Cubical 4−Twistoids: II

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The Fields Institute

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with Isabel Hubard, Daniel Pellicer, Asia Ivić Weiss

Outline

- Introduction to Twistoids
- Twistoids from the group 333 $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+$
- Twistoids from the group 22 $*\ \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+$

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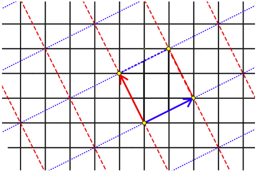
  - In this talk $\mathcal{U}$ is always the tessellation by cubes
- Let G be a subgroup of $Aut(\mathcal{U})$.
  - G has no fixed points.
- Twistoid $\mathcal{T} = \mathcal{U}/G$ has faces which are orbits of faces of $\mathcal{U}$ under the action of G

Example in 2-d



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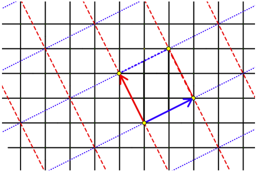

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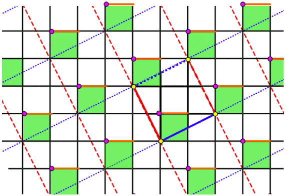

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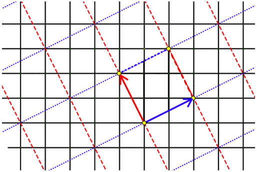

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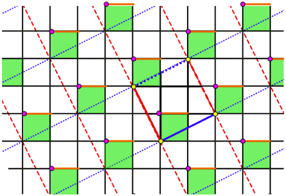

- In general, topologically, this quotient gives us a flat riemannian manifold.

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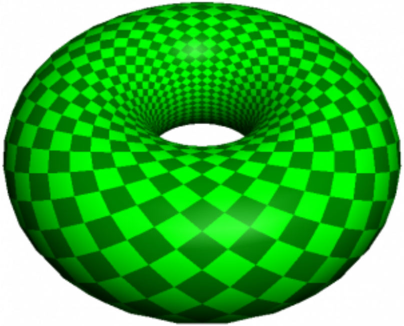

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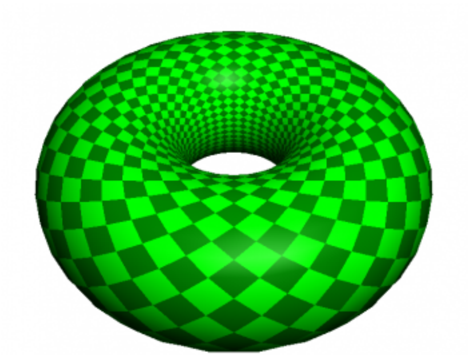
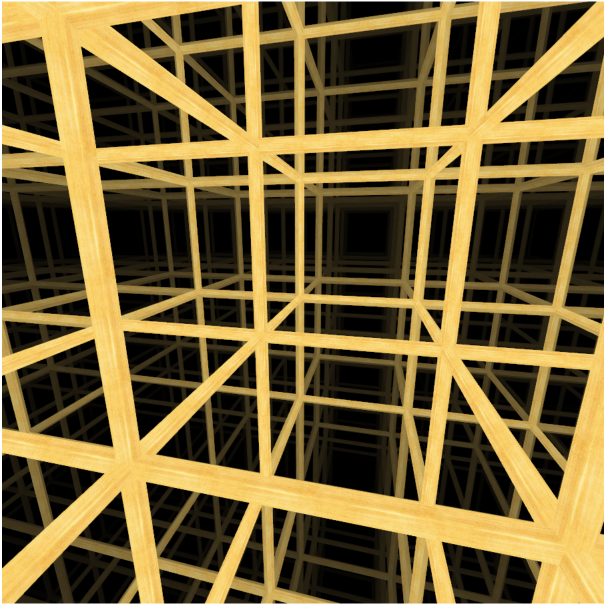


- Maps on the torus and Klein bottle are 3-twistoids.

Symmetries of $\{4, 3, 4\}$



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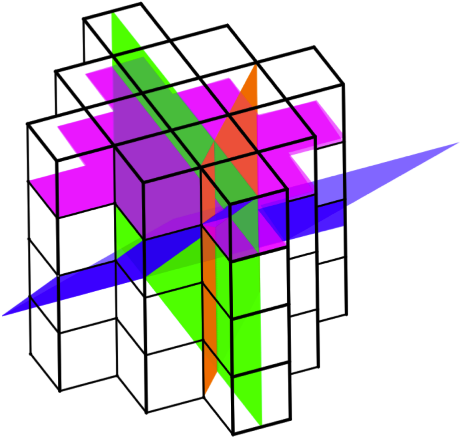

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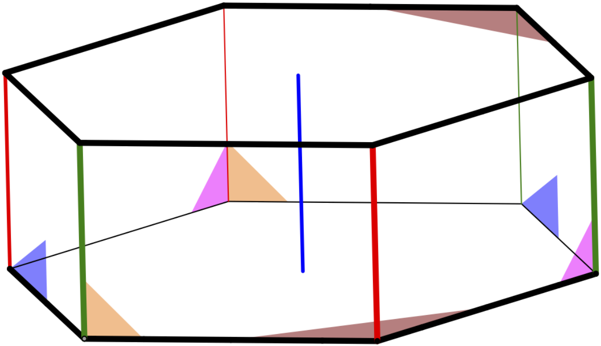
    $\Leftrightarrow h(F)$ is in the same orbit as $h(g(F))$ for all $g \in G$ and F in $\mathcal{F}(U)$.
    $\Leftrightarrow g'h = hg$
    $\Leftrightarrow hgh^{-1} \in G$
- We are looking for symmetries of the underlying tessellation which normalize the group that we are using for the quotient.
- $Aut(\mathcal{T}) = Sym(\mathcal{T})/G$

Twistoids from the group $333\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+$



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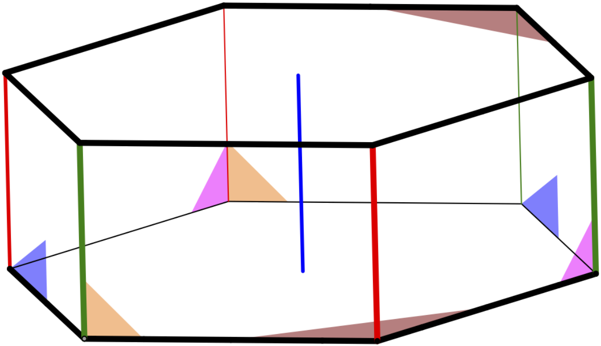

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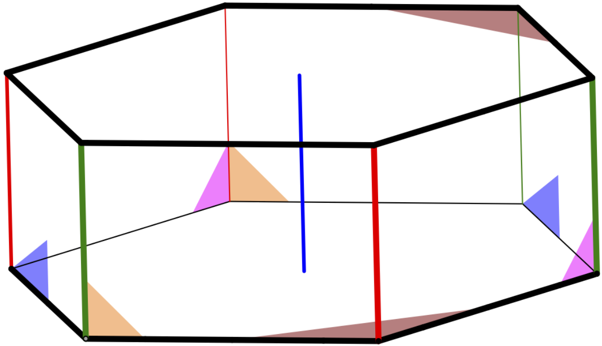

- G is generated by two 3-fold twists (screw motions) in parallel axis with equal translational component.
  - $\alpha\beta^{-2} = \gamma^{-1}$
- How can we place this fundamental region into a fixed cubical lattice so that G is a subgroup of the lattice's automorphisms?

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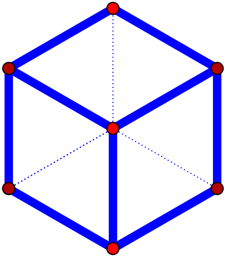
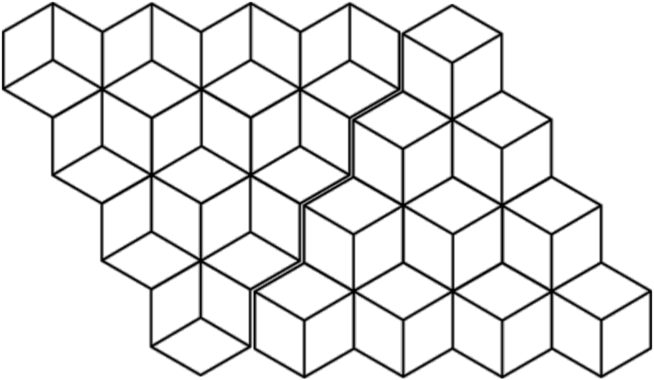


Projecting onto a plane perpendicular to twist axis



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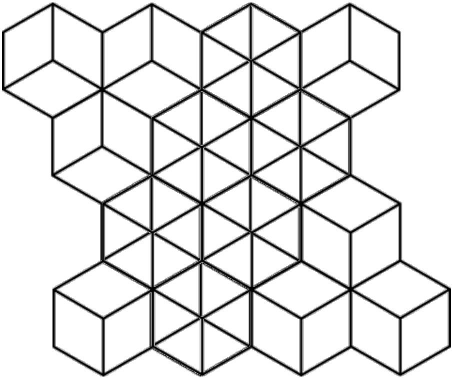

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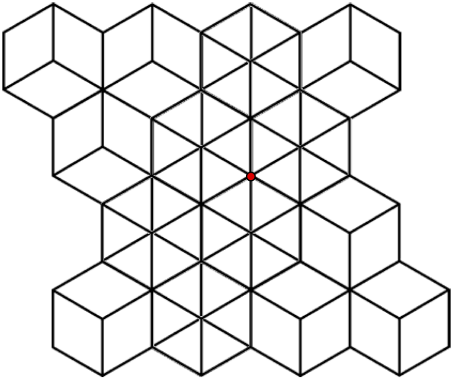

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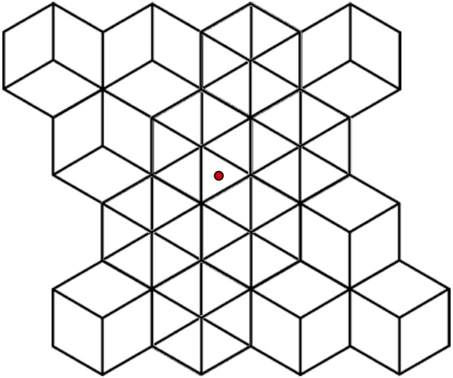
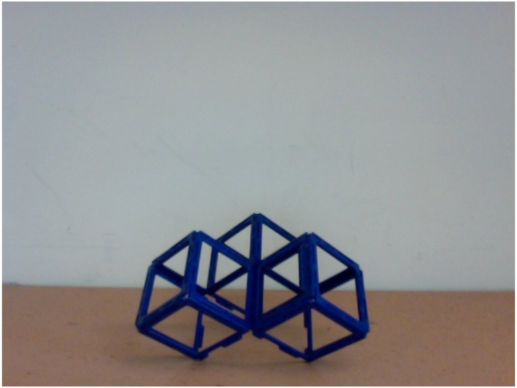


Example of cubes twisting through petrie motion



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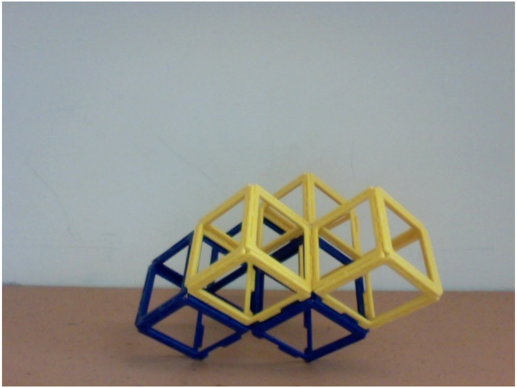

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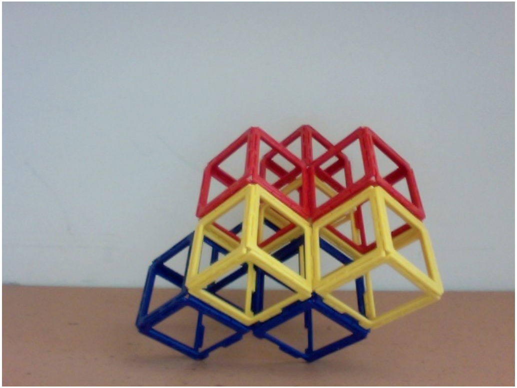

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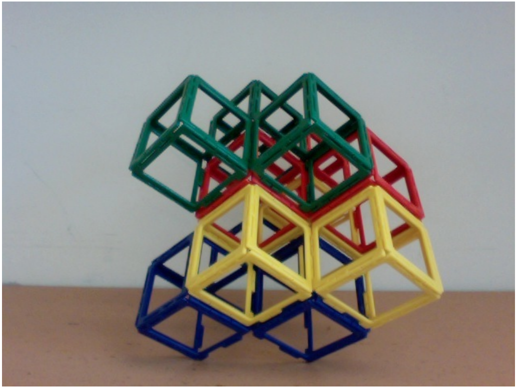

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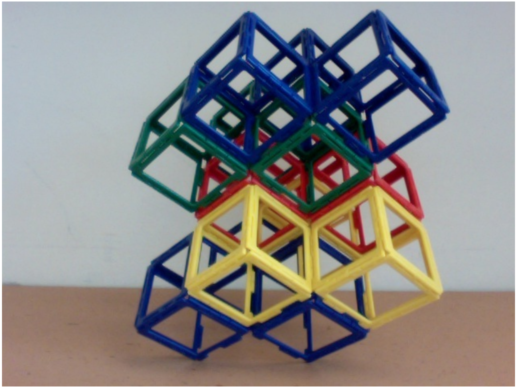
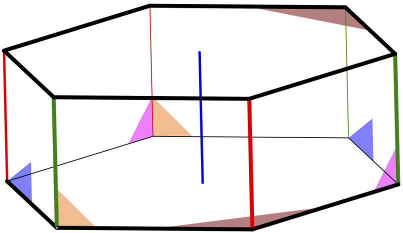


Placing a fundamental region for $333\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+$



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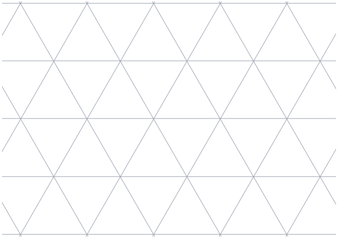

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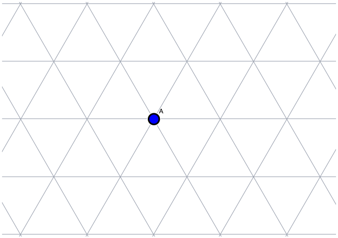

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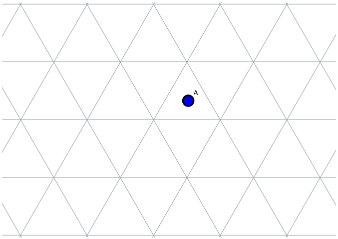

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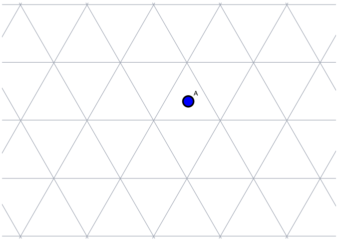

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- Thus you either have your twists all in diagonals of cubes, or all in axis of petrie motions.

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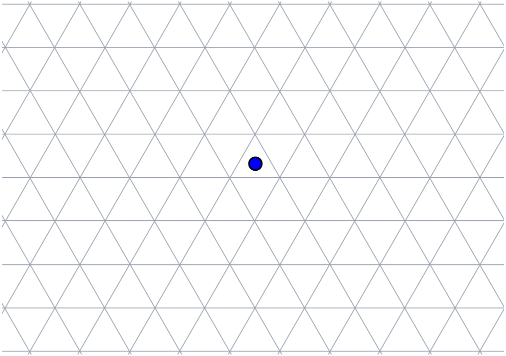

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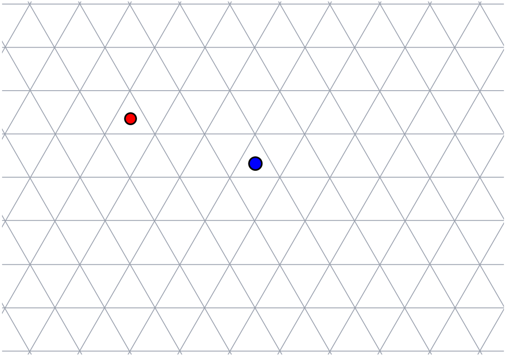

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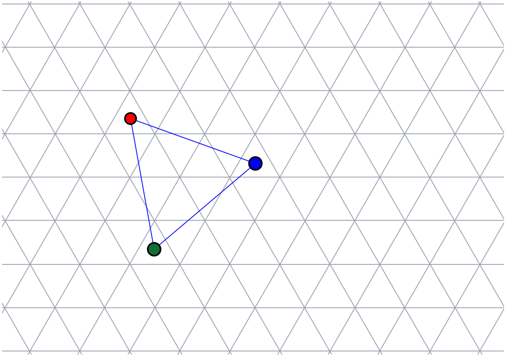

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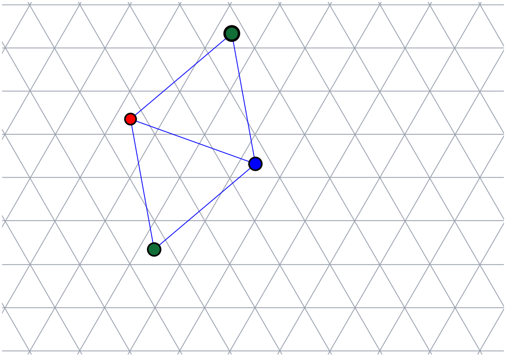

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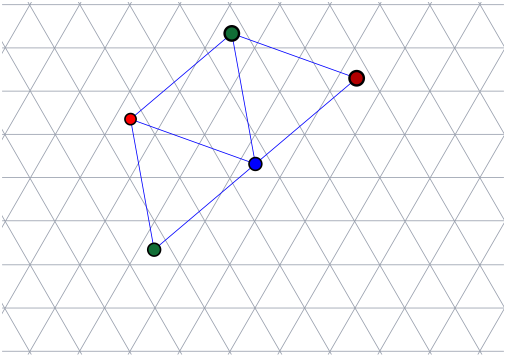

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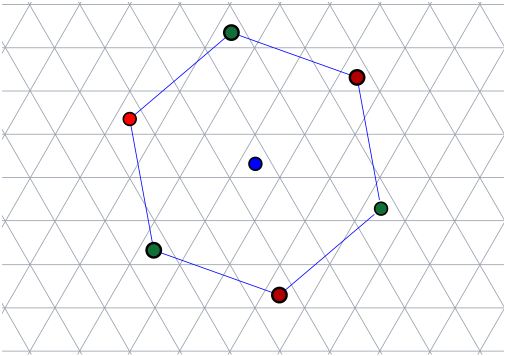

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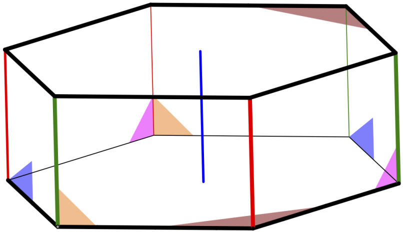

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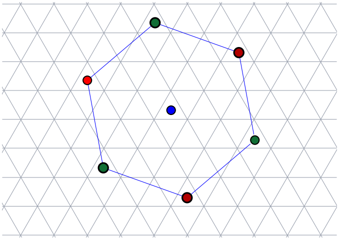

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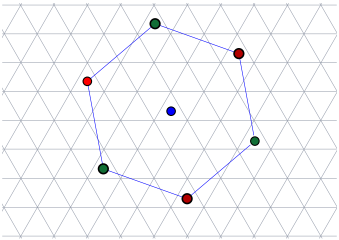

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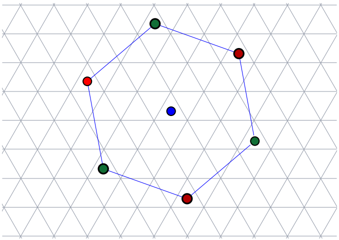

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  - A vector (a, b) which determines the position of the fundamental region in the projection.

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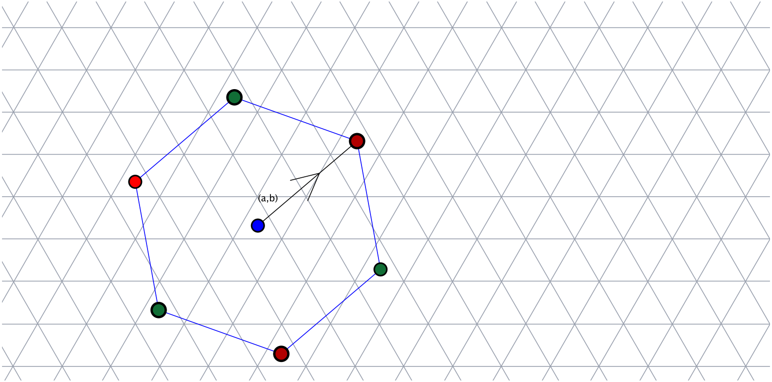
- The parameters also determine a map in the projection, and from the symmetries of this map we can determine the symmetries of the Twistoid.



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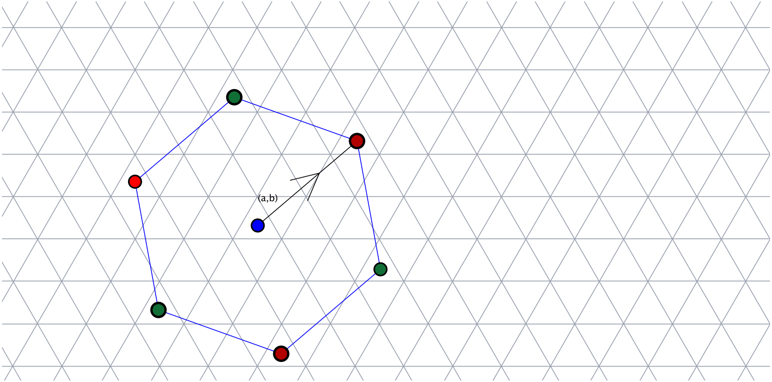

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- Translations in $\mathrm{Aut}(\mathcal{T})$ reduce the possible number of flag orbits to $16(a^2 + b^2 + ab)$
- If the map in the projection is regular then there are $8(a^2 + b^2 + ab)$ in the petrie axis case, and $4(a^2 + b^2 + ab)$ when the axis of twists lie on diagonals of cubes.

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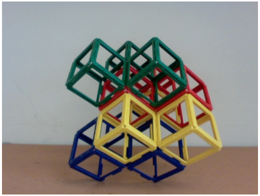

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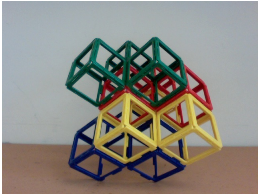

  - has 24 orbits on flags.
- $3 < k \equiv 0 \pmod 3$, $(a, b) = (1, 1)$ gives a polytopal example with 12 flag orbits.

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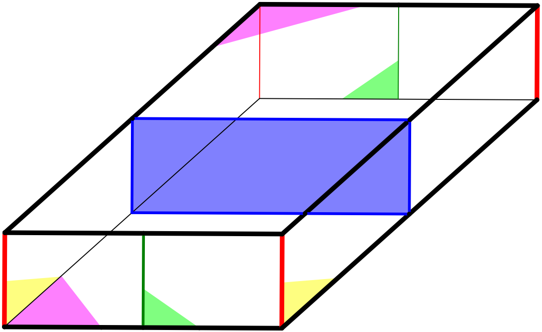

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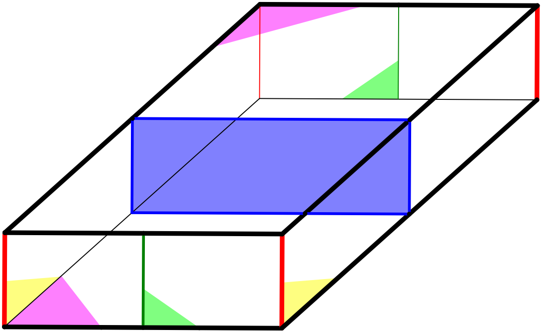

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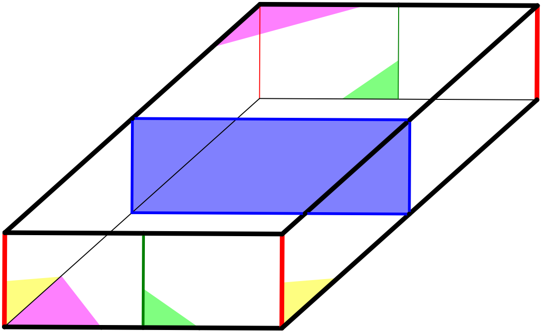

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- How can we place this fundamental region into a fixed cubical lattice so that G is a subgroup of the lattice's automorphisms?

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    - We treat both cases separately, and I will only mention the first case.

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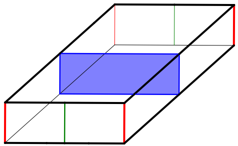
- There are $192kad$ flags in the twistoid.

Symmetries of a Twistoid



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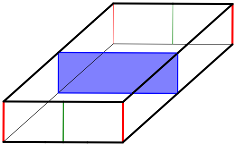

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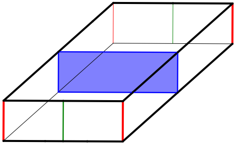

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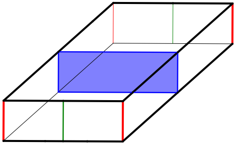

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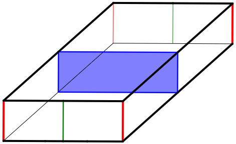

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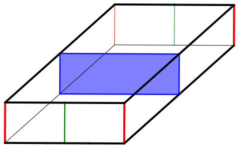

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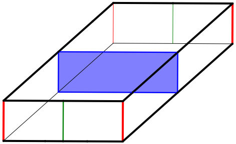

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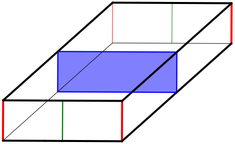

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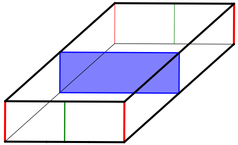

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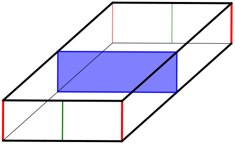

- In the non-orientable setting, strange things happen!
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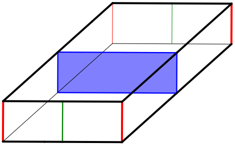
  - reflection in the same wall as the generating glide refection
  - half turn in the axis of the red twist
  - half turn in the axis of the green twist

Symmetries of a Twistoid



- There are two other types of automorphisms of the twistoid that are not induced by translations of the tessellation.
  - half turn with axis parallel to twists, in the center of the fundamental region
  - half turn with axis perpendicular to twists, in the center of the "front" of the fundamental region.
- Thus the automorphism group of the twistoid can have at most 16k elements.
- From $192kad$ flags, we get $12ad$ flag orbits.

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The End