Quadratic Solvability of Rigid Frameworks

Joint work with Bill Jackson

Motivation

A framework (G,p) defines a set of edge distances $d_G = d_1, \ldots d_E$ where $d_e = (p(u)-p(v))^2$ for uv in E(G).

Often we are given edge distances and wish to find some (or all) of the (G,p) which satisfy these edge distances – find the inverse of the edge map.

This requires us to solve the system of polynomial equations defined by G and the edge distances.

If G is minimally rigid we can consider generic distances. If G has more edges we must start with edge distances which are consistent. We use (G,p) to define a consistent set of edge distances and use these.

Equations are most easily understood using complex variables. Thus:

(Complex) Rigid Frameworks

Consider (complex) frameworks (G,p) in the plane.

A framework is a map from p:V(G) -> $\boldsymbol{C}^2$. Framework points are complex vectors $p(v)=(p(v)_x,p(v)_y)$ where $p(v)_x$ and $p(v)_y$ are complex numbers.

A framework defines a set of edge distances $d_G(p)=d_1,\ldots,d_E$ where $d_e=(p(u)_x-p(v)_x)^2+(p(u)_y-p(v)_y)^2$ for e=uv in E(G). d_e is a complex number.

Consider pinned (normal) frameworks which have $p(v_1)=(0,0)$ and $p(v_2)=(0,p(v_2)_y)$ for some v_1,v_2 in V(G). This takes care of (most) congruencies.

A framework (G,x) ~ (G,p) if $d_G(x)=d_G(p)$.

(G,p) is (complex) rigid if $W_p=\{(G,x) : d_G(x)=d_G(p)\}$ is finite.
(G,p) is (complex) globally rigid if $|W_p|=4$.

Quadratically solvable frameworks

A framework (G,p) defines an extension field [Q(p):Q]

(G,p) also defines an extension field $[Q(d_G(p)):Q]$

$Q(d_G(p))$ is contained in Q(p) so $[Q(p): Q(d_G(p))]$ is an extension of fields.

A framework (G,p) is quadratically solvable if $[Q(p): Q(d_G(p))]$ is a sequence of degree 2 (quadratic) extensions i.e. $\mathrm{degree}([Q(p): Q(d_G(p))])=2^s$

There is a related notion of radically solvable. Can also define the Galois group of (G,p).

Generic Properties

(G,p) is generic if p are algebraicall independent over Q.

(Complex) rigidity, global rigidity, quadratic solvability and radical solvability are all generic properties. This follows easily from the isomorphism of the algebraic closures of Q(p) and Q(p’) when p and p’ are generic.

(Complex) rigidity, global rigidity, quadratic solvability and radical solvability are all properties of a graph.

Real/Complex

- For any graph G:
- Real Rigidity = Complex Rigidity(for all d)
- Real Global Rigidity = Complex Global Rigidity (for d=2). Conjecture: true for all d.

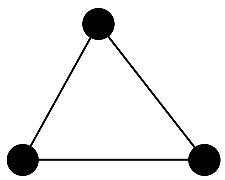
What does Quadratic Solvability mean?

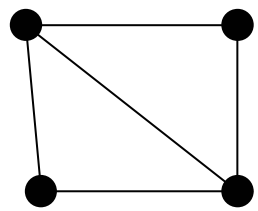
If a graph G is quadratically solvable we can start with *any* set of consistent edge distances and compute one (and usually all) of the frameworks (G,x) which satisfy these edge distances simply by solving quadratic equations. This is often a very desirable property. Many “practical” frameworks have this property.

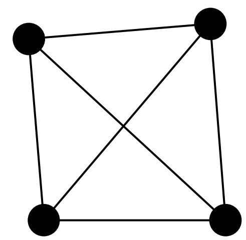
Which graphs are quadratically solvable?

Is there a good algorithm to compute (G,x)?

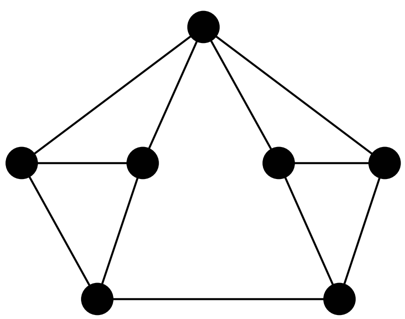
Quadratically solvable graphs

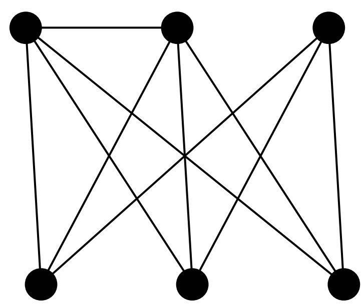


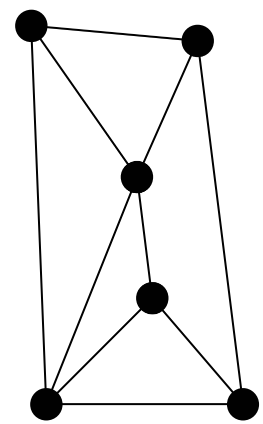




all K_n

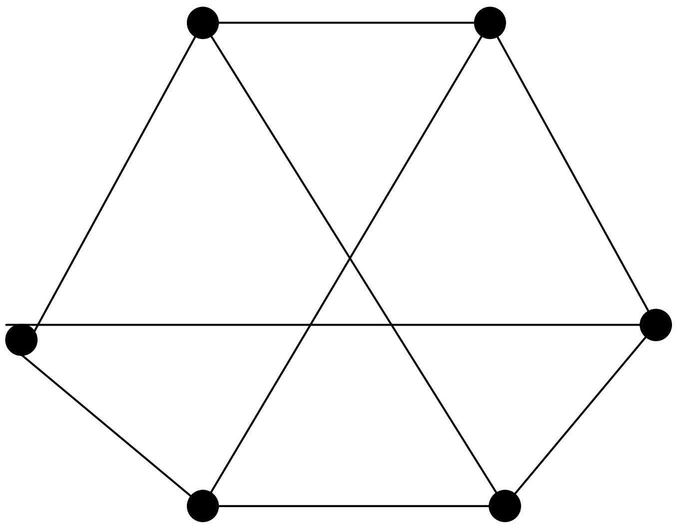


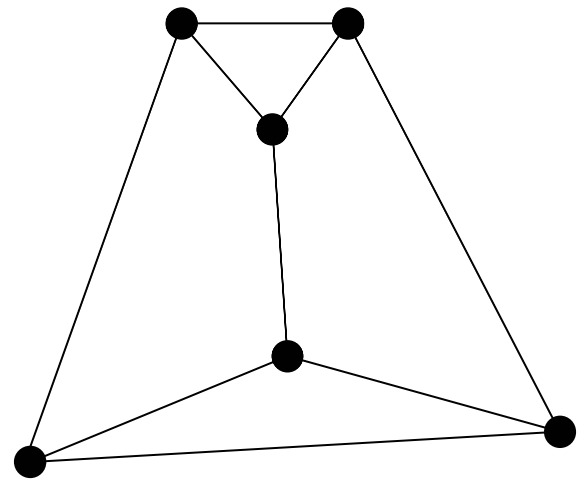


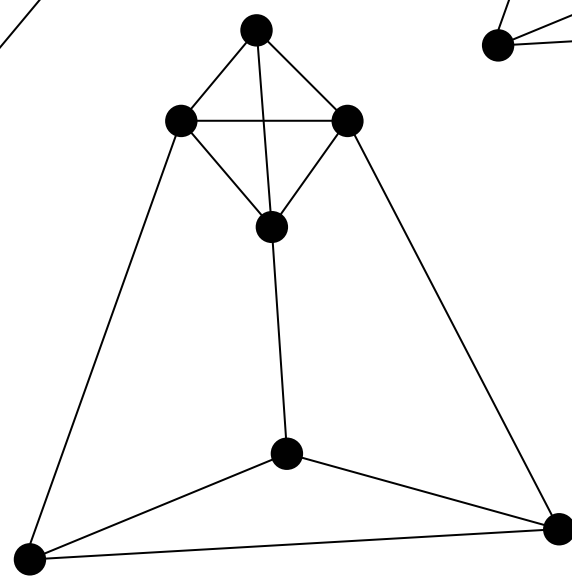


Not Quadratically Solvable Graphs

these are the only two with $|V(G)|<7$







This is the smallest graph which is not radically solvable and not isostatic

Globally Linked Vertices

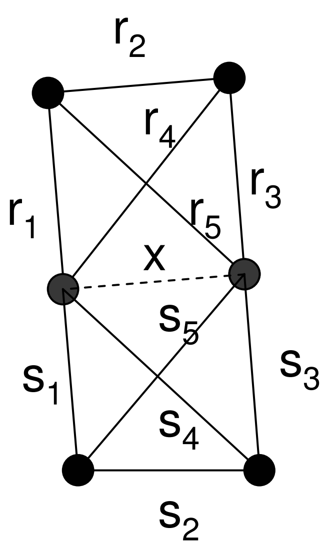
Vertices v_1 and v_2 in V(G) are Globally Linked in (G,p) if $d(q(v_1)-q(v_2)) = d(p(v_1)-p(v_2))$ for all (G,x) ~ (G,p).

(Complex) global linkedness is a generic property.

Theorem: If v_1 and v_2 are globally linked in (G,p) then $d(p(v_1)-p(v_2))$ is in $Q(d_G(p))$.

This means that $d(p(v_1)-p(v_2))$ is a rational function of $d_G(p)$.

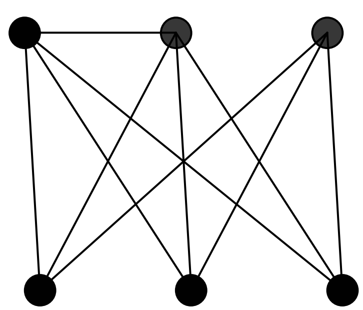
Example of Globally Linked Vertices



We have $a_r x^2+b_r x+c_r=0$ and
$a_s x^2+b_s x+c_s=0$
where a_r,b_r,c_r are polynomials in r_1,r_2,r_3,r_4,r_5
and a_s,b_s,c_s are polynomials in s_1,s_2,s_3,s_4,s_5

Therefore

$x = (a_r c_s-a_s c_r)/(a_s b_r-a_r b_s) = L(r_i,s_i)$ in $Q(r_i,s_i)$



Can we recognise globally linked pairs of vertices in any rigid graph G? Yes (in C^2).

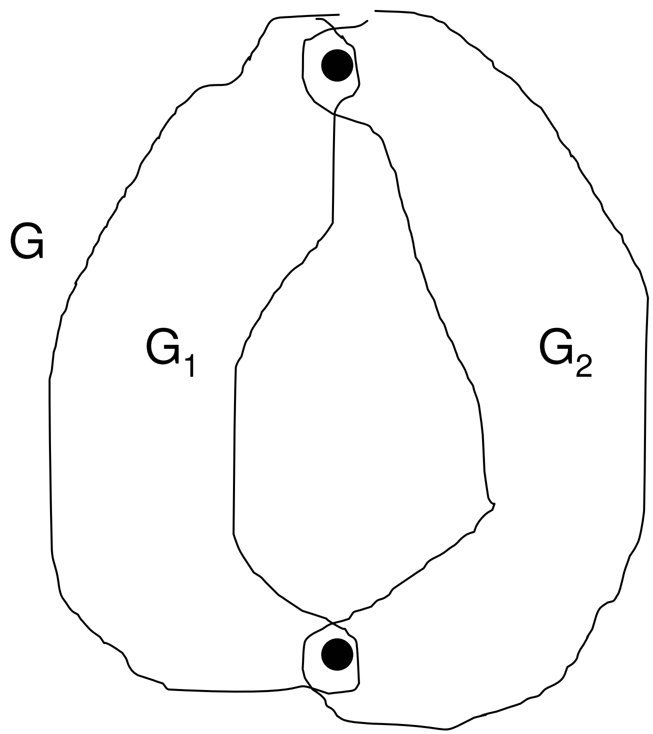
How do we find $L(d_G)$ for any globally linked pair of vertices in any G?????

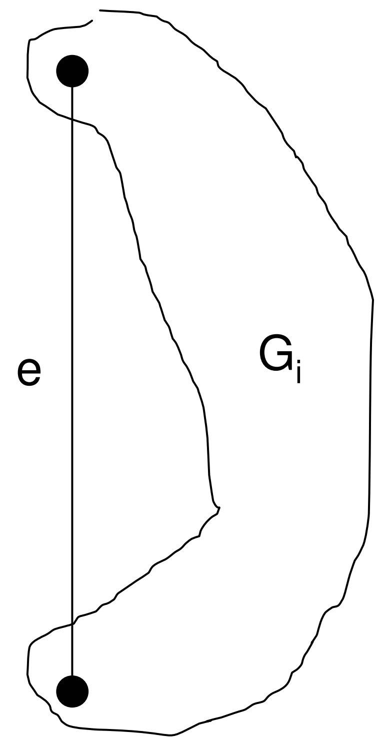
Global Rigidity

Theorem: If G is (complex) globally rigid then G is quadratically solvable.

Proof: Every pair of vertices is globally linked in G. K_n is quadratically solvable for all n.

Splitting at vertex 2-separations





$H_1 = G_1 + e$ if G_2 rigid else G_1
$H_2 = G_2 + e$ if G_1 rigid else G_2

G is quadratically solvable if and only if H_1 and H_2 are quadratically solvable

Theorem: If G is an ***RM-***connected graph then G is quadratically solvable

3-connected Graphs

We can determine if G is quadratically solvable by determining if its (recursively defined) 3-connected components are quadratically solvable.

We need consider only 3-connected graphs.

Conjecture : If G is isostatic and 3-connected then G is not quadratically solvable.

Proved (Owen and Power) when G is planar.

Rigid 3-connected Graphs

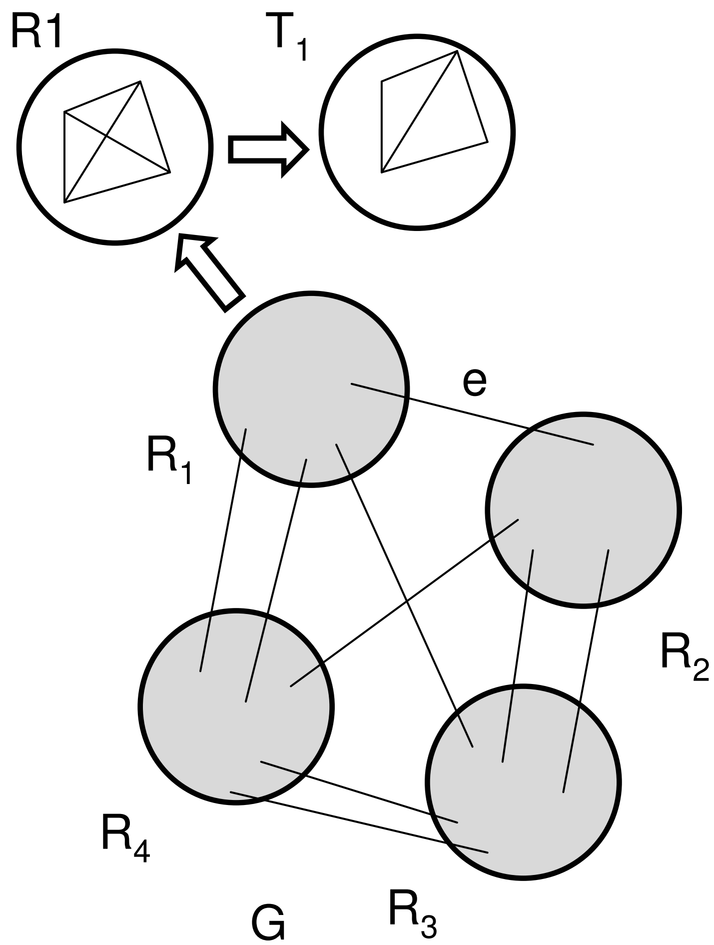
Theorem: The following are equivalent (for d=2):

(1) If G is isostatic and 3-connected then G is not quadratically solvable.

(2) If G is rigid and 3-connected then G is quadratically solvable if and only if it is globally rigid.

(3) If G is rigid and 3-connected then G is quadratically solvable if and only if it is redundantly rigid.

Sketch Proof



G is 3-connected and
not redundantly rigid

1.G\e is not rigid. G\e has rigid components R_i
2. Replace each R_i by T_i in G where
 a. T_i is isostatic and quadratically solvable.
 b. New graph G(R,T) is 3-connected.
3. Then G(R,T) is 3-connected and isostatic.
4. Can show: if G is quadratically solvable
 then
 any G(R,T) is quadratically solvable.
5. A contradiction: so G is not radically solvable
6. The only escape is for G to be redundantly rigid.

Summary and Problems

1. For $d=2$: Conjecture: If G is rigid and 3-connected then G is quadratically solvable if and only if G is globally (or redundantly) rigid.

   Proved for G planar. Prove for G non-planar.

2. For $d>2$ Conjecture: If G is rigid and $(d+1)$-connected then G is quadratically solvable if and only if G is globally (or redundantly) rigid.

   Conjecture: Every G which is radically solvable is also quadratically solvable.

3. If v_1 and v_2 are globally linked in (G,p) generate (in polynomial time) the formula L for $d(p(v_1)-p(v_2))$.

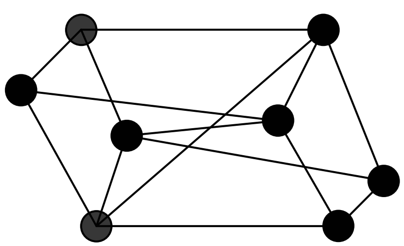
Specialisations

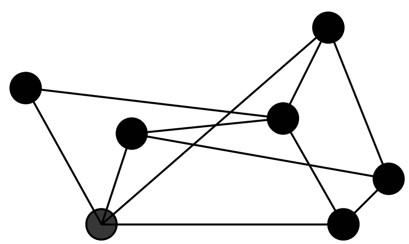
Specialisation conjecture:

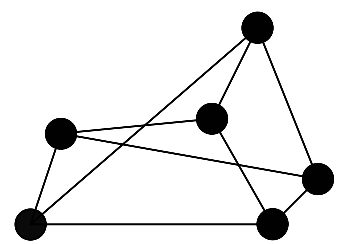
If G is isostatic and quadratically solvable and a specialisation (G,p’) is rigid then (G,p’) is quadratically solvable.

Generally: Galois(G,p’) is a subgroup of Galois(G).

Note: This may fail if G is not isostatic. True for K_n







Quadratically solvable

Not Quadratically solvable, K(,3,3) subgraph