Restricting the Degree/Diameter and Cage Problems to Vertex-Transitive Graphs

Robert Jajcay,
Indiana State University
robert.jajcay@indstate.edu

October 21, 2011

- A (Δ, D)-**graph** is a (finite) graph of *maximum degree* Δ and *diameter* D.

- A (Δ, D)-**graph** is a (finite) graph of *maximum degree* Δ and *diameter* D.
- A (k, g)-**graph** is a (finite) graph of *degree* k and *girth* g.

Moore Bound(s)

$$n(\Delta, D) \leq M(\Delta, D) = \begin{cases} 1 + \Delta\frac{(\Delta-1)^D-1}{\Delta-2}, & \text{if } \Delta > 2 \\ 2D+1, & \text{if } \Delta = 2 \end{cases}$$

Moore Bound(s)

$$n(\Delta, D) \leq M(\Delta, D) = \begin{cases} 1 + \Delta\frac{(\Delta-1)^D-1}{\Delta-2}, & \text{if } \Delta > 2 \\ 2D+1, & \text{if } \Delta = 2 \end{cases}$$

$$n(k, g) \geq M(k, g) = \begin{cases} 1 + k\frac{(k-1)^{(g-1)/2}-1}{k-2}, & g \text{ odd} \\ 2\frac{(k-1)^{g/2}-1}{k-2}, & g \text{ even} \end{cases}$$

Moore Bound(s)

$$n(\Delta, D) \leq M(\Delta, D) = \begin{cases} 1 + \Delta\frac{(\Delta-1)^D - 1}{\Delta - 2}, & \text{if } \Delta > 2 \\ 2D + 1, & \text{if } \Delta = 2 \end{cases}$$

$$n(k, g) \geq M(k, g) = \begin{cases} 1 + k\frac{(k-1)^{(g-1)/2} - 1}{k-2}, & g \text{ odd} \\ 2\frac{(k-1)^{g/2} - 1}{k-2}, & g \text{ even} \end{cases}$$

Any (k, g)-graph whose order matches this "naive" bound is called a

Moore graph.

Cages and Extremal (Δ, D)-Graphs

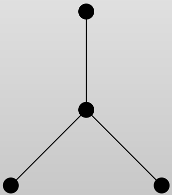
- $n(\Delta, D) =$ the order of a *largest* (Δ, D)-graph;
- $n(k, g) =$ the order of a *smallest* (k, g)-graph;

Cages and Extremal (Δ, D)-Graphs

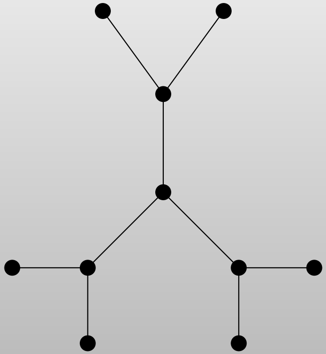
- $n(\Delta, D) =$ the order of a *largest* (Δ, D)-graph;
- $n(k, g) =$ the order of a *smallest* (k, g)-graph;
- the largest (Δ, D)-graph is an **extremal (Δ, D)-graph**
- the smallest (k, g)-graph is a **cage**

Constructing a $(3, 5)$-cage

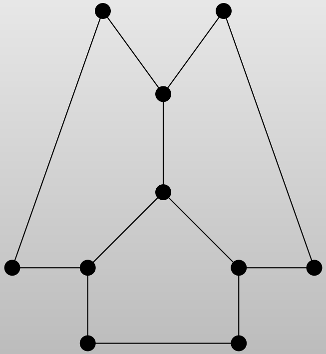
Constructing a $(3, 5)$-cage



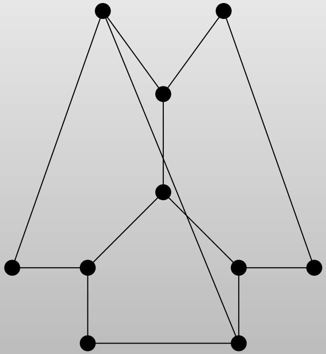
Constructing a $(3,5)$-cage



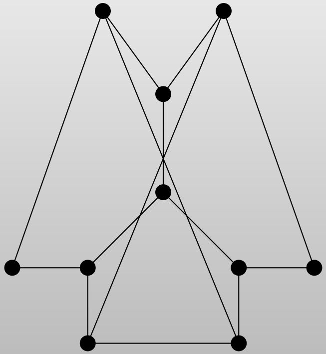
Constructing a $(3, 5)$-cage



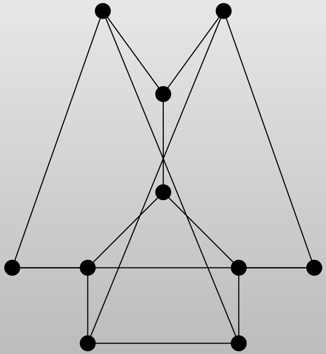
Constructing a $(3, 5)$-cage



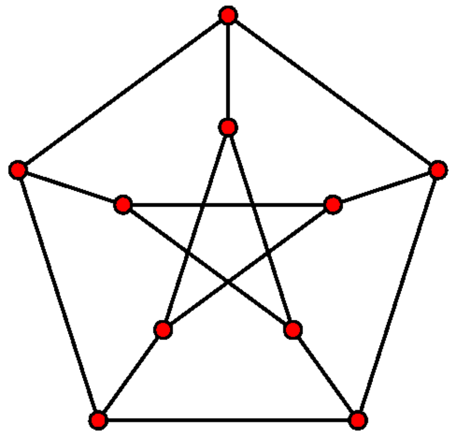
Constructing a $(3, 5)$-cage



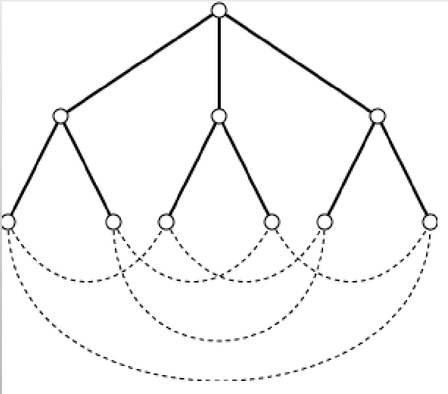
Constructing a $(3,5)$-cage



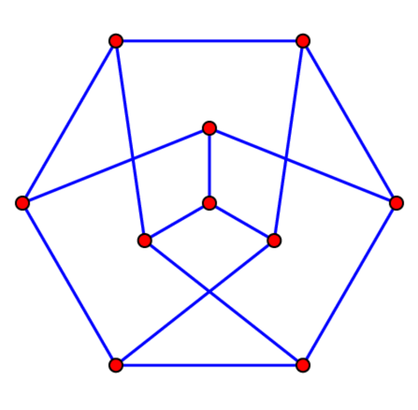
$(3,5)$-cage



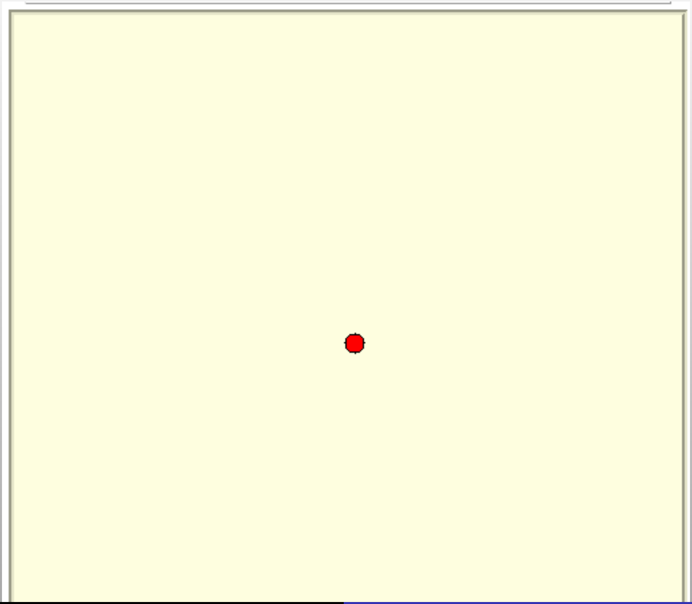
$(3, 5)$-cage



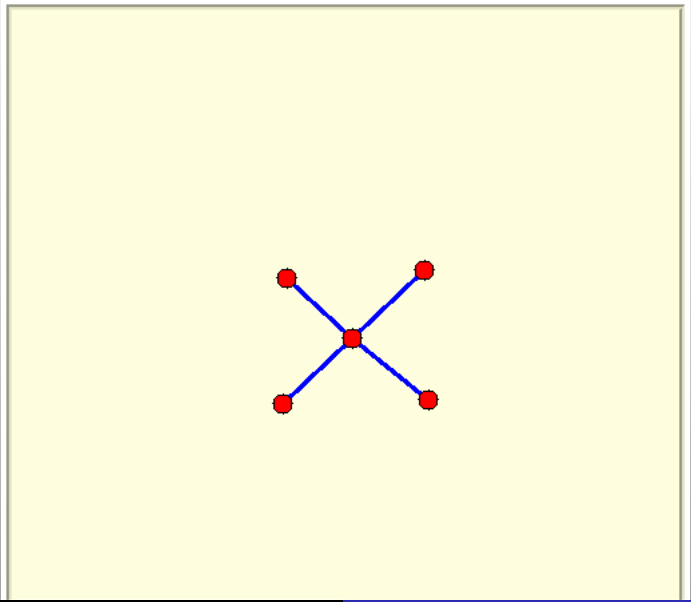
$(3,5)$-cage



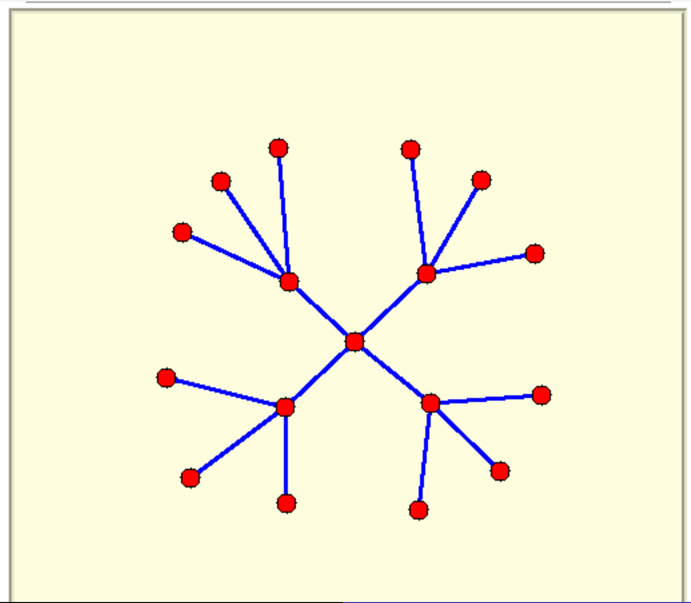
Constructing a $(4, 7)$-cage



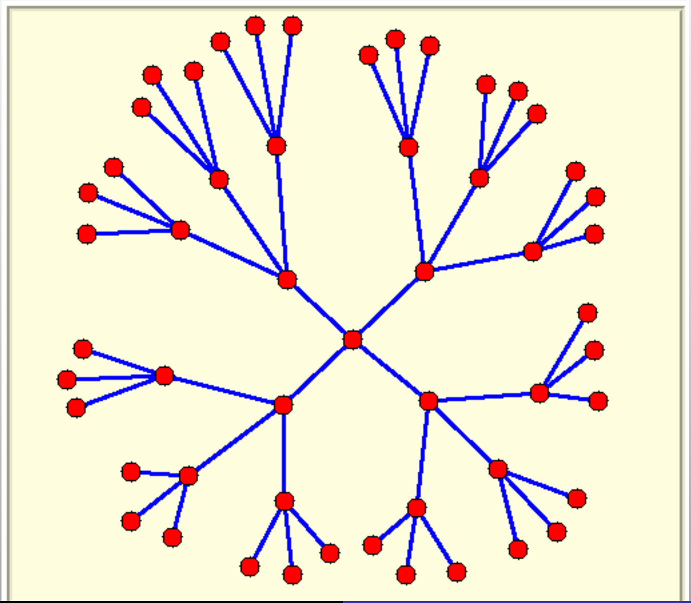
Constructing a $(4, 7)$-cage



Constructing a $(4, 7)$-cage



Constructing a $(4, 7)$-cage



Does Efficiency Necessarily Imply Beauty?

Does Efficiency Necessarily Imply Beauty?

Are cages necessarily highly symmetric?

Does Efficiency Necessarily Imply Beauty?

Are cages necessarily highly symmetric?

The case of the Petersen graph suggests that cages should be both vertex-transitive and have large vertex-stabilizers.

Known Cages of Degree 3

girth	**5**	**6**	7	**8**	9	10	11	**12**
order	**10**	**14**	24	**30**	58	70	112	**126**
# of cages	**1**	**1**	1	**1**	18	3	1	**1**
# of sym's	120	336	32	1440	≤ 24	≤ 120	64	12,096

Small Cayley (k, g)-Graphs

girth	*degree 3*			*degree 4*		
	Lower Bound	Best Graph	Cayley Cage	Lower Bound	Best Graph	Cayley Cage
5	10	10	50	19	19	24
6	14	14	14	26	26	26
7	24	24	30	67	67	72
8	30	30	42	80	80	96
9	58	58	60	162	275	
10	70	70	96	243	384	410
11	112	112	192	486		
12	126	126	162	728	728	
13	202	272	272			
14	258	384	406			

Theorem (Nedela, Škoviera)

For every $k \geq 2, g \geq 3$, there exists a **vertex-transitive** *(k, g)-graph.*

Vertex-Transitive (k, g)-Graphs

Theorem (Nedela, Škoviera)

For every $k \geq 2, g \geq 3$, there exists a **vertex-transitive** *(k, g)-graph.*

Theorem (Exoo, RJ, Širáň)

For every $k \geq 2, g \geq 3$, there exists a **Cayley** *(k, g)-graph.*

Theorem (Nedela, Škoviera)

For every $k \geq 2, g \geq 3$, there exists a **vertex-transitive** *(k, g)-graph.*

Theorem (Exoo, RJ, Širáň)

For every $k \geq 2, g \geq 3$, there exists a **Cayley** *(k, g)-graph.*

Proof.

- using (infinite) Cayley maps of Šiagiová and Watkins in combination with the fact that automorphism groups of Cayley maps are residually finite

Vertex-Transitive (k, g)-Graphs

Theorem (Nedela, Škoviera)

For every $k \geq 2, g \geq 3$, there exists a **vertex-transitive** *(k, g)-graph.*

Theorem (Exoo, RJ, Širáň)

For every $k \geq 2, g \geq 3$, there exists a **Cayley** *(k, g)-graph.*

Proof.

- using (infinite) Cayley maps of Šiagiová and Watkins in combination with the fact that automorphism groups of Cayley maps are residually finite
- adding a g-cycle to a graph of girth *at least* g constructed by Biggs

Vertex-Transitive (k, g)-Graphs

Theorem (Nedela, Škoviera)

For every $k \geq 2, g \geq 3$, there exists a **vertex-transitive** *(k, g)-graph.*

Theorem (Exoo, RJ, Širáň)

For every $k \geq 2, g \geq 3$, there exists a **Cayley** *(k, g)-graph.*

Proof.

- using (infinite) Cayley maps of Šiagiová and Watkins in combination with the fact that automorphism groups of Cayley maps are residually finite
- adding a g-cycle to a graph of girth *at least* g constructed by Biggs
- constructing Cayley graphs from 1- and 2-factorizations of (k, g)-graphs

Theorem

Let G be a k-regular graph of girth g whose edge set can be partitioned into a family $\mathcal{F}$ of k_1 1-factors, F_i, $1 \leq i \leq k_1$, and k_2 oriented 2-factors F_i, $k_1 + 1 \leq i \leq k_1 + k_2$ (where $k_1 + 2k_2 = k$). If $\Gamma_{\mathcal{F}}$ is the finite permutation group acting on the set $V(G)$ generated by the set

$$X = \{\delta_{F_i} \mid 1 \leq i \leq k_1\} \cup \{\sigma_{F_i} \mid k_1 + 1 \leq i \leq k_1 + k_2\} \cup \{\sigma_{F_i}^{-1} \mid k_1 + 1 \leq i \leq k_1 + k_2\},$$

then the Cayley graph $Cay(\Gamma_{\mathcal{F}}, X)$ is k-regular of girth at least g.

Theorem

Let G be a k-regular graph of girth g whose edge set can be partitioned into a family $\mathcal{F}$ of k_1 1-factors, F_i, $1 \leq i \leq k_1$, and k_2 oriented 2-factors F_i, $k_1 + 1 \leq i \leq k_1 + k_2$ (where $k_1 + 2k_2 = k$). If $\Gamma_{\mathcal{F}}$ is the finite permutation group acting on the set $V(G)$ generated by the set

$$X = \{\delta_{F_i} \mid 1 \leq i \leq k_1\} \cup \{\sigma_{F_i} \mid k_1 + 1 \leq i \leq k_1 + k_2\} \cup \{\sigma_{F_i}^{-1} \mid k_1 + 1 \leq i \leq k_1 + k_2\},$$

then the Cayley graph $Cay(\Gamma_{\mathcal{F}}, X)$ is k-regular of girth at least g.

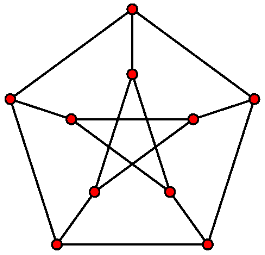
Graphs to Groups

Theorem

Let G be a k-regular graph of girth g whose edge set can be partitioned into a family $\mathcal{F}$ of k_1 1-factors, F_i, $1 \leq i \leq k_1$, and k_2 oriented 2-factors F_i, $k_1 + 1 \leq i \leq k_1 + k_2$ (where $k_1 + 2k_2 = k$). If $\Gamma_{\mathcal{F}}$ is the finite permutation group acting on the set $V(G)$ generated by the set

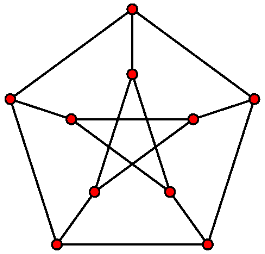
$$X = \{\delta_{F_i} \mid 1 \leq i \leq k_1\} \cup \{\sigma_{F_i} \mid k_1 + 1 \leq i \leq k_1 + k_2\} \cup \{\sigma_{F_i}^{-1} \mid k_1 + 1 \leq i \leq k_1 + k_2\},$$

then the Cayley graph $Cay(\Gamma_{\mathcal{F}}, X)$ is k-regular of girth at least g.



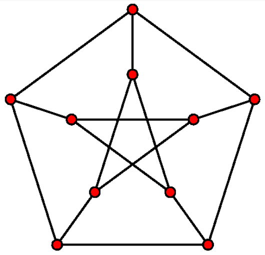
$F_1 : \{1,6\}, \{2,7\}, \{3,8\}, \{4,9\}, \{5,10\}$

Graphs to Groups



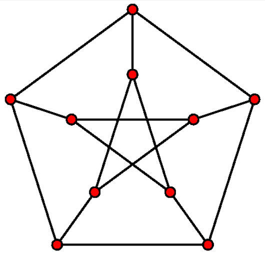
$$F_1 : \{1,6\},\{2,7\},\{3,8\},\{4,9\},\{5,10\}$$
$$\sigma_{F_1} = (1,6)\,(2,7)\,(3,8)\,(4,9)\,(5,10)$$

Graphs to Groups



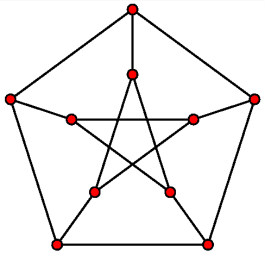
$$F_1 : \{1,6\},\{2,7\},\{3,8\},\{4,9\},\{5,10\}$$
$$\sigma_{F_1} = (1,6)\,(2,7)\,(3,8)\,(4,9)\,(5,10)$$
$$F_2 : \{1,2,3,4,5\},\{6,8,10,7,9\}$$

Graphs to Groups



$$F_1 : \{1,6\}, \{2,7\}, \{3,8\}, \{4,9\}, \{5,10\}$$
$$\sigma_{F_1} = (1,6)\,(2,7)\,(3,8)\,(4,9)\,(5,10)$$
$$F_2 : \{1,2,3,4,5\}, \{6,8,10,7,9\}$$
$$\sigma_{F_2} = (1,2,3,4,5)\,(6,8,10,7,9)$$

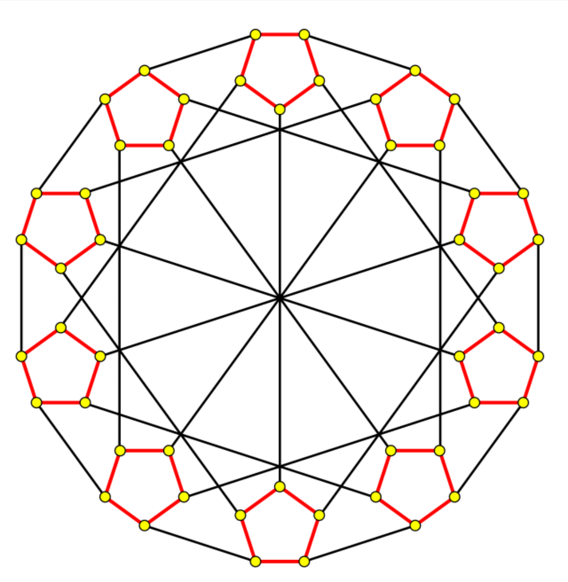
Graphs to Groups



$$F_1 : \{1,6\}, \{2,7\}, \{3,8\}, \{4,9\}, \{5,10\}$$
$$\sigma_{F_1} = (1,6)\,(2,7)\,(3,8)\,(4,9)\,(5,10)$$
$$F_2 : \{1,2,3,4,5\}, \{6,8,10,7,9\}$$
$$\sigma_{F_2} = (1,2,3,4,5)\,(6,8,10,7,9)$$

$$\Gamma_{\mathcal{F}} = \langle \sigma_{F_1}, \sigma_{F_2} \rangle \cong \mathbb{Z}_5 \times \mathbb{D}_5$$

Smallest $(3, 5)$-Cayley Graph



Theorem

Let $Cay(\Gamma, X)$ *be a* k*-regular graph of girth* g*. Suppose that* Γ *has a permutation representation* $\gamma \to \sigma_\gamma$, $\gamma \in \Gamma$, *on a set* V, *satisfying the property that no non-reversing product of the permutations* σ_x, $x \in X$, *of length smaller that* g *fixes a vertex* $v \in V$, *and for every* $v \in V$, *the images* $\sigma_x(v)$ *are all different. Then the graph* G_Γ *with vertex set* V *and edge set* $E = \{\{v, \sigma_x(v)\} \mid v \in V, x \in X\}$ *is* k*-regular of girth* g*.*

Theorem

Let $Cay(\Gamma, X)$ be a k-regular graph of girth g. Suppose that Γ has a permutation representation $\gamma \to \sigma_\gamma$, $\gamma \in \Gamma$, on a set V, satisfying the property that no non-reversing product of the permutations σ_x, $x \in X$, of length smaller that g fixes a vertex $v \in V$, and for every $v \in V$, the images $\sigma_x(v)$ are all different. Then the graph G_Γ with vertex set V and edge set $E = \{\{v, \sigma_x(v)\} \mid v \in V, x \in X\}$ is k-regular of girth g.

Theorem

Let $Cay(\Gamma, X)$ *be a* k*-regular graph of girth* g*. Suppose that* Γ *has a permutation representation* $\gamma \to \sigma_\gamma$, $\gamma \in \Gamma$*, on a set* V*, satisfying the property that no non-reversing product of the permutations* σ_x*,* $x \in X$*, of length smaller that* g *fixes a vertex* $v \in V$*, and for every* $v \in V$*, the images* $\sigma_x(v)$ *are all different. Then the graph* G_Γ *with vertex set* V *and edge set* $E = \{\{v, \sigma_x(v)\} \mid v \in V, x \in X\}$ *is* k*-regular of girth* g*.*

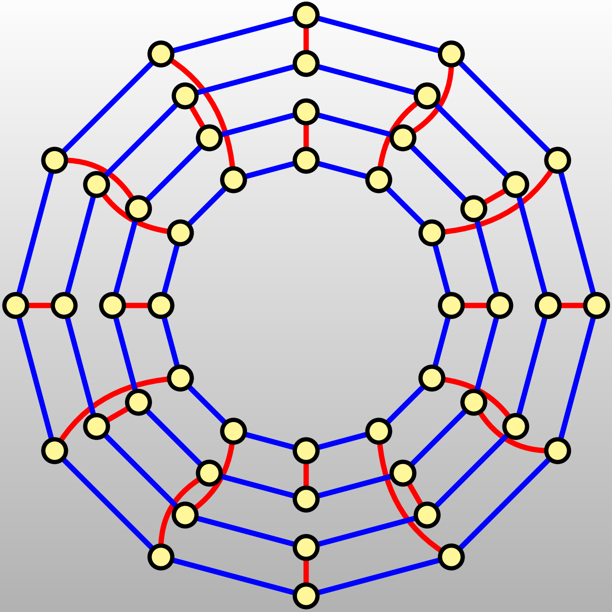


Figure: A Cayley Graph of the group $(\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\phi \mathbb{Z}_3$

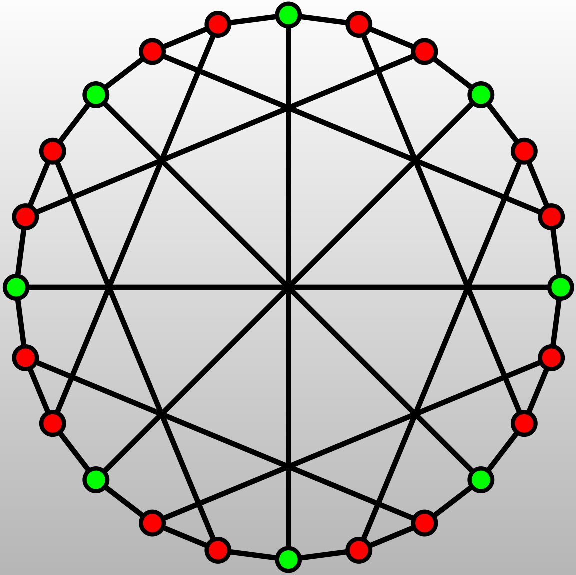


Figure: The McGee Graph

Theorem (Conder, Exoo,RJ)

If Γ is a solvable group with derived series of length n, then the girth g of any Cayley graph $Cay(\Gamma, X)$ of degree at least 3 is bounded from above as follows:

$$\begin{aligned} g &\leq 4, && \text{if } n = 1, \\ g &\leq 14 \cdot 4^{n-2}, && \text{if } n \geq 2. \end{aligned}$$

Girth of Cayley Graphs of Solvable Groups

Theorem (Conder, Exoo, RJ)

If Γ is a solvable group with derived series of length n, then the girth g of any Cayley graph $Cay(\Gamma, X)$ of degree at least 3 is bounded from above as follows:

$$\begin{array}{ll} g \leq 4, & \text{if } n = 1, \\ g \leq 14 \cdot 4^{n-2}, & \text{if } n \geq 2. \end{array}$$

Specifically, if $Cay(\Gamma, X)$ is a Cayley graph of a solvable group Γ of derived length n and of degree at least 3, $|X| \geq 3$. Then

$$\begin{array}{ll} g \leq 44, & \text{if } n = 3 \text{ and } X \text{ contains at least three inv's,} \\ g \leq 48, & \text{if } n = 3, \text{ and } X \text{ contains at least two distinct non-inv's,} \\ g \leq 50, & \text{if } n = 3, \text{ and } X \text{ consists of one inv and one non-inv,} \\ g \leq 148, & \text{if } n = 4 \text{ and and } X \text{ contains at least three inv's,} \\ g \leq 168, & \text{if } n = 4, \text{ and } X \text{ contains at least two distinct non-inv's.} \end{array}$$

Girth of Cayley Graphs of Nilpotent Groups

Theorem (Conder, Exoo, RJ)

If Γ is a nilpotent group of nilpotency class n, then the girth g of any Cayley graph $Cay(\Gamma, X)$ of degree at least 3 is bounded from above as follows:

$$g \le 4, \quad \text{if } n = 1,$$
$$g \le 8, \quad \text{if } n = 2,$$
$$g \le (n+1)^2, \quad \text{if } n \ge 3.$$

Non-Solvable Groups

- If G has a faithful representation on n vertices than the girth of any $Cay(G, X)$ such that X contains a non-involution is bounded from above by the maximal order of elements in $\mathbb{S}_n$.

Non-Solvable Groups

- If G has a faithful representation on n vertices than the girth of any $Cay(G, X)$ such that X contains a non-involution is bounded from above by the maximal order of elements in $\mathbb{S}_n$.
- If G has a faithful representation on n vertices than the girth of any $Cay(G, X)$ such that X contains an involution is bounded from above by $6n$ (almost proved).

The Order of Vertex-Transitive (k, g)-Graphs

Theorem (RJ, Širáň)

Let G be a vertex-transitive graph of degree k and girth $g = p^r > k$, where p is an odd prime and $r \geq 1$. If G is not a Moore graph (that is, $|V(G)| > M(k,g)$), and g is relatively prime to all the integers in the union

$$\bigcup_{0 \leq i \leq k} \mathcal{L}(k, g, i),$$

*where $\mathcal{L}(k,g,0) = \{M(k,g)+1, M(k,g)+2, \ldots, M(k,g)+k\}$, and $\mathcal{L}(k,g,i) =$
$\{k(k-1)^{(g-1)/2} - ik,\ k(k-1)^{(g-1)/2} - ik + 1, \ldots,$
$k(k-1)^{(g-1)/2} - ik + i - 1\}$, $i > 0$,
then the order of G is at least $M(k,g) + k + 1$.*

Corollary

For every $\sigma > 0$, there exists a pair of parameters (k, g) with the property that every Cayley (k, g)-graph $Cay(G, X)$ satisfies

$$|V(Cay(G, X))| - M(k, g) > \sigma$$

Corollary

For every $\sigma > 0$, there exists a pair of parameters (k, g) with the property that every Cayley (k, g)-graph $Cay(G, X)$ satisfies

$$|V(Cay(G, X))| - M(k, g) > \sigma$$

But the degree may have to be increasing.

Corollary

For every $\sigma > 0$, there exists a pair of parameters (k, g) with the property that every Cayley (k, g)-graph $Cay(G, X)$ satisfies

$$|V(Cay(G, X))| - M(k, g) > \sigma$$

But the degree may have to be increasing.

So far.

Cayley (Δ, D)-Graphs

Recall:
A (Δ, D)-graph is of maximum degree Δ and diameter d.

Cayley (Δ, D)-Graphs

Recall:
A (Δ, D)-graph is of maximum degree Δ and diameter d.

If Γ is vertex-transitive, Δ is the degree of all the vertices of Γ.

Cayley (Δ, D)-Graphs

Recall:

A (Δ, D)-graph is of maximum degree Δ and diameter d.

If Γ is vertex-transitive, Δ is the degree of all the vertices of Γ.

Theorem (Exoo, RJ, Mačaj, Širáň)

For every $\Delta \geq 2, D \geq 1$, there exists a **Cayley** *(Δ, D)-graph.*

Cayley (Δ, D)-Graphs

Recall:

A (Δ, D)-graph is of maximum degree Δ and diameter d.

If Γ is vertex-transitive, Δ is the degree of all the vertices of Γ.

Theorem (Exoo, RJ, Mačaj, Širáň)

For every $\Delta \geq 2, D \geq 1$, *there exists a* **Cayley** (Δ, D)*-graph.*

All the examples are **circulants**, i.e., Cayley graphs of cyclic groups of girth 4.

- The difference $\delta = M(\Delta, D) - |V(G)|$ is ≥ 1 for all parameters (Δ, D) not allowing for a Moore graph

- The difference $\delta = M(\Delta, D) - |V(G)|$ is ≥ 1 for all parameters (Δ, D) not allowing for a Moore graph
- It is called the **defect** of G

The Defect of Cayley (Δ, D)-Graphs

Theorem (Exoo, RJ, Mačaj, Širáň)

Let $\sigma > 0$ and $\Delta > 2$ be fixed. Then there exist infinitely many (Δ, D) pairs with the property that each vertex-transitive (Δ, D)-graph Γ satisfies

$$\delta(\Gamma) > \sigma$$

The Defect of Cayley (Δ, D)-Graphs

Theorem (Exoo, RJ, Mačaj, Širáň)

Let $\sigma > 0$ and $\Delta > 2$ be fixed. Then there exist infinitely many (Δ, D) pairs with the property that each vertex-transitive (Δ, D)-graph Γ satisfies

$$\delta(\Gamma) > \sigma$$

Corollary

There exists a sequence of extremal cubic Cayley $(3, d)$-graphs whose defects increase to infinity.

The Defect of Cayley (Δ, D)-Graphs

Theorem (Exoo, RJ, Mačaj, Širáň)

Let $\sigma > 0$ and $\Delta > 2$ be fixed. Then there exist infinitely many (Δ, D) pairs with the property that each vertex-transitive (Δ, D)-graph Γ satisfies

$$\delta(\Gamma) > \sigma$$

Corollary

There exists a sequence of extremal cubic Cayley $(3, d)$-graphs whose defects increase to infinity.

Corollary

There exists a sequence of cubic Cayley $(3, g)$-cages whose defects increase to infinity.

Diameter of Cayley Graphs from A Group Theoretical Point of View

Fact:

It follows from the Moore bound that the diameter of any regular graph is bounded from below by the log of its order.

Diameter of Cayley Graphs from A Group Theoretical Point of View

Fact:
It follows from the Moore bound that the diameter of any regular graph is bounded from below by the log of its order.

Theorem (Babai, Kantor and Lubotzky)

There is a constant C such that every non-abelian finite simple group G has a set S of seven generators for which $d(G,S) \leq C \log |G|$.

Diameter of Cayley Graphs from A Group Theoretical Point of View

Fact:

It follows from the Moore bound that the diameter of any regular graph is bounded from below by the log of its order.

Theorem (Babai, Kantor and Lubotzky)

There is a constant C such that every non-abelian finite simple group G has a set S of seven generators for which $d(G,S) \leq C \log |G|$.

Theorem (Kantor)

- *If $n \geq 10$, then there exists a trivalent Cayley graph for $G = PSL(n,q)$ whose diameter is $O(\log |G|)$.*
- *For n large enough, there exist trivalent Cayley graphs of $\mathbb{S}_n$ and $\mathbb{A}_n$ whose diameter is $O(\log n!)$.*

THANK YOU.