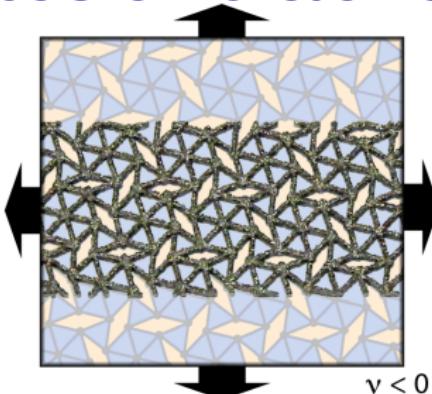


# Floppy Frameworks as Models for Auxetic Materials



Holger Mitschke<sup>1</sup>, Jan Schwerdtfeger<sup>2</sup>, Vanessa Robins<sup>3</sup>, Carolin Körner<sup>2</sup>,  
Robert F. Singer<sup>2</sup>, Klaus Mecke<sup>1</sup>, Gerd Schröder-Turk<sup>1</sup>

<sup>1</sup>Theoretische Physik,  
Friedrich-Alexander Universität, Erlangen-Nürnberg

<sup>2</sup>Institute of Advanced Materials and Processes,  
Friedrich-Alexander Universität, Erlangen-Nürnberg

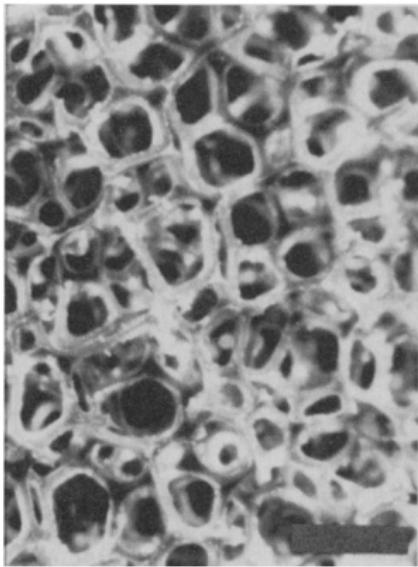
<sup>3</sup>Applied Maths, Research School of Physical Sciences & Engineering,  
Australian National University, Canberra

# Auxetic Deformation in Polymer Foams

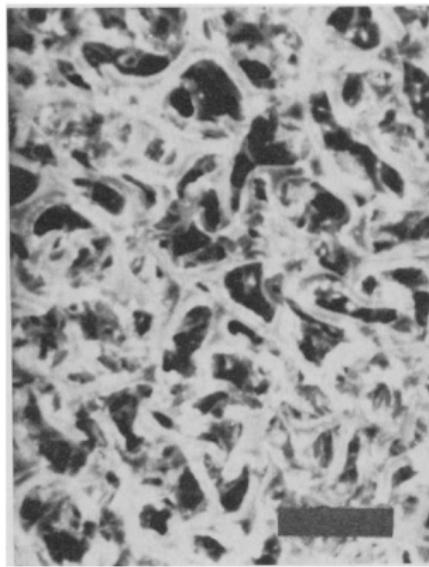
Negative Poisson's ratio due to changes in geometry

$$\text{Poisson's ratio } \nu = -\frac{\text{transversal strain}}{\text{longitudinal strain}}$$

$\nu > 0$



$\nu < 0$



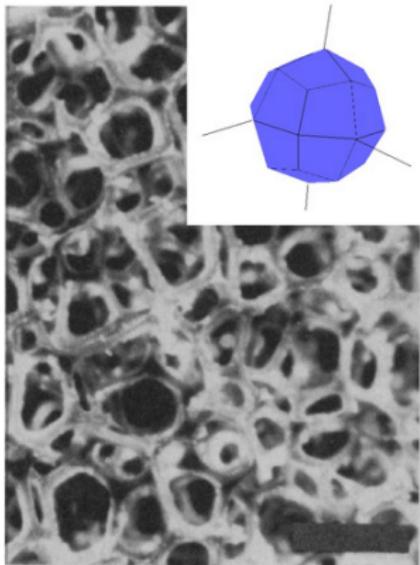
(R. Lakes, *Science*, 1987)

# Auxetic Deformation in Polymer Foams

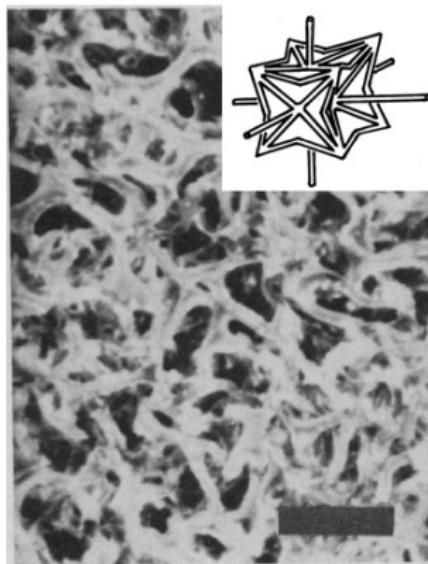
Negative Poisson's ratio due to changes in geometry

$$\text{Poisson's ratio } \nu = -\frac{\text{transversal strain}}{\text{longitudinal strain}}$$

$$\nu > 0$$



$$\nu < 0$$



(R. Lakes, *Science*, 1987)

# Cellular Solids

Examples: wood, cork, sponge, coral, solid foam

“An assembly of cells with solid edges or faces,  
packed so that they fill space”

Elastic regime:

linear stress-strain-relation for solid  
material:



$$\varepsilon_i = \sum_{j=1}^6 S_{ij} \sigma_j$$

BUT

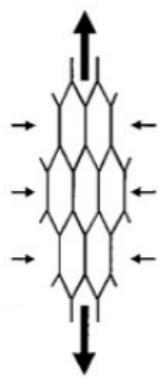
Relation of effective moduli to  
geometry complex!

Methods:

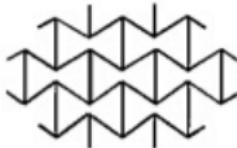
- ▶ Homogenization techniques
- ▶ Numerically: FEM-Simulation (PDE's)

# Inverted Honeycomb Pattern

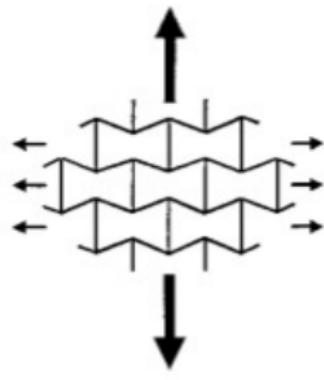
Most common picture of an auxetic framework



$$\nu > 0$$

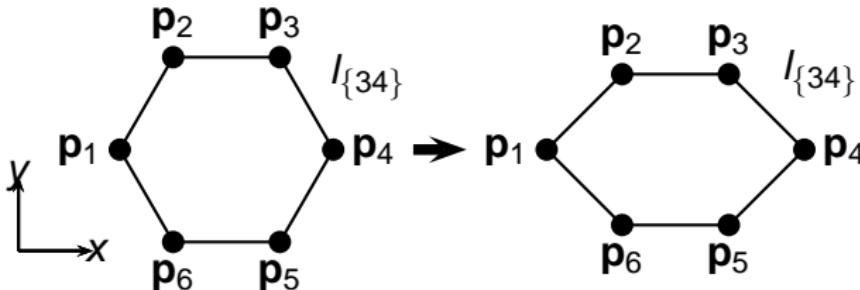


$$\nu < 0$$



# Framework

Pin-joint-and-bar framework



Edges are inextensible, incompressible rods which are joined but rotate freely at the vertices.

## Deformation of a framework:

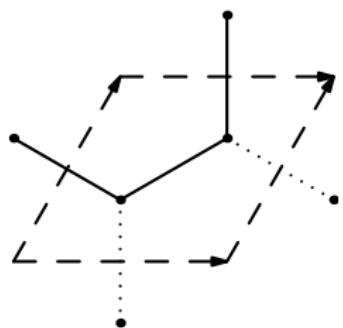
Continuous 1-parameter family  $\mathbf{P}(\delta) = (\mathbf{p}_1(\delta), \dots, \mathbf{p}_N(\delta))$  with

$$\blacktriangleright |\mathbf{p}_i(\delta) - \mathbf{p}_j(\delta)|^2 - I_{\{ij\}}^2 = 0 \quad \forall \{i, j\} \in B$$

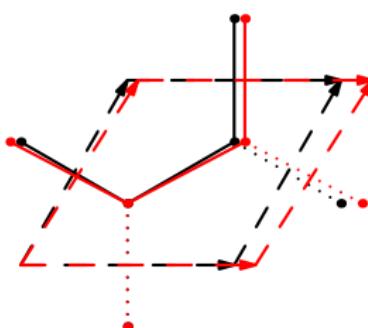
Numerical solution by Newton's method

# Numerical calculation of affine deformations

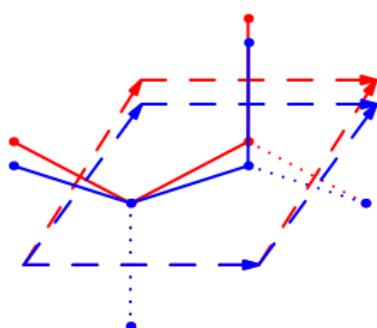
Initial configuration



Affine uniaxially deformed initial configuration



Calculated configuration by Newton method



Defines edge equations

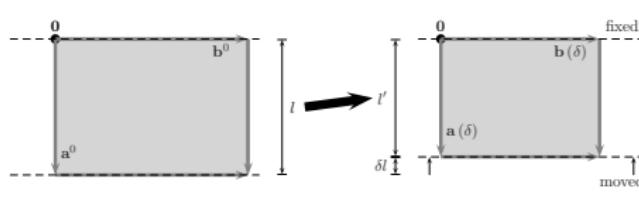
Edge equations violated

Edge equations fulfilled

- ▶ Multi-dimensional Newton-Raphson method
- ▶ Singular Value Decomposition

# Deformation of unit cell and its Poisson's ratio $\nu_{\text{periodic F.}}$

Deformation of a rectangular unit cell  
with constraint  $\mathbf{a}(\delta) \cdot \mathbf{b}(\delta) = 0$



Imposed deformation:

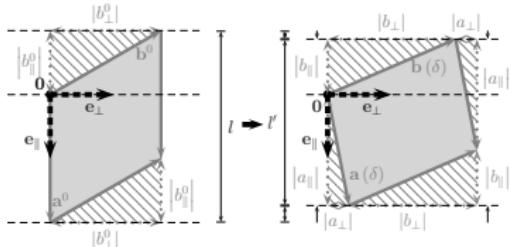
$$\mathbf{a}(\delta) = (1 + \delta) \mathbf{a}^0$$

Poisson's ratio:

$$\nu(\delta) = - \frac{\frac{|\mathbf{b}|(\delta) - |\mathbf{b}^0|}{|\mathbf{b}^0|}}{\frac{|\mathbf{a}|(\delta) - |\mathbf{a}^0|}{|\mathbf{a}^0|}} = - \frac{X}{\delta}$$

with  $\mathbf{b}(\delta) = (1 + X) \mathbf{b}^0$

Deformation of an oblique unit cell  
without any constraints



Imposed deformation:

$$\mathbf{a}(\delta) = a_{\parallel}^0 (1 + \delta) \mathbf{e}_{\parallel} + a_{\perp}^0 (\delta) \mathbf{e}_{\perp}$$

$$\mathbf{b}(\delta) = b_{\parallel}^0 (1 + \delta) \mathbf{e}_{\parallel} + b_{\perp}^0 (\delta) \mathbf{e}_{\perp}$$

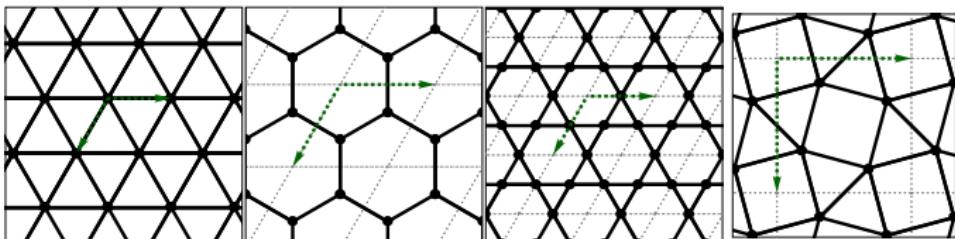
Poisson's ratio:

$$\nu(\delta) = - \frac{\frac{|a_{\perp}(\delta)| + |b_{\perp}(\delta)| - |a_{\perp}^0| - |b_{\perp}^0|}{|a_{\perp}^0| + |b_{\perp}^0|}}{\frac{|a_{\parallel}(\delta)| + |b_{\parallel}(\delta)| - |a_{\parallel}^0| - |b_{\parallel}^0|}{|a_{\parallel}^0| + |b_{\parallel}^0|}}$$

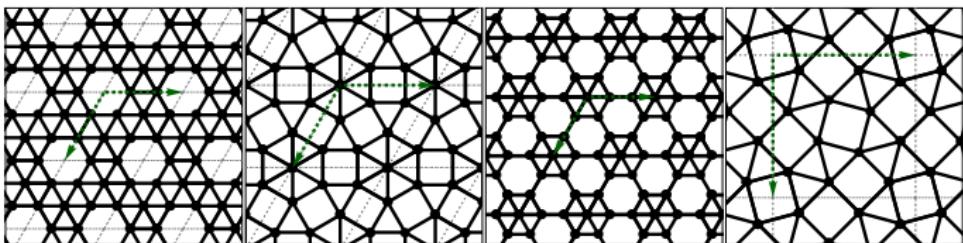
# Repositories of frameworks: **Tilings**

two-dimensional:

- ▶ 1-uniform



- ▶ 2-uniform

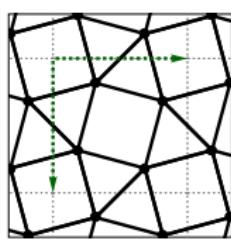
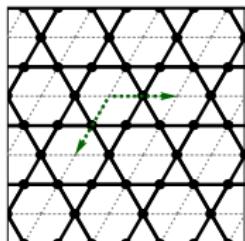
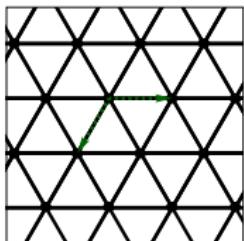


- ▶ :
- ▶ n-uniform

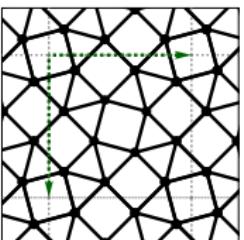
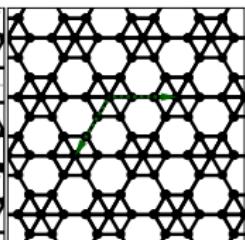
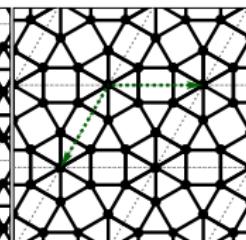
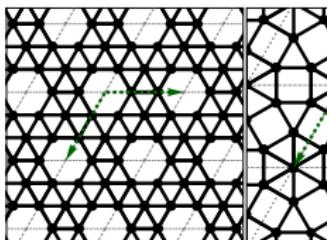
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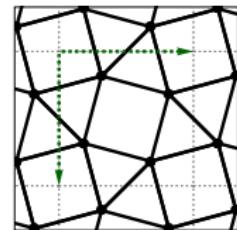
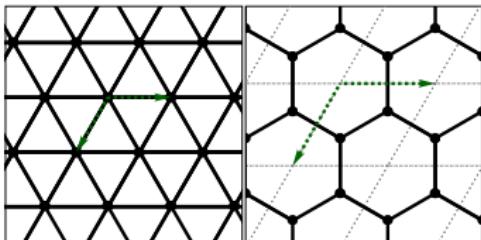
⋮

- ▶ n-uniform

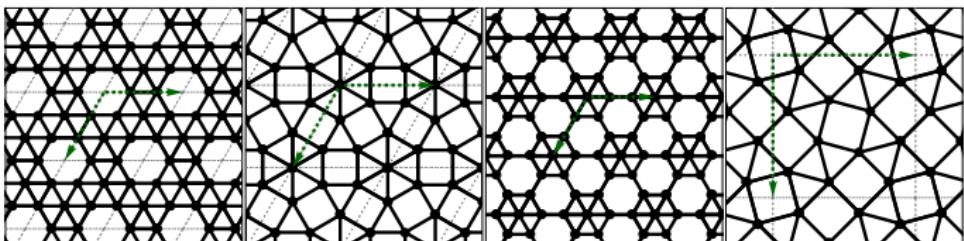
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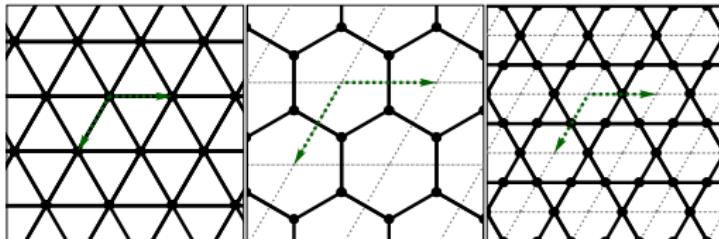


- ▶ :
- ▶ n-uniform

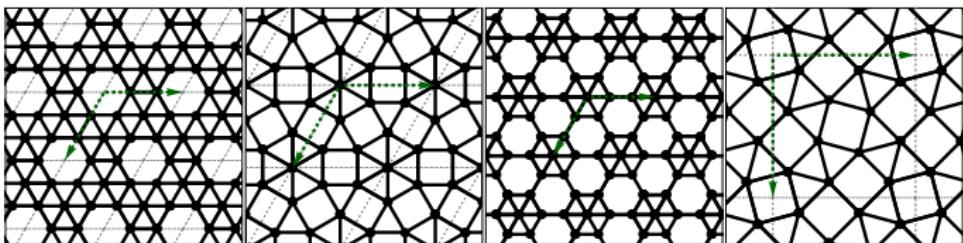
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two-dimensional:

- ▶ 1-uniform



- ▶ 2-uniform

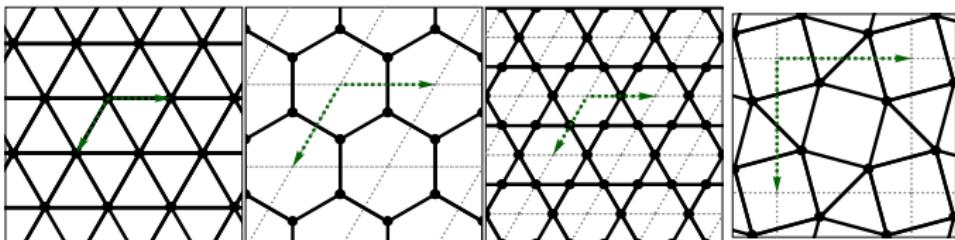


- ▶ :
- ▶ n-uniform

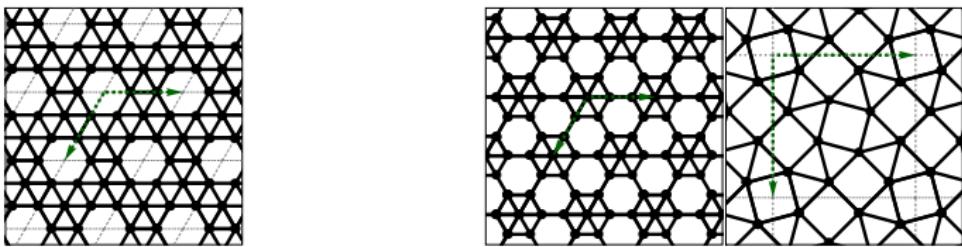
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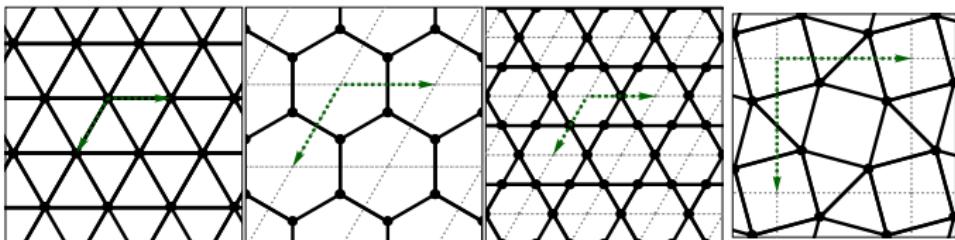


- ▶ :
- ▶ n-uniform

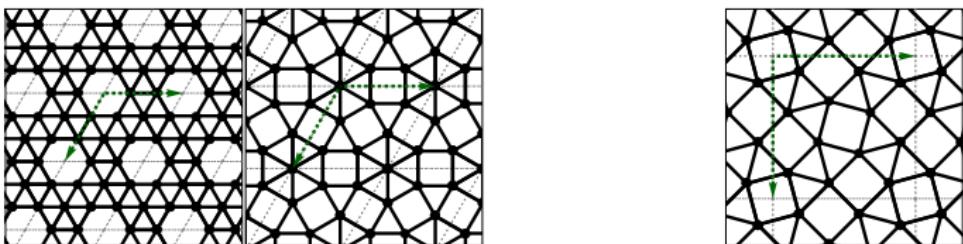
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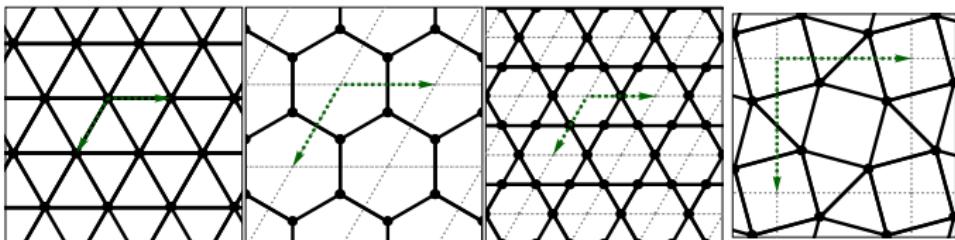


- ▶ :
- ▶ n-uniform

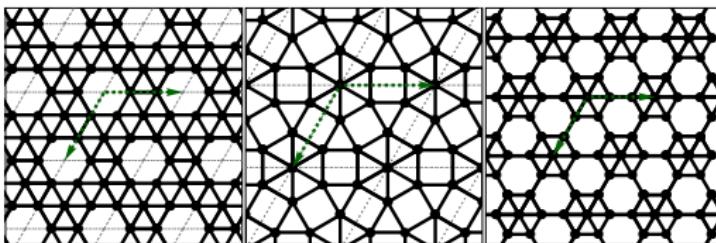
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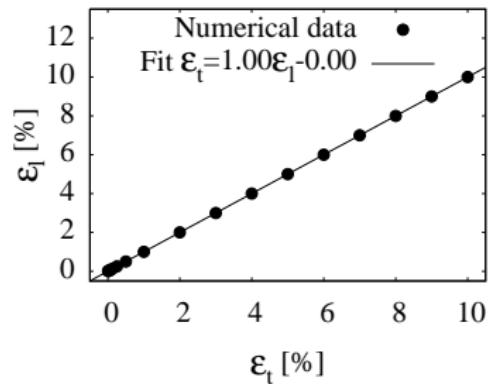
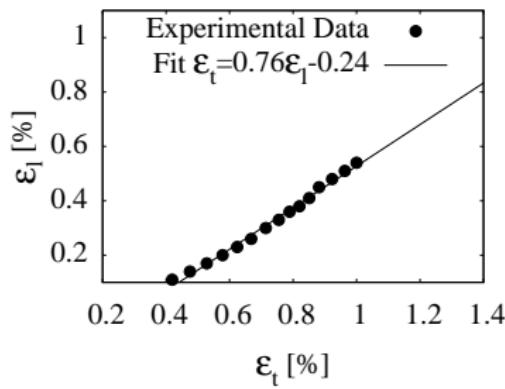
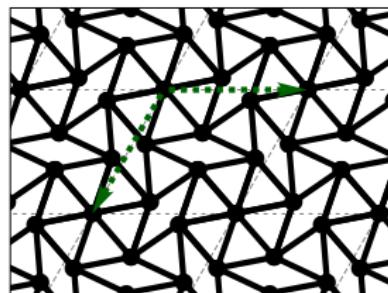


- ▶ 2-uniform



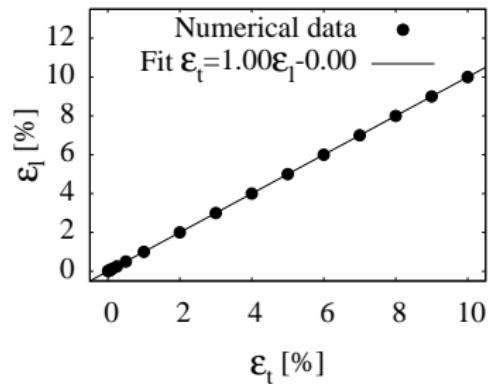
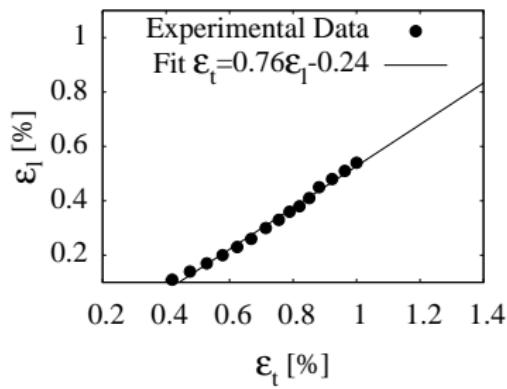
- ▶ :
- ▶ n-uniform

# Results of tensile experiments of Ti-6Al-4V cellular solid vs. numerical results of skeletal structures



Same sign and similar value of  $\nu$

# Results of tensile experiments of Ti-6Al-4V cellular solid vs. numerical results of skeletal structures

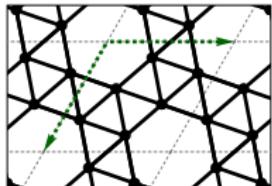


Same sign and similar value of  $\nu$

# Deformations of periodic frameworks

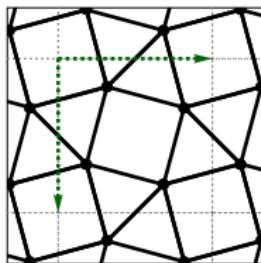
*Existence and uniqueness* illustrated by Archimedean tilings

$(3^4.6)$



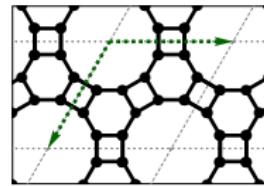
Rigid

$(3^2.4.3.4)$



Unique

$(4.6.12)$



Ambiguous

Any information for their **cellular solid** counterparts?

No

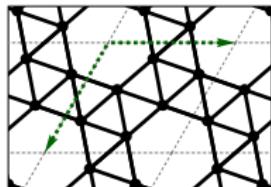
Possibly

Suitable extension  
necessary

# Deformations of periodic frameworks

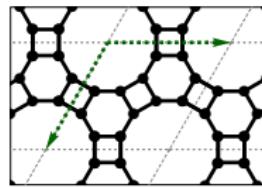
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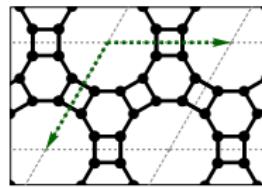
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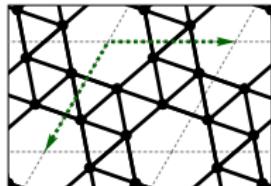
Possibly

Suitable extension  
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# Deformations of periodic frameworks

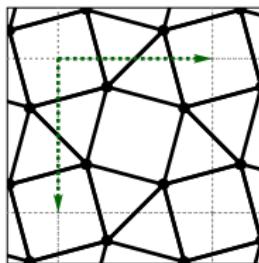
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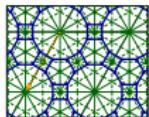
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No

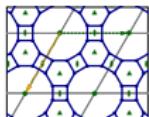
Possibly

Suitable extension  
necessary

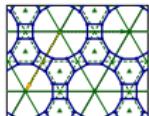
# Deformation retaining available symmetries



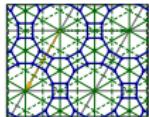
*p6mm*



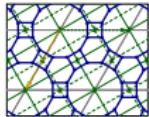
*p6*



*p31m*

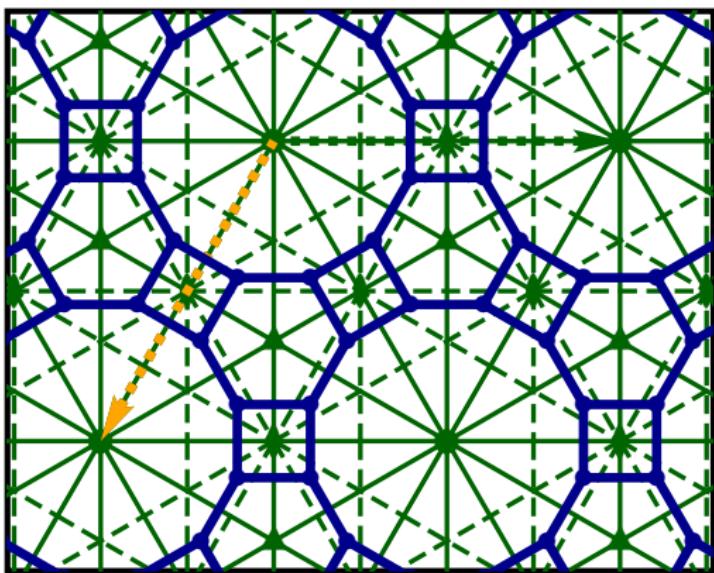


*p3m1*

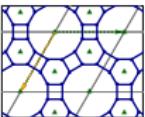


*p2mm<sub>r</sub>*

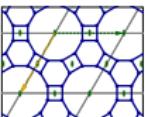
Archimedean tiling (4.6.12)



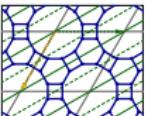
Group *p6mm*: rigid



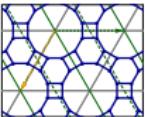
*p3*



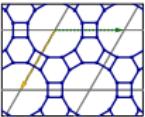
*p2*



*p11m<sub>r</sub>*

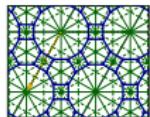


*pm<sub>r</sub>*

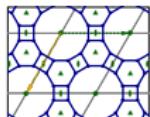


*p1*

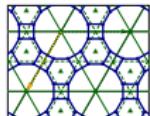
# Deformation retaining available symmetries



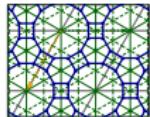
$p6mm$



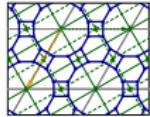
$p6$



$p31m$

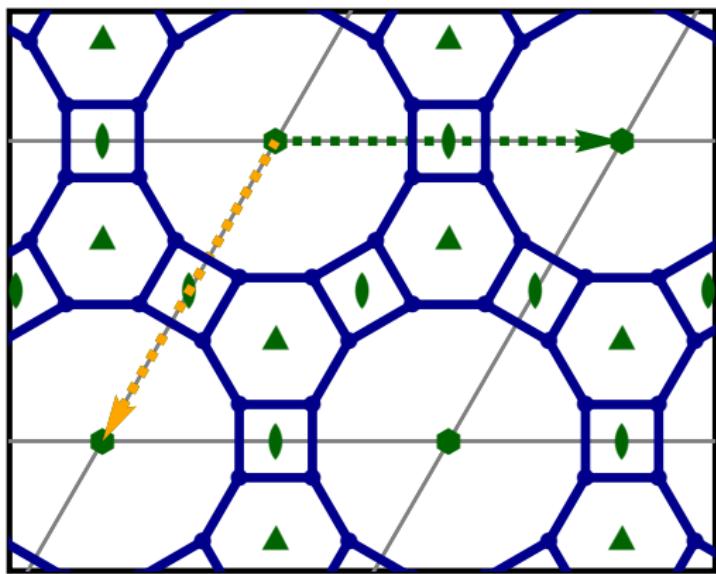


$p3m1$

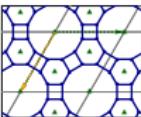


$p2mm_r$

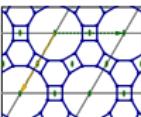
Archimedean tiling (4.6.12)



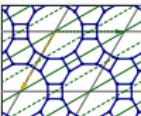
Subgroup  $p6$ : unique ( $\nu = -1$ )



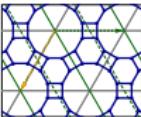
$p3$



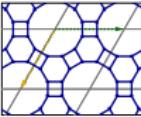
$p2$



$p11m_r$

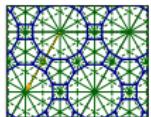


$pm_r$

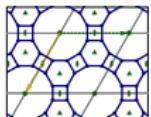


$p1$

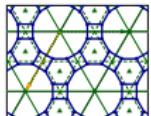
# Deformation retaining available symmetries



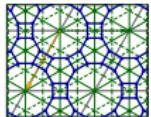
*p6mm*



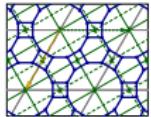
*p6*



*p31m*

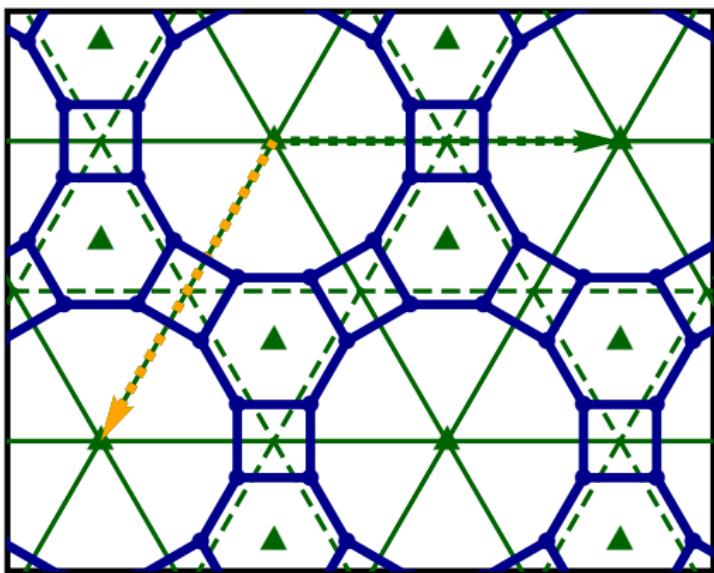


*p3m1*

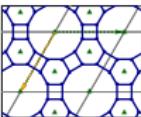


*p2mm<sub>r</sub>*

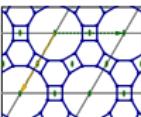
Archimedean tiling (4.6.12)



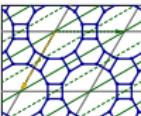
Subgroup *p31m*: unique ( $\nu = -1$ )



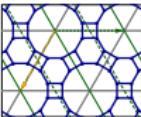
*p3*



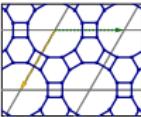
*p2*



*p11m<sub>r</sub>*

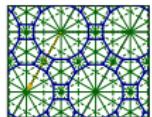


*pm<sub>r</sub>*

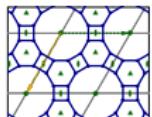


*p1*

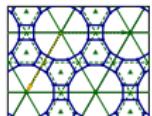
# Deformation retaining available symmetries



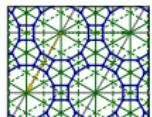
$p6mm$



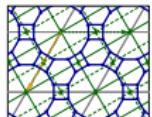
$p6$



$p31m$

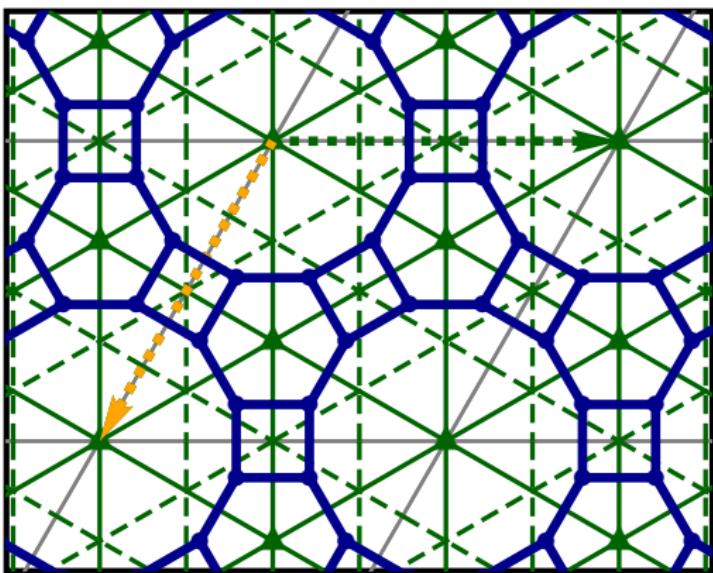


$p3m1$  rigid

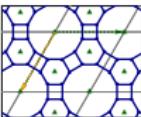


$p2mm_r$

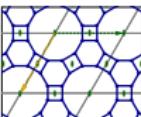
Archimedean tiling (4.6.12)



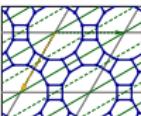
Subgroup  $p3m1$ : rigid



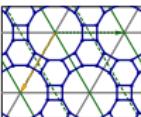
$p3$



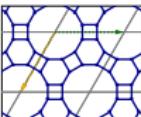
$p2$



$p11m_r$

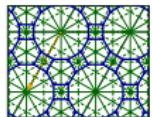


$pm_r$

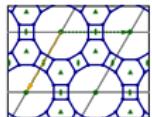


$p1$

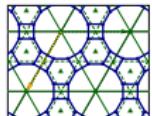
# Deformation retaining available symmetries



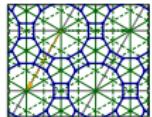
$p6mm$



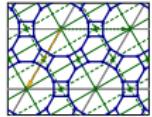
$p6$



$p31m$

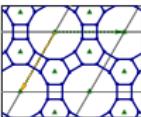
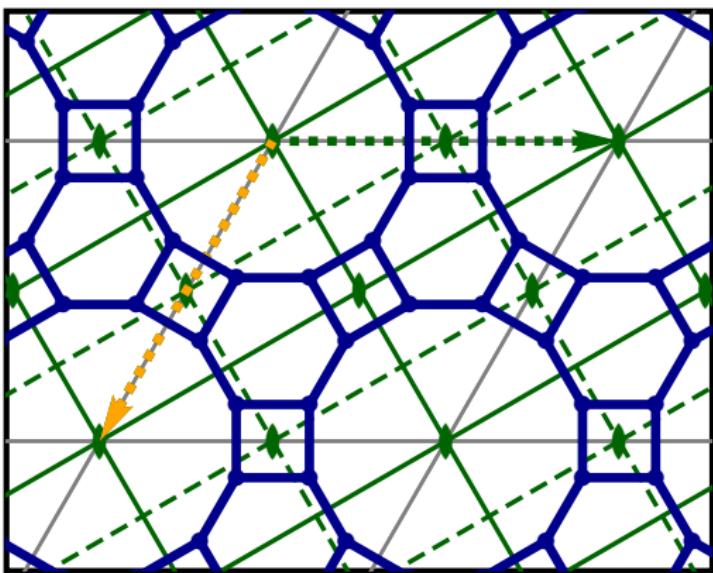


$p3m1$

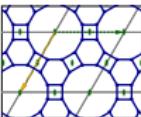


$p2mm_r$

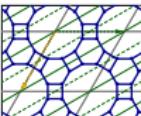
Archimedean tiling (4.6.12)



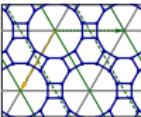
$p3$



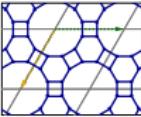
$p2$



$p11m_r$



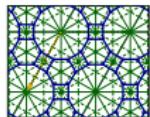
$pm_r$



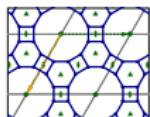
$p1$

Subgroup  $p2mm_r$ : ambiguous deformation

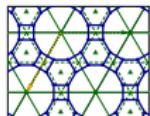
# Deformation retaining available symmetries



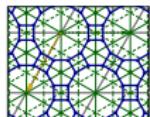
$p6mm$



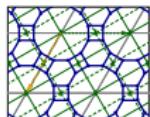
$p6$



$p31m$

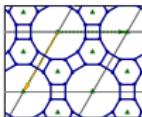
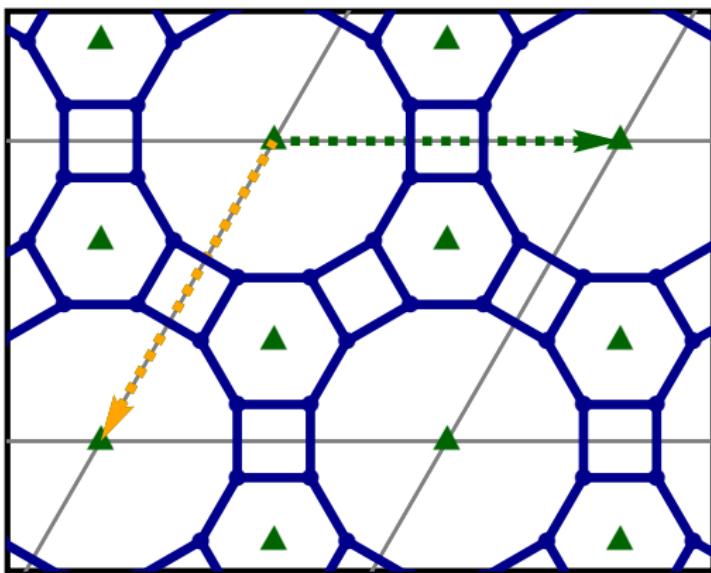


$p3m1$

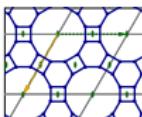


$p2mm_r$

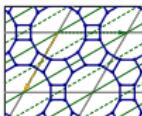
Archimedean tiling (4.6.12)



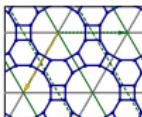
$p3$



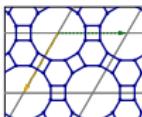
$p2$



$p11m_r$



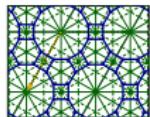
$pm_r$



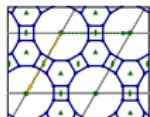
$p1$

Subgroup  $p3$ : ambiguous deformation

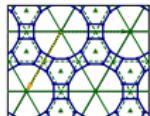
# Deformation retaining available symmetries



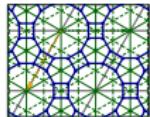
$p6mm$



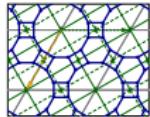
$p6$



$p31m$

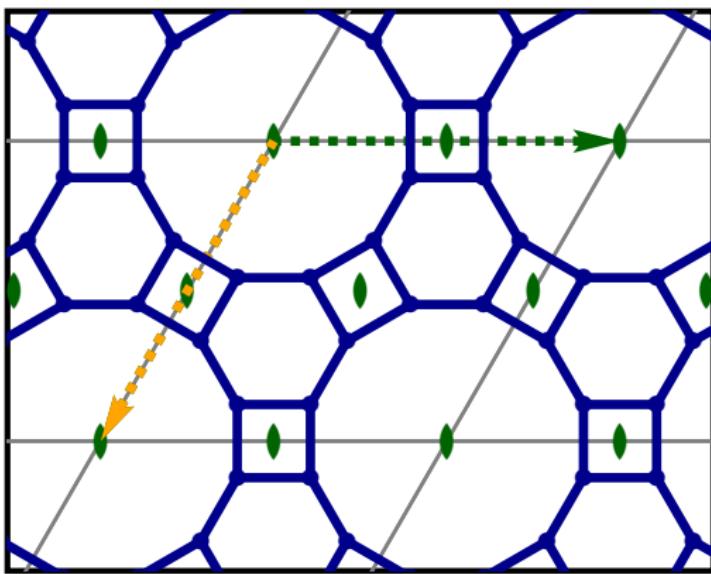


$p3m1$

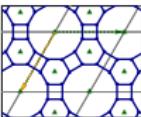


$p2mm_r$

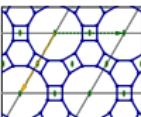
Archimedean tiling (4.6.12)



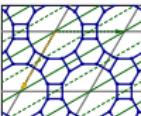
Subgroup  $p2$ : ambiguous deformation



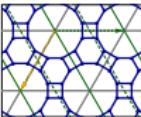
$p3$



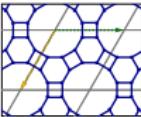
$p2$



$p11mr$

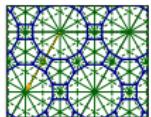


$pm_r$

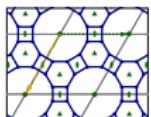


$p1$

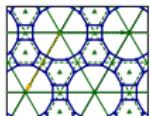
# Deformation retaining available symmetries



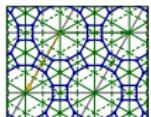
$p6mm$



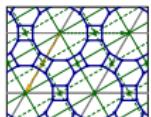
$p6$



$p31m$

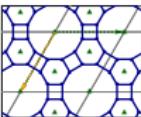
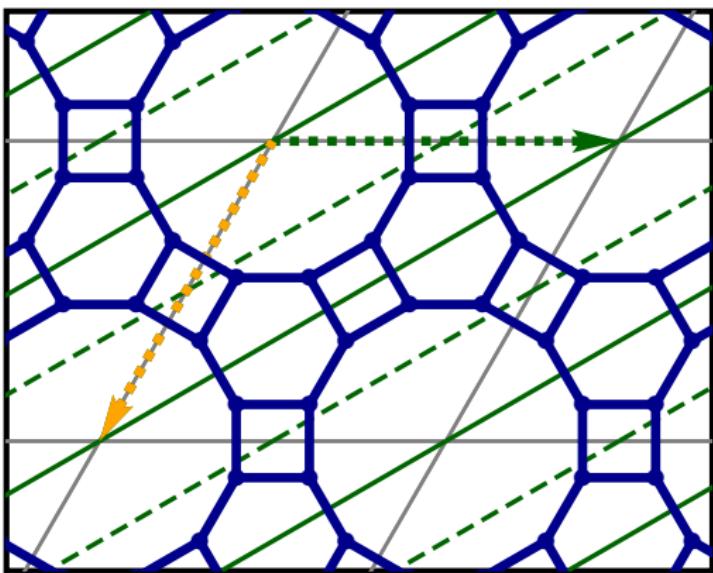


$p3m1$

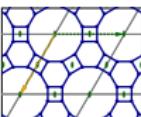


$p2mm_r$

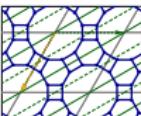
Archimedean tiling (4.6.12)



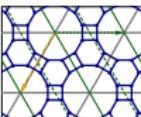
$p3$



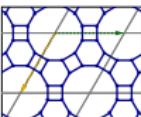
$p2$



$p11m_r$



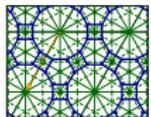
$pm_r$



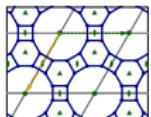
$p1$

Subgroup  $p11m_r$ : ambiguous deformation

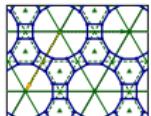
# Deformation retaining available symmetries



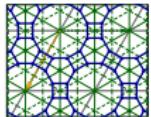
$p6mm$



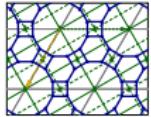
$p6$



$p31m$

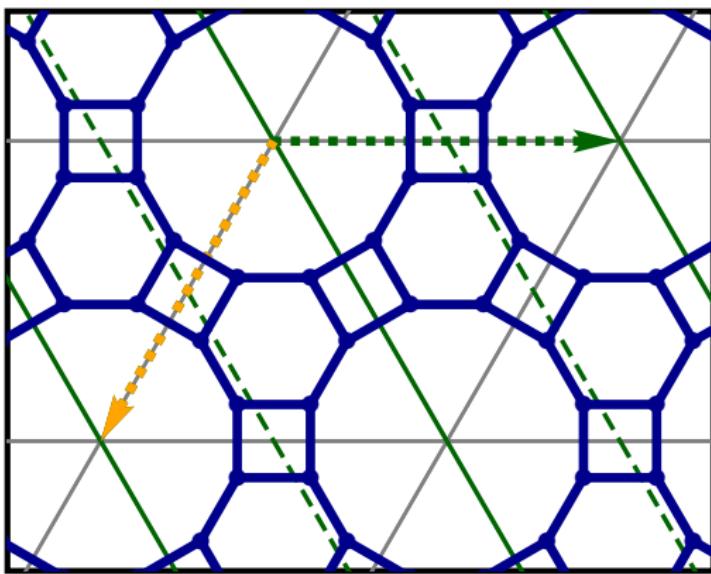


$p3m1$

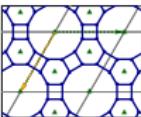


$p2mm_r$

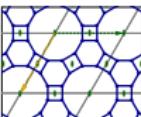
Archimedean tiling (4.6.12)



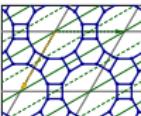
Subgroup  $pm_r$ : ambiguous deformation



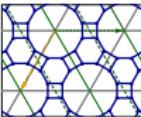
$p3$



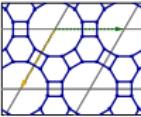
$p2$



$p11m_r$

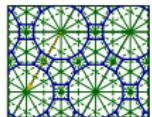


$pm_r$

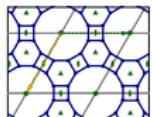


$p1$

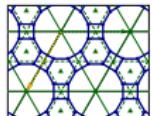
# Deformation retaining available symmetries



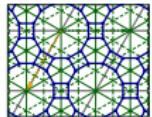
$p6mm$



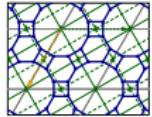
$p6$



$p31m$

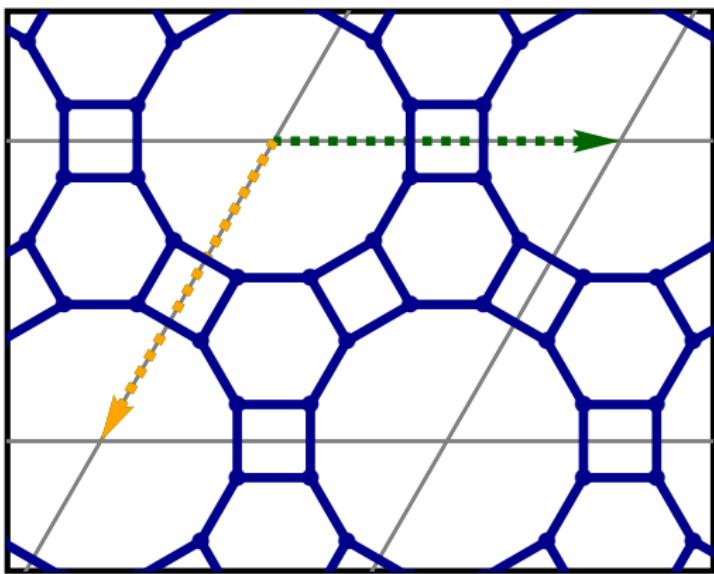


$p3m1$

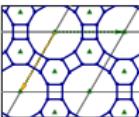


$p2mm_r$

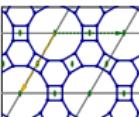
Archimedean tiling (4.6.12)



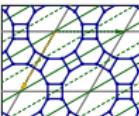
Subgroup  $p1$ : ambiguous deformation



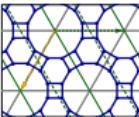
$p3$



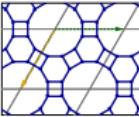
$p2$



$p11m_r$



$pm_r$



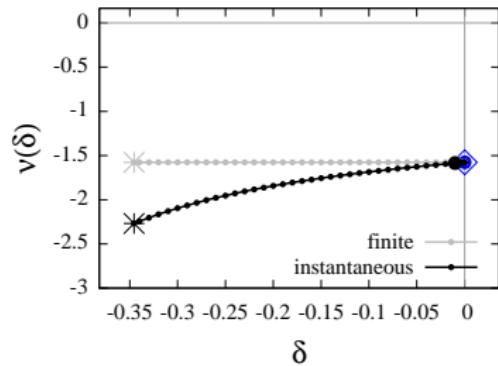
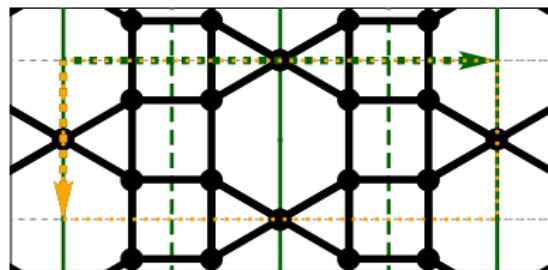
$p1$

# Strain amplification: $|\nu| > 1$

[Baughman et al, "Negative Poisson's Ratios for Extreme States of Matter", Science (2000)]



$(3.4^2.6; 3.6.3.6)_2$  in  $cm \subset c2mm$ :



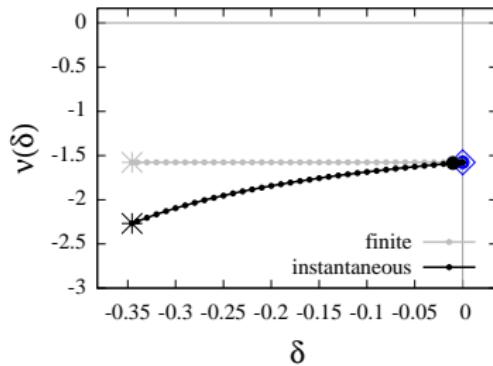
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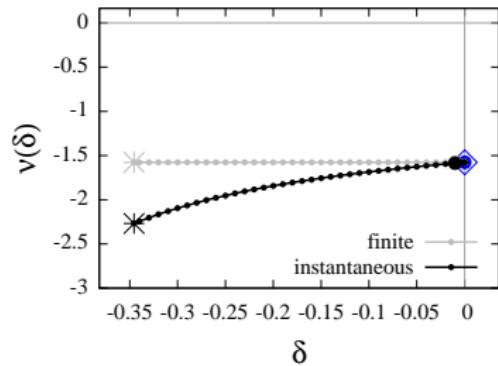
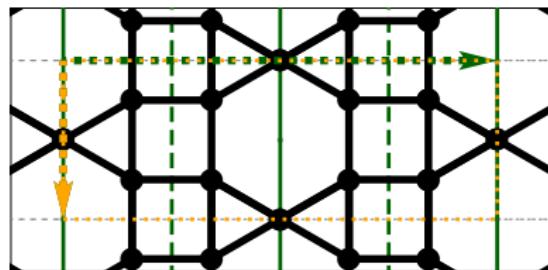
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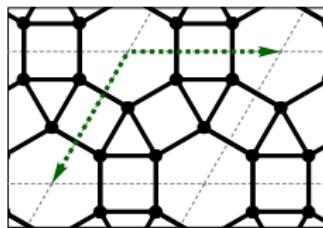
Physical realisation of glide plane symmetries?

# Ambiguous deformations of frameworks

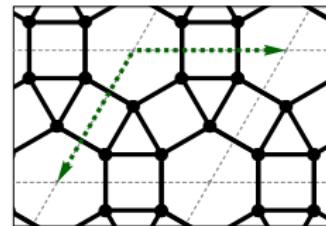
Conservation of symmetries and energy minimization

E.g. harmonic angular springs:

$$H[P(\delta)] = \sum_{\{i;jk\}} (\alpha(P(0))_{\{i;jk\}} - \alpha(P(\delta))_{\{i;jk\}})^2$$



arbitrary  
deformation mode



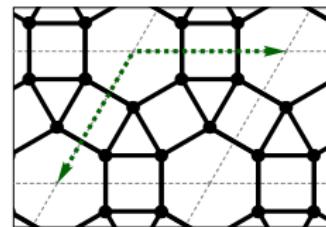
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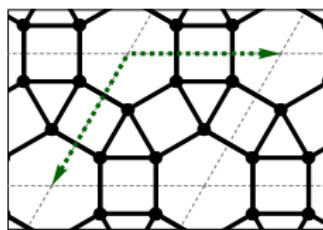
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deformation mode  
with minimal energy

## ff. Strain Amplification and Energy Minimization

2-uniform tiling  $(3.4^2.6; 3.6.3.6)_2$

$c11m$

$c2$

$c1 + \min(E)$

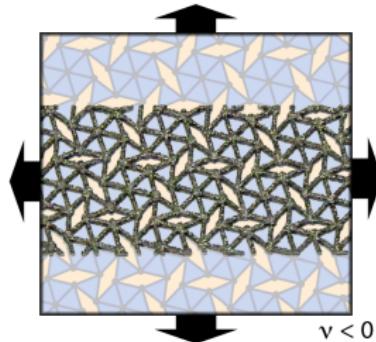
$p1 + \min(E)$

# Summary

## Results

- ▶ Qualitative similar result in TS-wheels structure between framework and cellular solid
- ▶ Find new auxetic mechanisms in repositories of tilings
- ▶ Deformation modes depend on symmetries

Can floppy frameworks  
predict auxetic cellular materials?



[Mitschke et al, "Finding auxetic frameworks in periodic tessellations", Adv.Mat., 2011]