

Equivelar Maps on the Klein Bottle, and a Higher Dimensional Generalization

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To my advisors,

Special thanks to my advisors:
Isabel Hubbard, Mark Mixer, Daniel Pellicer, Asia Weiss

Tessellations on the Klein Bottle

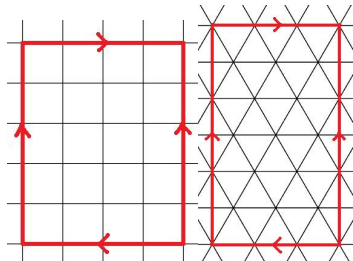
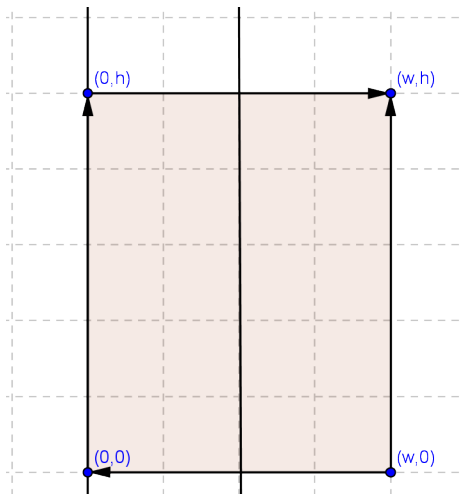


Figure: Fundamental regions of some tessellations by regular polygons on the Klein bottle

Glide Reflections and the Klein Bottle

A glide reflection is a reflection through an axis followed by a translation along the axis. The quotient of \mathbb{R}^2 under two glide reflections through parallel axes and with equal translational components is a Klein bottle



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- If \mathcal{T} is a tessellation and G is its group of symmetries, and $H \leq G$, then we can form the quotient \mathcal{T}/H
- The normalizer, N , of H in G is closely related to the symmetries of \mathcal{T}/H . If $\gamma \in N$ then γ induces a well defined action on \mathcal{T}/H which is a symmetry, and $\text{Aut}(\mathcal{T}/H) \cong N/H$.

Glide Reflections Explicitly Written

The glide reflections from two slides ago are:

$$t_1(x) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} (x) + \begin{pmatrix} 0 \\ h \end{pmatrix}$$

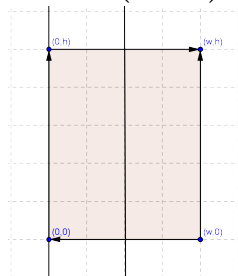
$$t_2(x) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} (x) + \begin{pmatrix} w \\ h \end{pmatrix}$$

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- We first calculate the groups generated by t_1 and t_2 :

$$\langle t_1, t_2 \rangle = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^\beta (x) + \begin{pmatrix} \alpha w \\ \beta h \end{pmatrix} \mid \alpha, \beta \in \mathbb{Z} \right\}$$

Isometries of \mathbb{R}^2 That Trivially Conjugate H

Next we calculate the isometries which conjugate $H = \langle t_1, t_2 \rangle$ trivially. We find they are:

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} (x) + \begin{pmatrix} \frac{nw}{2} \\ v_2 \end{pmatrix}$$

Note: these are all isometries of \mathbb{R}^2 which satisfy $\gamma H \gamma^{-1} = H$. It's more economical to calculate this and restrict down to specific isometries of tessellations.

Plan of Attack

We have enough now to begin finding symmetry groups of quotients of tessellations. So if we have a tessellation \mathcal{T} and a subgroup of its symmetry group, $H \leq \text{Sym}(\mathcal{T})$, we can go on to find the structure of $\text{Aut}(\mathcal{T}/H)$. To do so:

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- Find the normalizer, N , of H in $\text{Sym}(\mathcal{T})$
- Find representatives for the quotient N/H
- Make an isomorphism between N/H and some well known group, or a product of well known groups

Glide Reflections Through Hyperplanes

The group of glide reflections from before can be generalized to higher dimensions in an easy way to glide reflections through hyperplanes:

$$t_{2i-1}(x) = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & -1 & \\ & & & \ddots \\ 0 & & & & 1 \end{pmatrix} (x) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ h \end{pmatrix}$$

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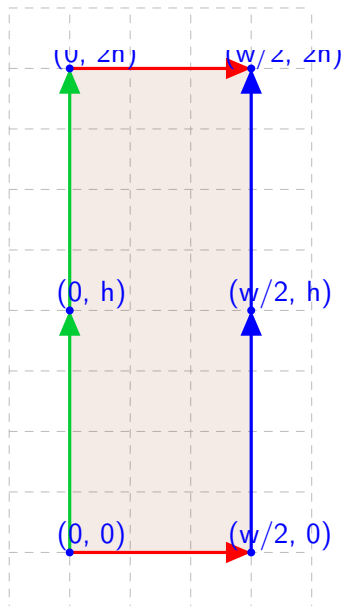
The General Element of H

The general $t \in H$ has the form:

$$\begin{pmatrix} (-1)^{\epsilon_1} & & & & \\ & (-1)^{\epsilon_2} & & & \\ & & \ddots & & \\ & & & (-1)^{\epsilon_{n-1}} & \\ & & & & 1 \end{pmatrix} (x) + \begin{pmatrix} \alpha_1 w_1 \\ \alpha_2 w_2 \\ \vdots \\ \alpha_{n-1} w_{n-1} \\ (2\beta + \chi(\epsilon))h \end{pmatrix}$$

where, $\chi(\epsilon) = \sum \epsilon_i \bmod 2$ or $\frac{1 - \det A}{2}$

Another Look at the Klein Bottle



Fundamental Regions in Higher Dimensions

The fundamental region is:

$$R = \{(\lambda_1 w_1, \dots, \lambda_{n-1} w_{n-1}, \lambda_n h) \mid \lambda_1, \dots, \lambda_{n-1} \in [0, \frac{1}{2}], \lambda_n \in [0, 2]\}$$

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it's difficult to convey the identifications on the boundary even in 3 dimensions on a slide - to the black board

Calculating the Symmetry Group

We begin by finding the trivial conjugators of the matrix group that appeared in $H = \langle t_1, \dots, t_{2n-2} \rangle$

$$\begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \\ & & & & 1 \end{pmatrix}$$

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$$\text{Aut}(\mathcal{T}/H) \cong D_{2(2h)} \times (C_2^k \wr \Omega)$$

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Thanks

Thank you for listening