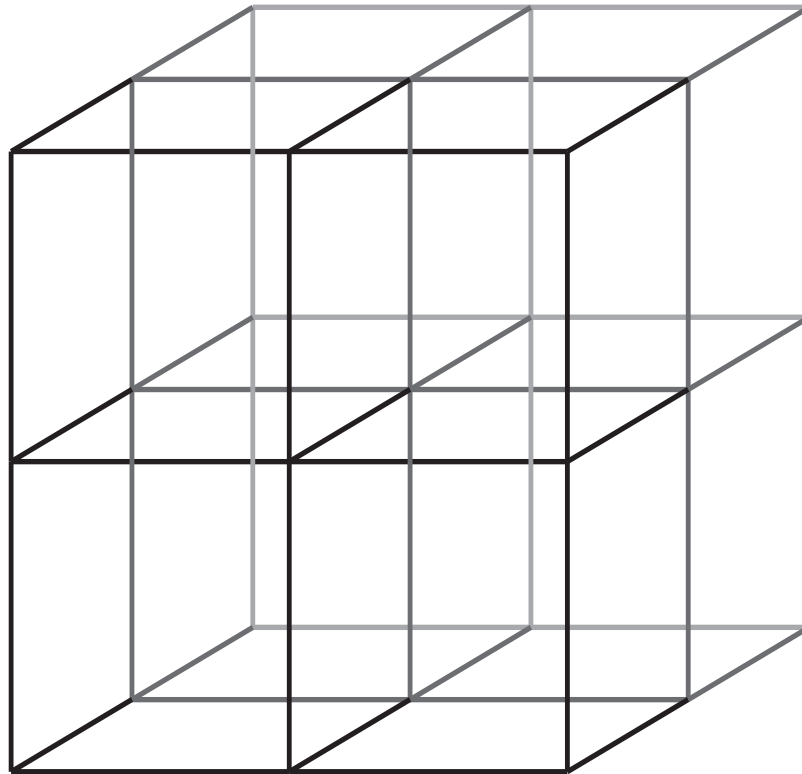
Enumeration



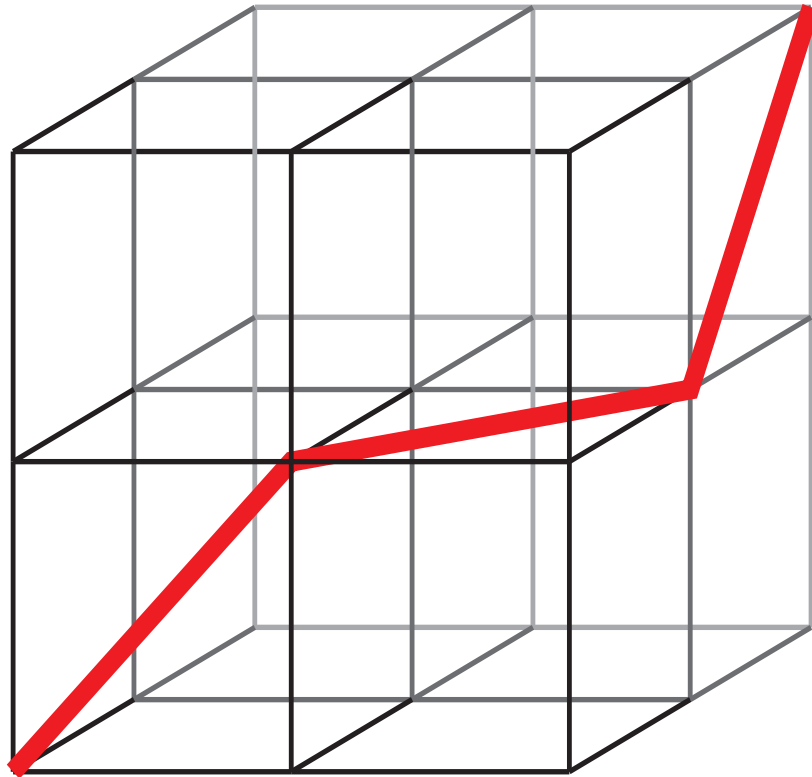
Enumeration



Enumeration



Enumeration





... E N D ...