Aperiodic Tilings: Notions and Properties

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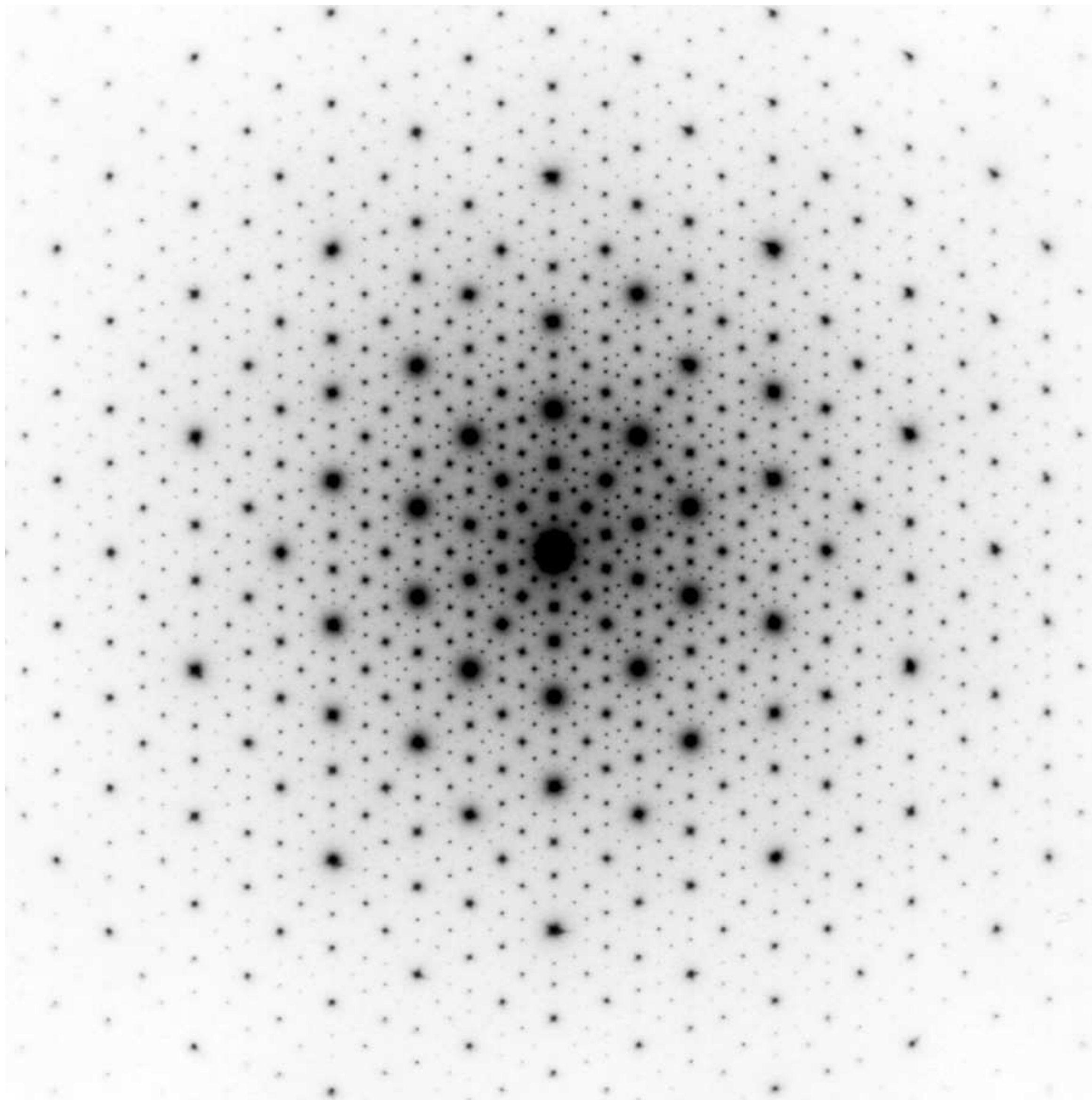
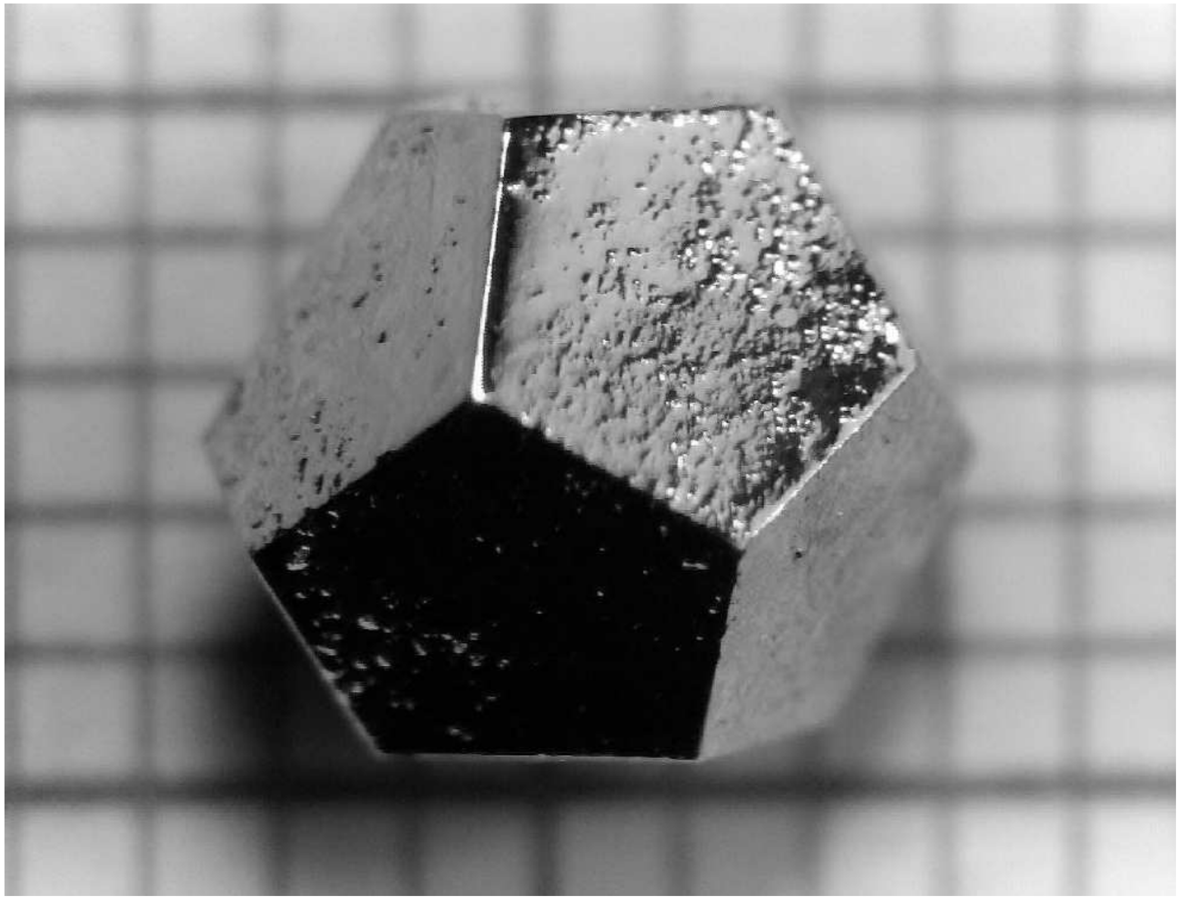
Quasicrystals



Quasicrystals



Dan Shechtman

Wolf Prize in Physics 1999

Nobel Prize in Chemistry 2011

Periodic point sets

Definition: A (discrete) point set $\varLambda \subset \mathbb{R}^d$ is called *periodic*, when $t + \varLambda = \varLambda$ holds for some $t \neq 0$. It is called *crystallographic* when the group of periods, $\mathrm{per}(\varLambda) = \{t \in \mathbb{R}^d \mid t + \varLambda = \varLambda\}$, is a lattice.

Crystallographic restriction: If (t, M) is a Euclidean motion that maps a crystallographic point set $\varLambda \subset \mathbb{R}^d$ onto itself, the characteristic polynomial of M has integer coefficients only.

In particular, for $d \in \{2, 3\}$, the possible rotation symmetries have order $1, 2, 3, 4$ or 6.

Non-periodic point sets

Definition: A discrete point set $\varLambda \subset \mathbb{R}^d$ is called *non-crystallographic* when $\mathrm{per}(\varLambda)$ is not a lattice, and *non-periodic* when $\mathrm{per}(\varLambda) = \{0\}$.

Examples: $\mathbb{Z} \setminus \{0\}$
$(\mathbb{Z} \setminus \{0\}) \times \mathbb{Z}$

Definition: The *hull* of a discrete point set $\varLambda$ is defined as

$$\mathbb{X}(\varLambda) \;:=\; \overline{\{t + \varLambda \mid t \in \mathbb{R}^d\}},$$

where the closure is taken in the local (rubber) topology.

Non-periodic point sets

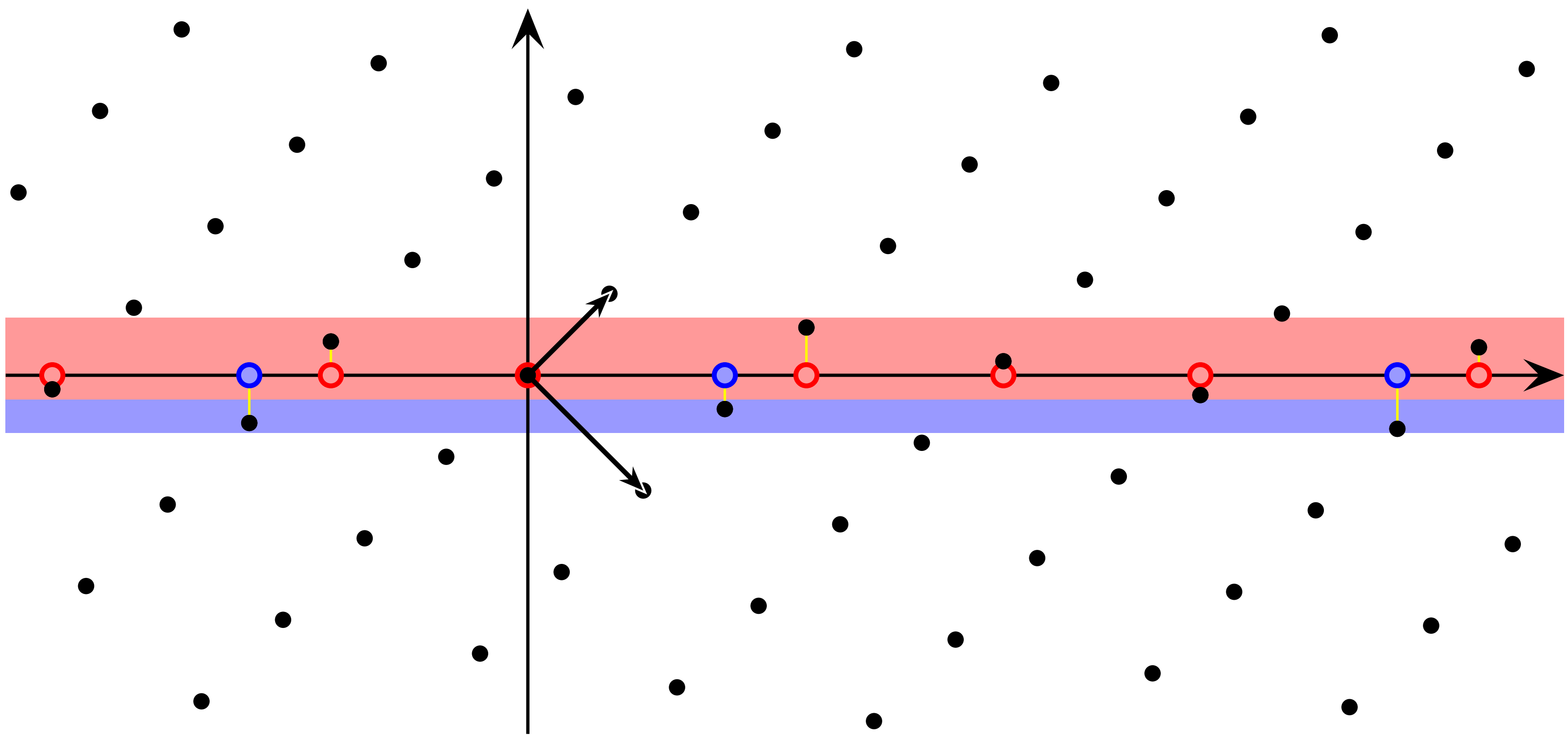
Definition: A discrete point set $\varLambda \subset \mathbb{R}^d$ is called *aperiodic* when $\mathbb{X}(\varLambda)$ contains only non-periodic elements.

It is called *strongly aperiodic* when the remaining symmetry group of the hull is a finite group.

Aperiodic point sets

Silver mean substitution: $a \mapsto aba,\ b \mapsto a \qquad (\lambda_{\mathrm{PF}} = 1 + \sqrt{2})$

Silver mean point set: $\varLambda = \big\{ x \in \mathbb{Z}[\sqrt{2}\,] \mid x' \in [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\,] \big\}$



Model sets

CPS:

$$
\begin{array}{ccccc}
\mathbb{R}^d & \xleftarrow{\;\pi\;} & \mathbb{R}^d \times \mathbb{R}^m & \xrightarrow{\;\pi_{\mathrm{int}}\;} & \mathbb{R}^m \\
\cup & & \cup & & \cup \ \text{dense} \\
\pi(\mathcal{L}) & \xleftarrow{\;1-1\;} & \mathcal{L} & \longrightarrow & \pi_{\mathrm{int}}(\mathcal{L}) \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\qquad\star\qquad\qquad} & & L^{\star}
\end{array}
$$

Model set: $\boxed{\varLambda = \{ x \in L \mid x^{\star} \in W \}}$ (assumed regular)

with $W \subset \mathbb{R}^m$ compact, $\lambda(\partial W) = 0$

Diffraction: $\boxed{\widehat{\gamma} = \sum_{k \in L^{\circledast}} |A(k)|^2 \, \delta_k}$

with $L^{\circledast} = \pi(\mathcal{L}^{*})$ (Fourier module of $\varLambda$)

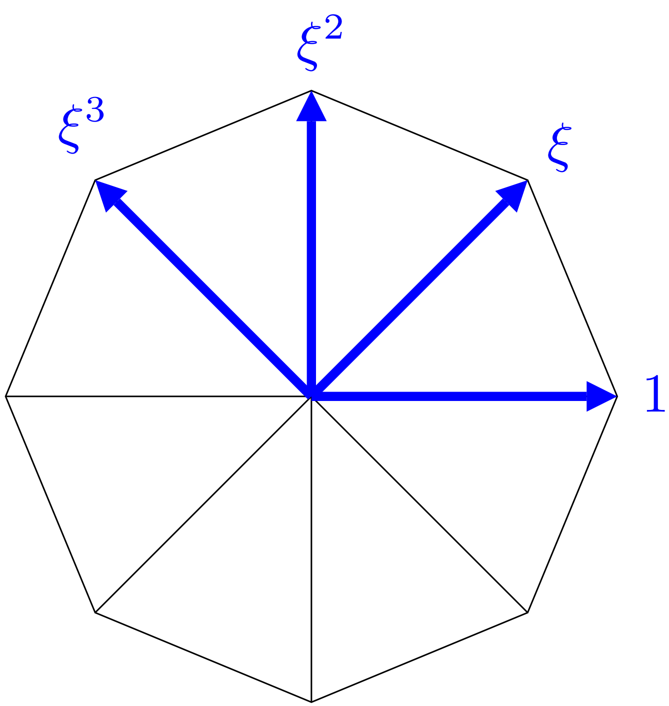
and amplitude $A(k) = \frac{\mathrm{dens}(\varLambda)}{\mathrm{vol}(W)} \, \widehat{1_W}(-k^{\star})$

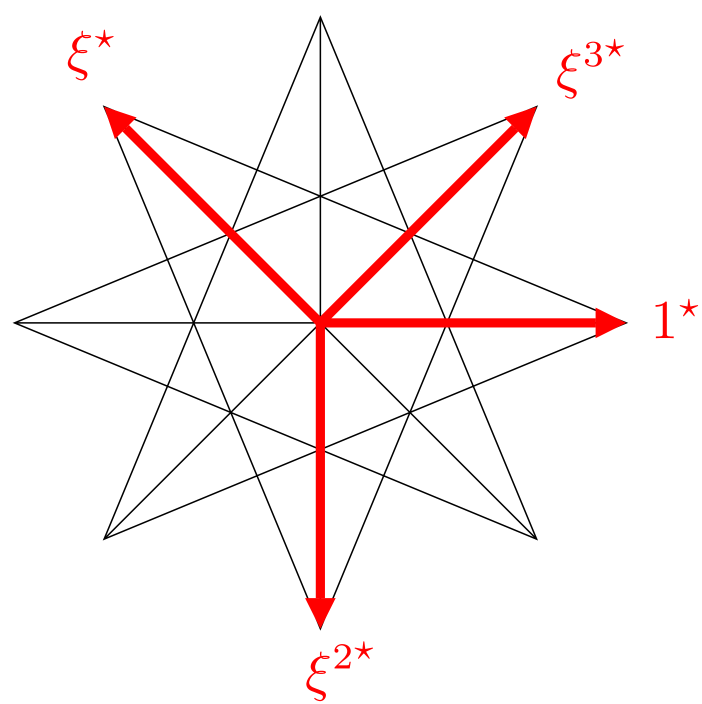
Ammann-Beenker tiling

$L = \mathbb{Z}[\xi]$ $\qquad \mathcal{L} \sim \mathbb{Z}^4 \subset \mathbb{R}^2 \times \mathbb{R}^2$ $\qquad O$: octagon

$\xi = \exp(2\pi i/8)$ $\qquad \phi(8) = 4$ $\qquad \star$-map: $\xi \mapsto \xi^3$

$$\Lambda_{\mathrm{AB}} = \left\{ x \in \mathbb{Z}1 + \mathbb{Z}\xi + \mathbb{Z}\xi^2 + \mathbb{Z}\xi^3 \mid x^\star \in O \right\}$$





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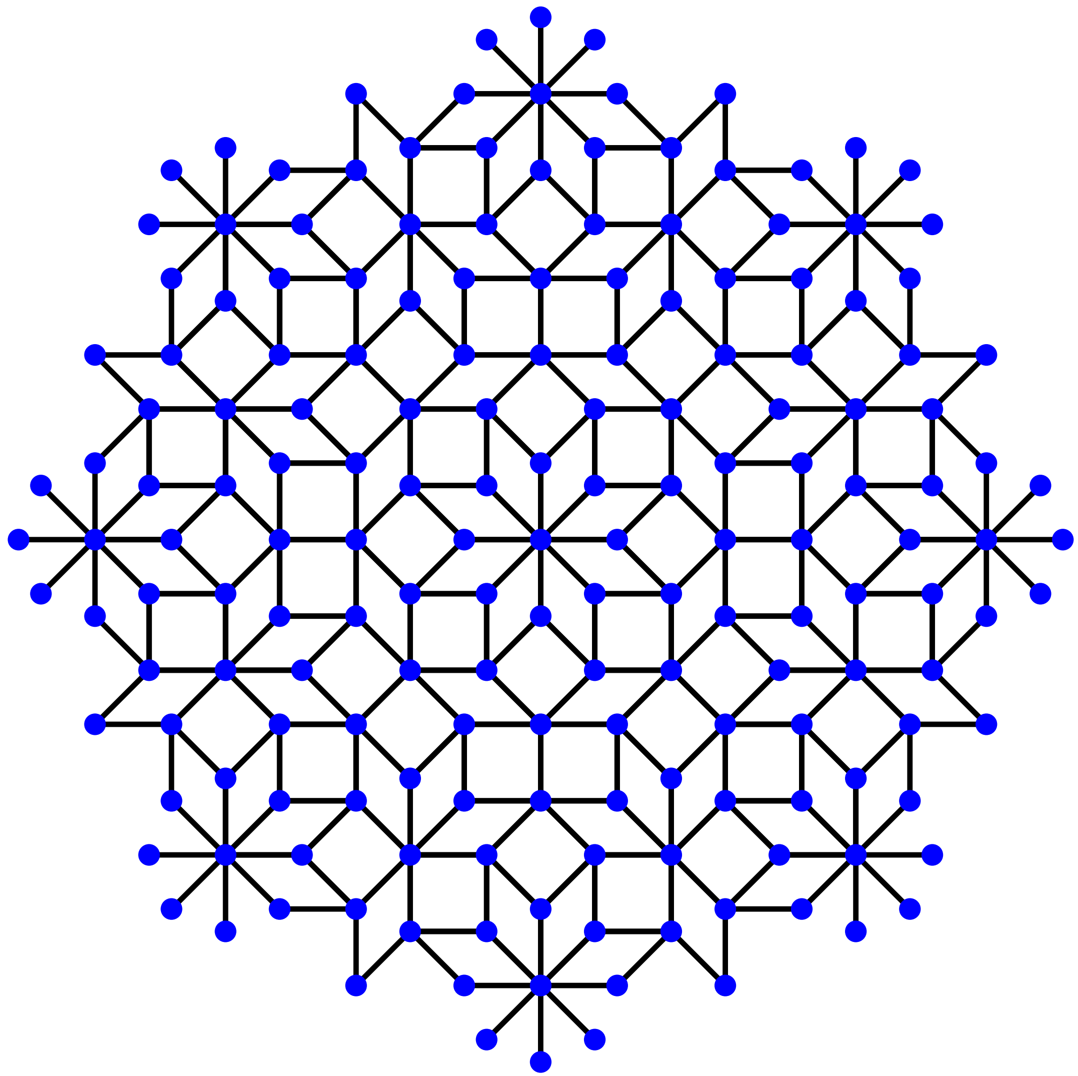
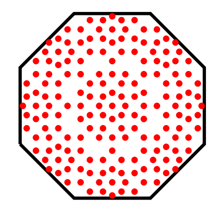
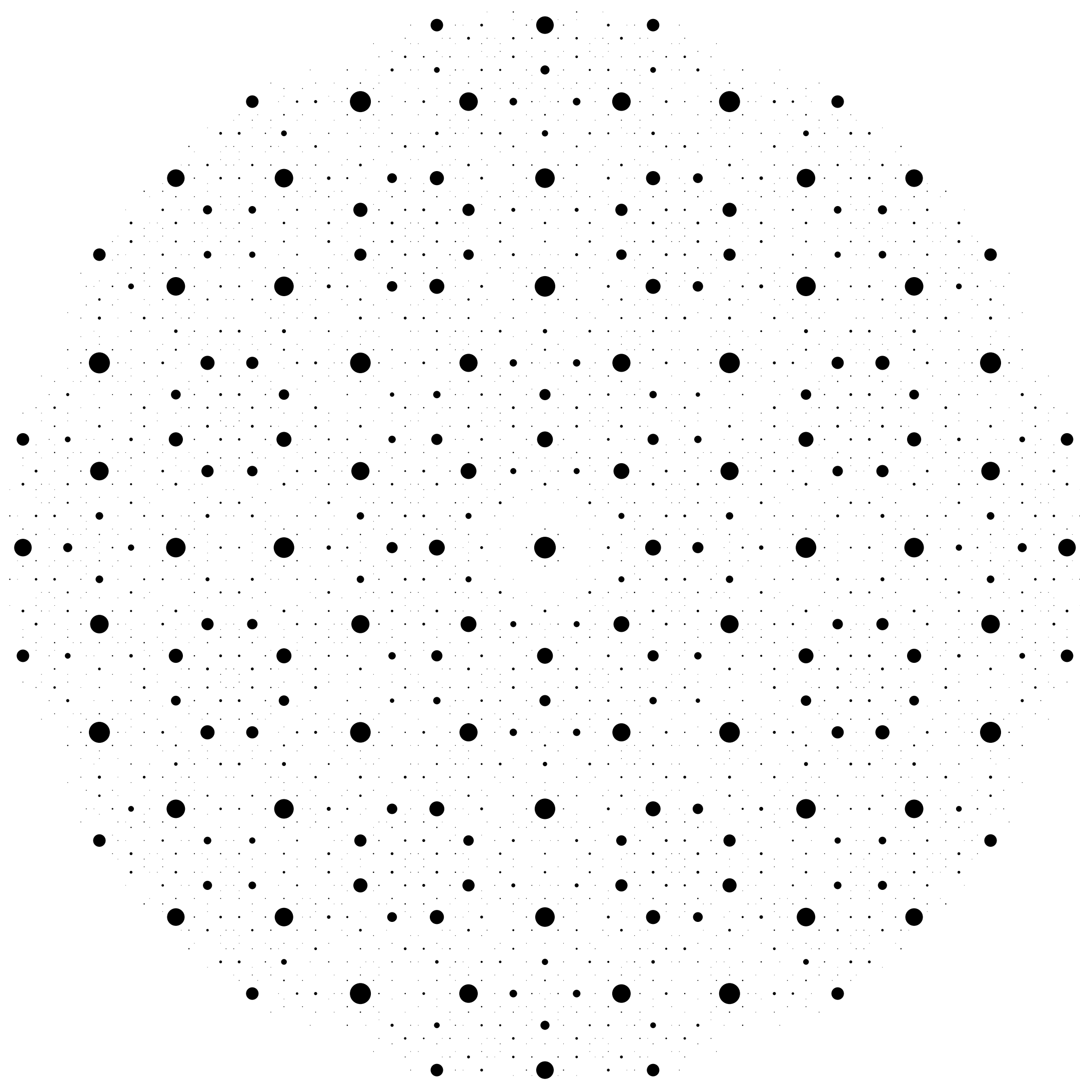


physical space

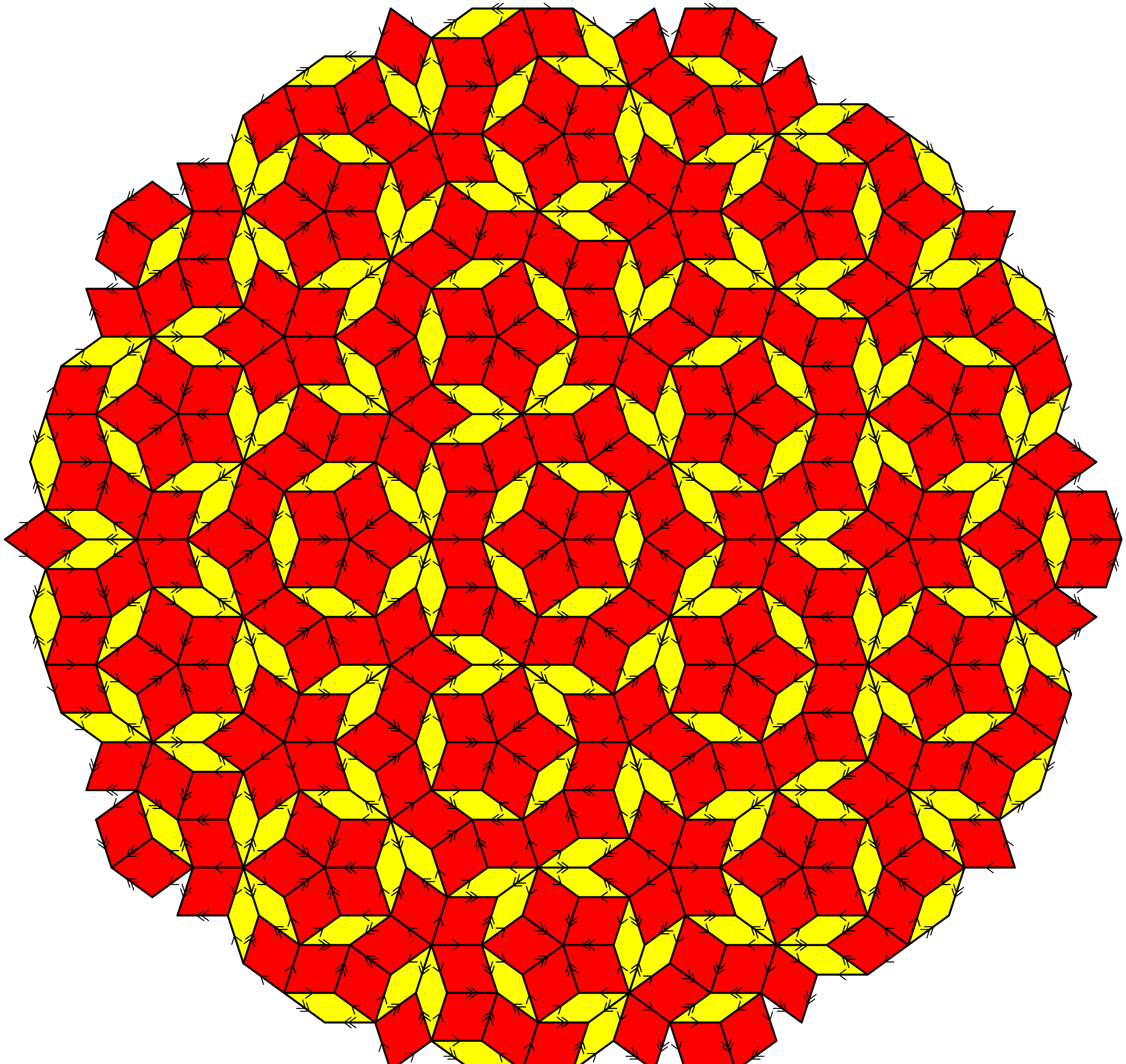


internal space

Ammann-Beenker tiling



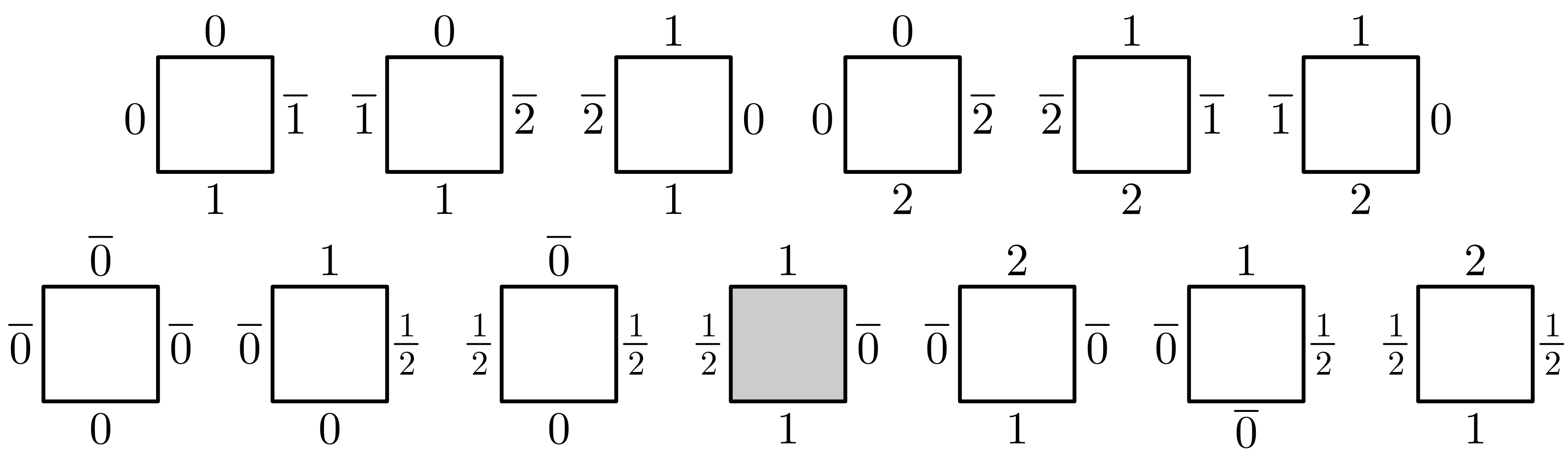
Aperiodic tilings



Aperiodic tilings

Many examples with hierarchical structure (see below).

Exception: The Kari-Culik prototile set



Question

Is there a single shape that tiles space without gaps or overlaps, but does not admit any periodic tiling?

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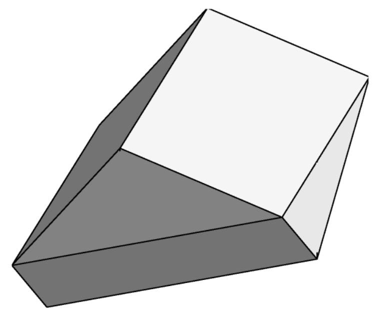
3D: Schmitt-Conway-Danzer ‘einstein’



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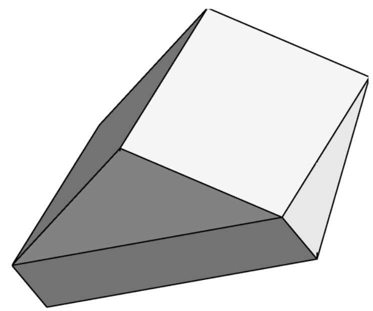
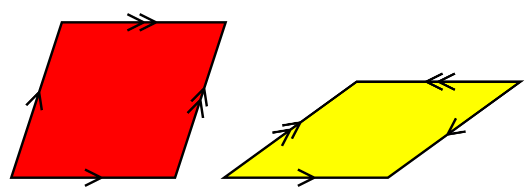


2D: Penrose tiling (two tiles)



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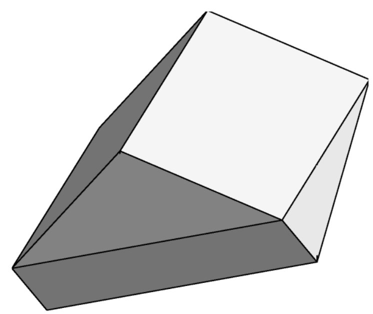

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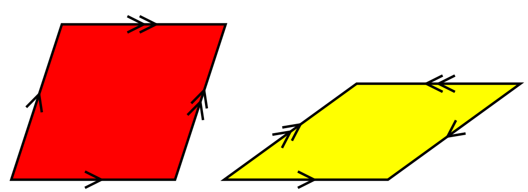

No monotile known — but Penrose's $1 + \varepsilon + \varepsilon^2$ tiling

The Taylor Tiling: Story

19 Feb 2010: Email from Joshua Socolar announcing
An aperiodic hexagonal tile
(joint preprint with Joan M. Taylor)

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28 Feb 2010: Visit Joan Taylor in Burnie, Tasmania

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19 Feb 2010: Email from Joshua Socolar announcing
An aperiodic hexagonal tile
(joint preprint with Joan M. Taylor)

based on Joan's unpublished manuscript

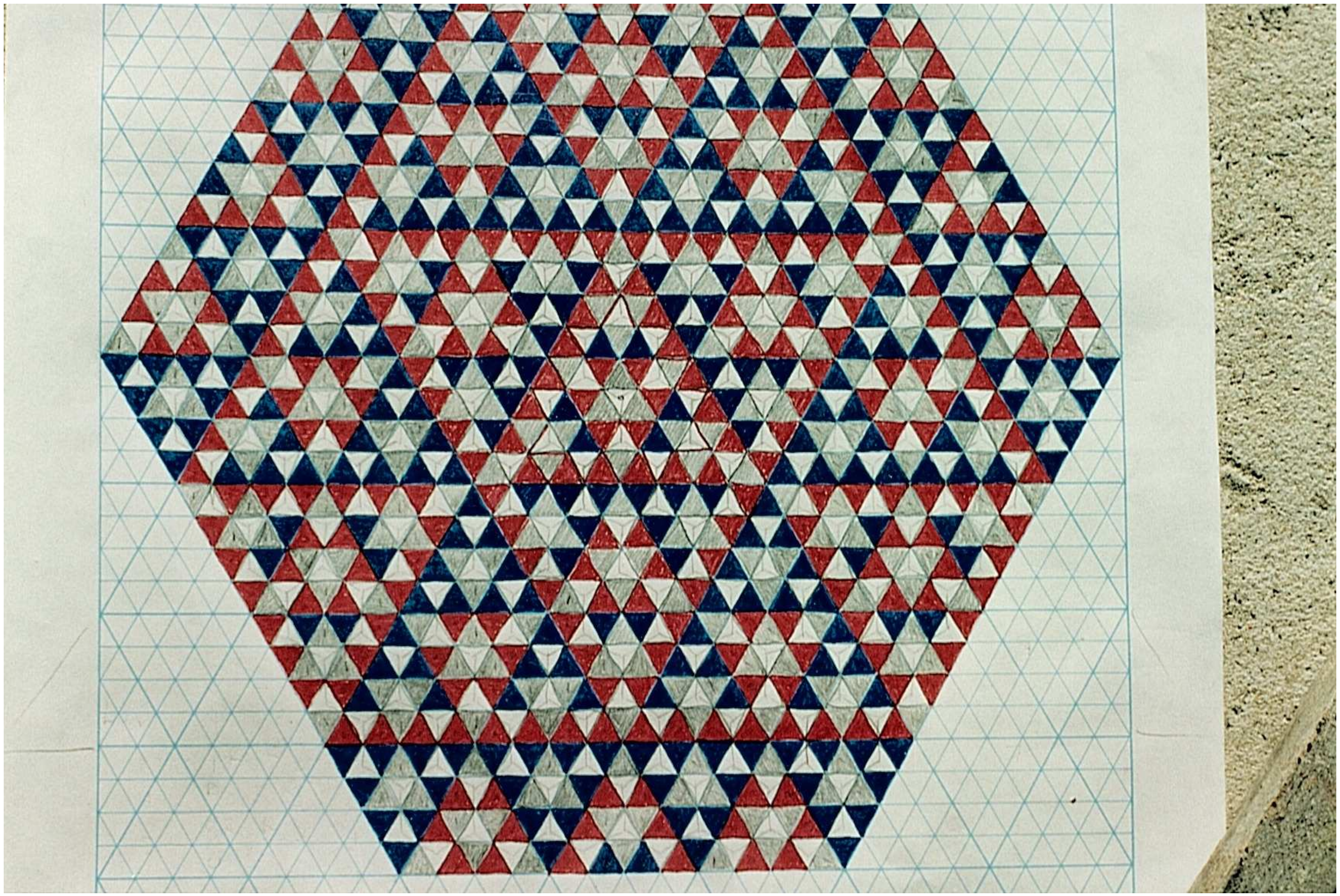
Aperiodicity of a functional monotile

which is available (with hand-drawn diagrammes) from
`http://www.math.uni-bielefeld.de/sfb701/preprints/view/420`

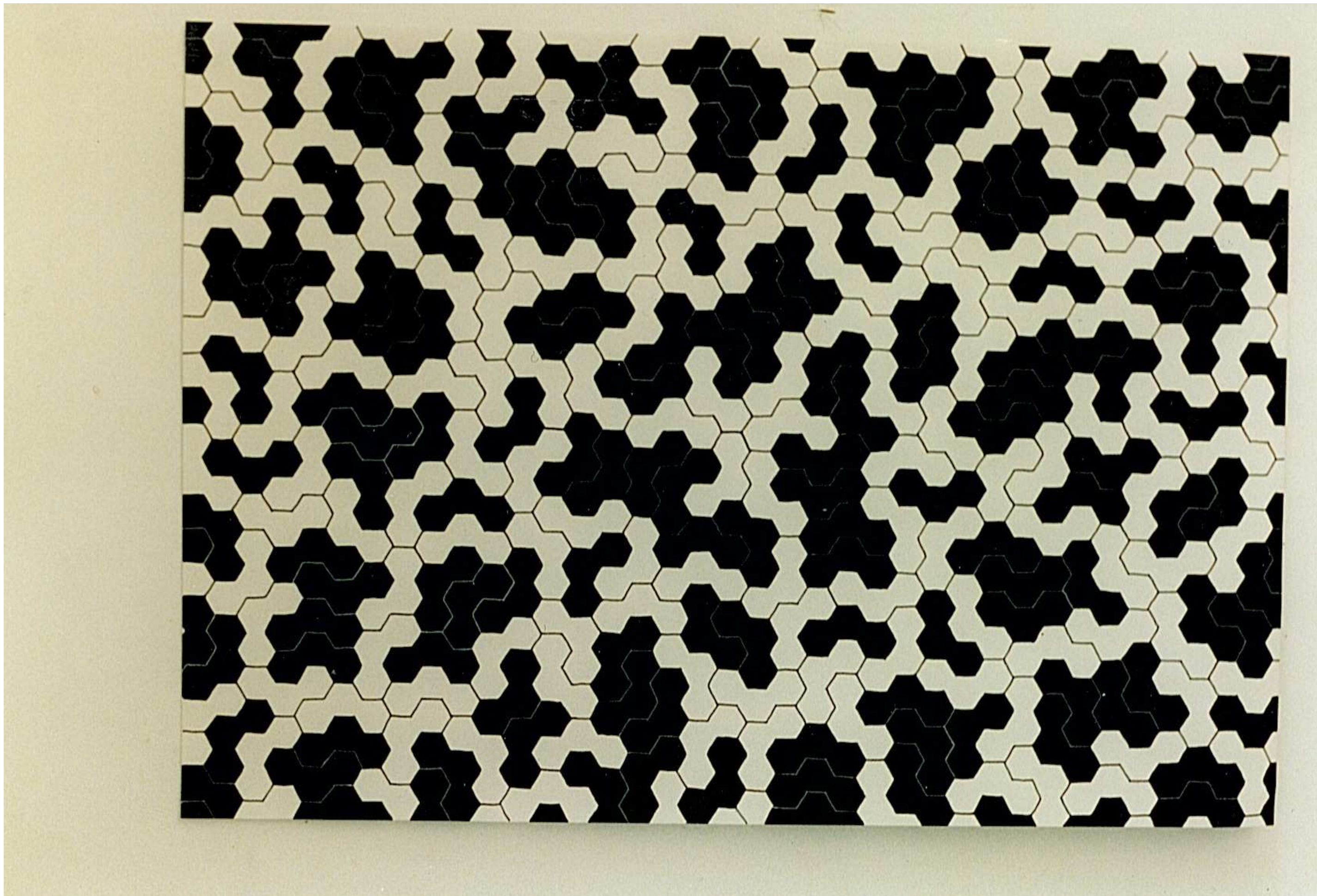
(slight difference in definition of matching rules)

Joan Taylor

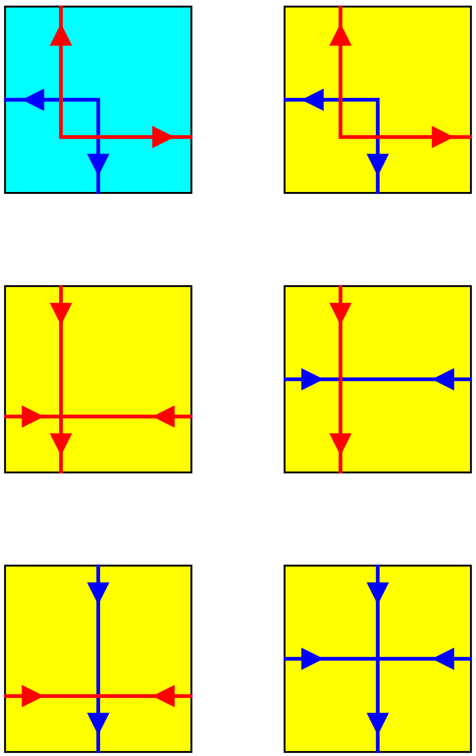
Joan Taylor

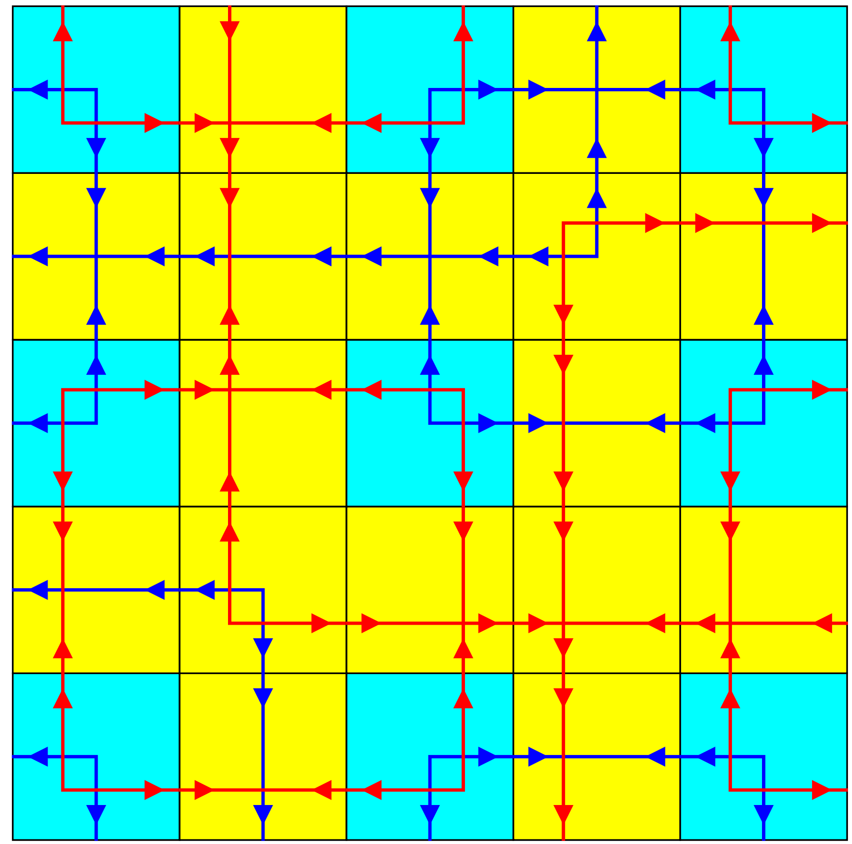


Joan Taylor



Robinson's tiling





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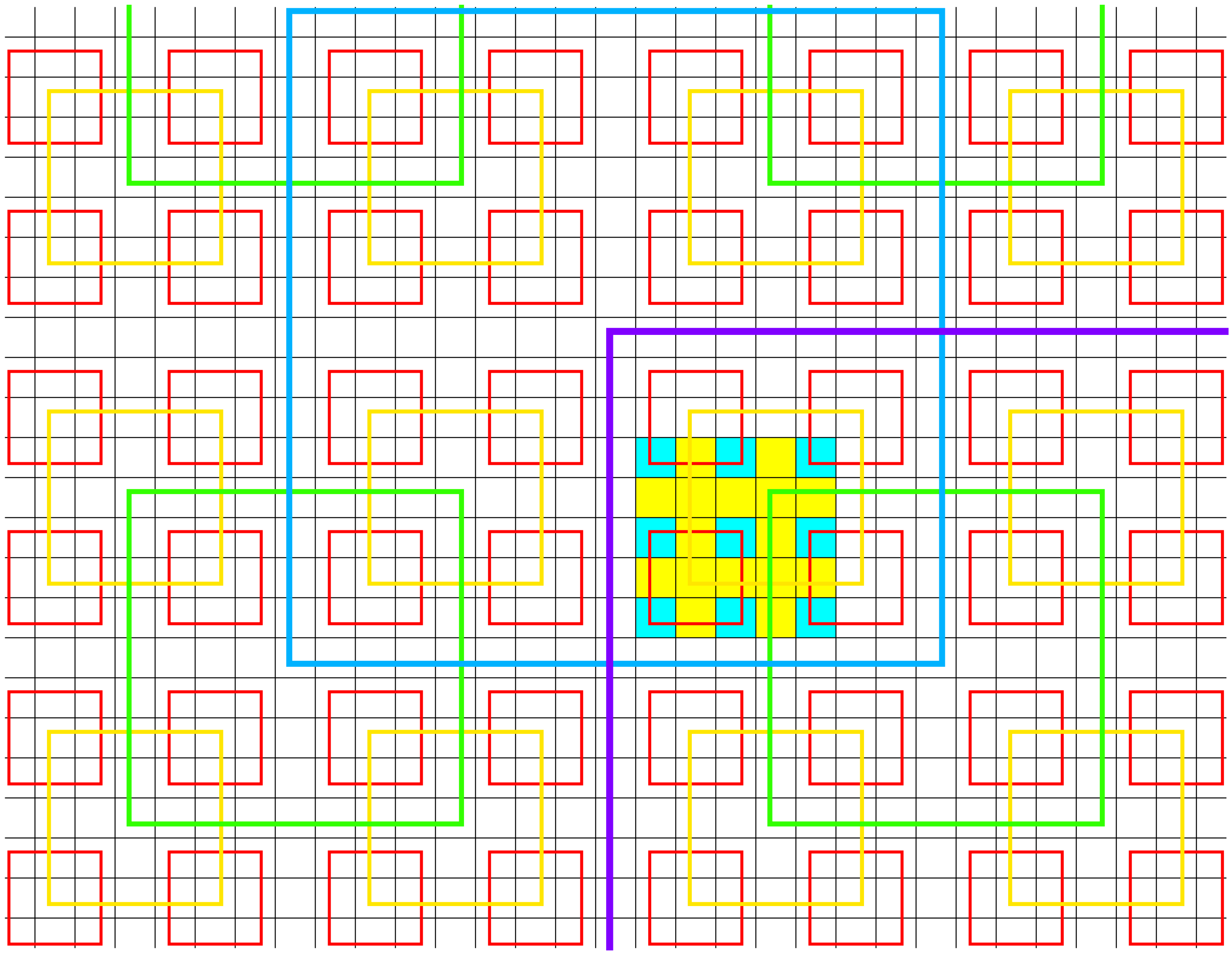
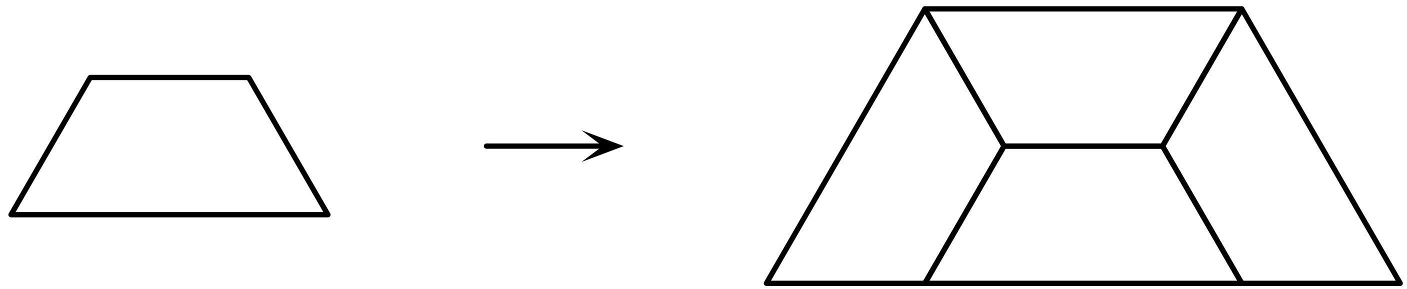


Half-hex tiling



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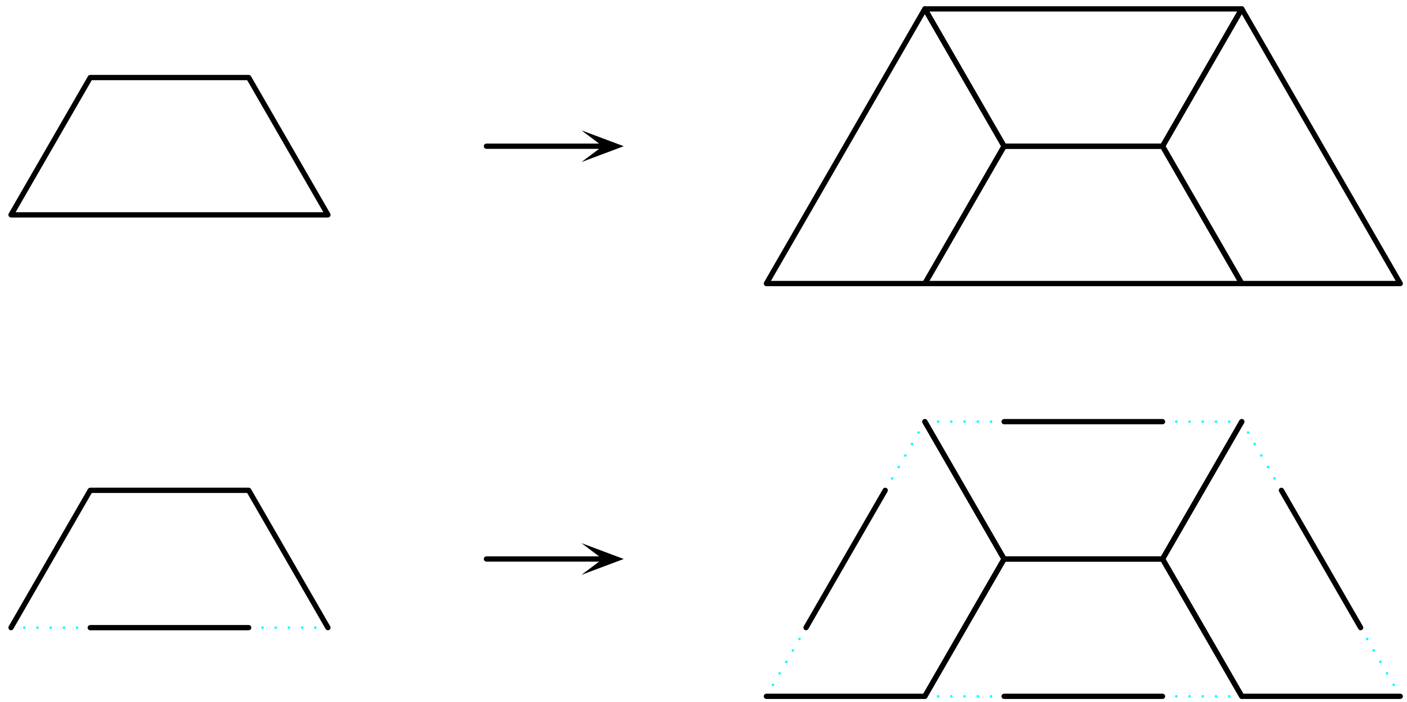

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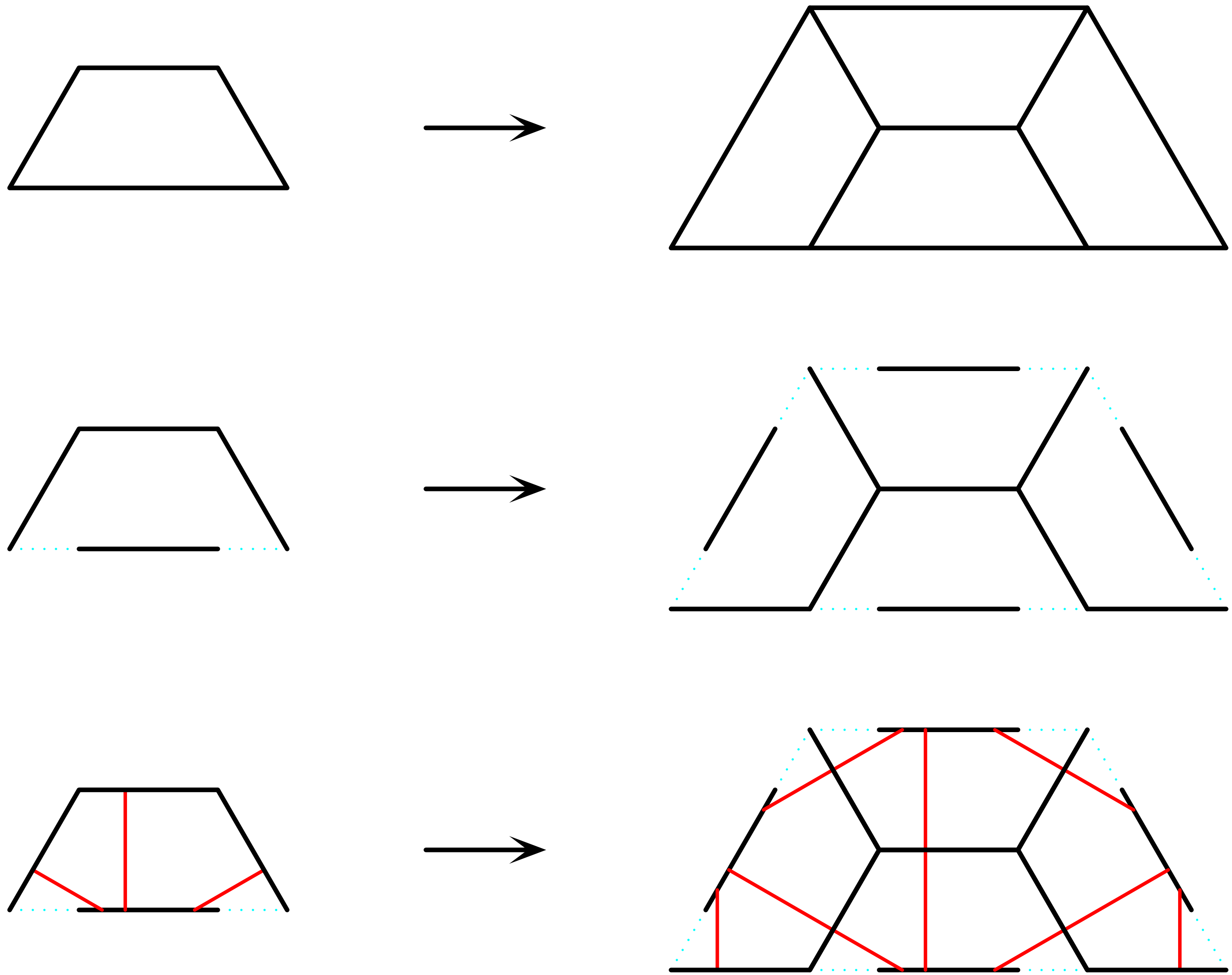

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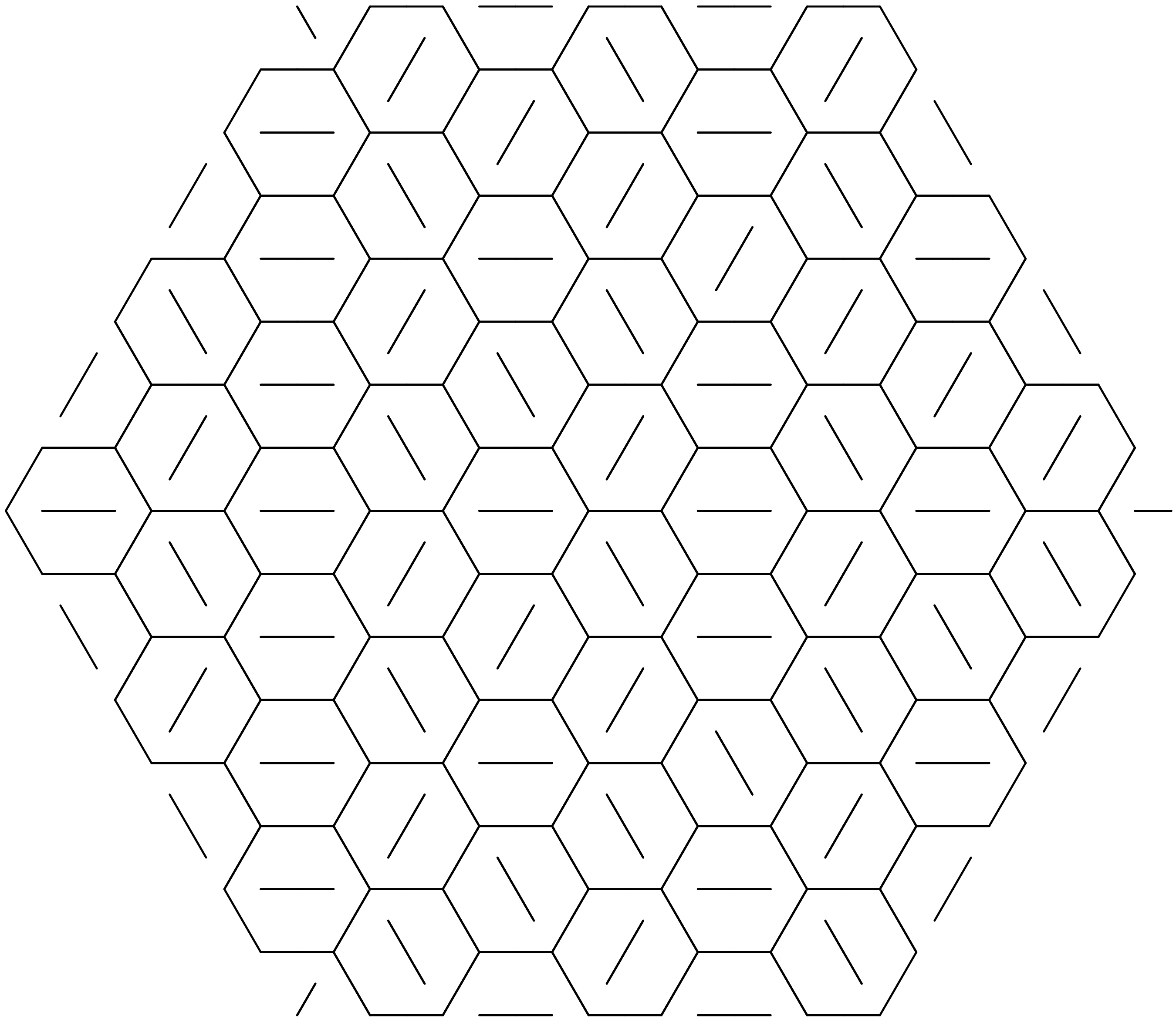

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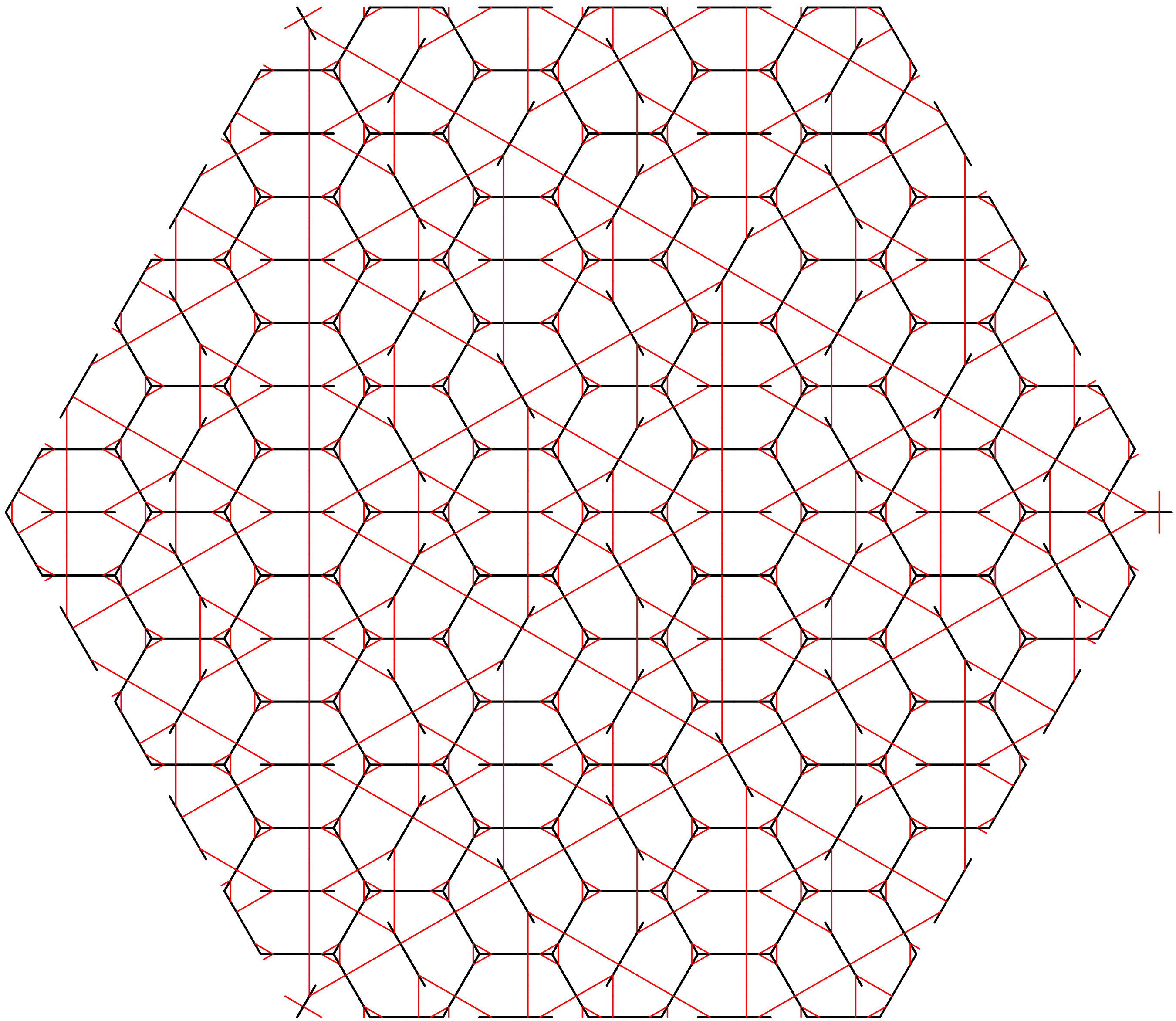

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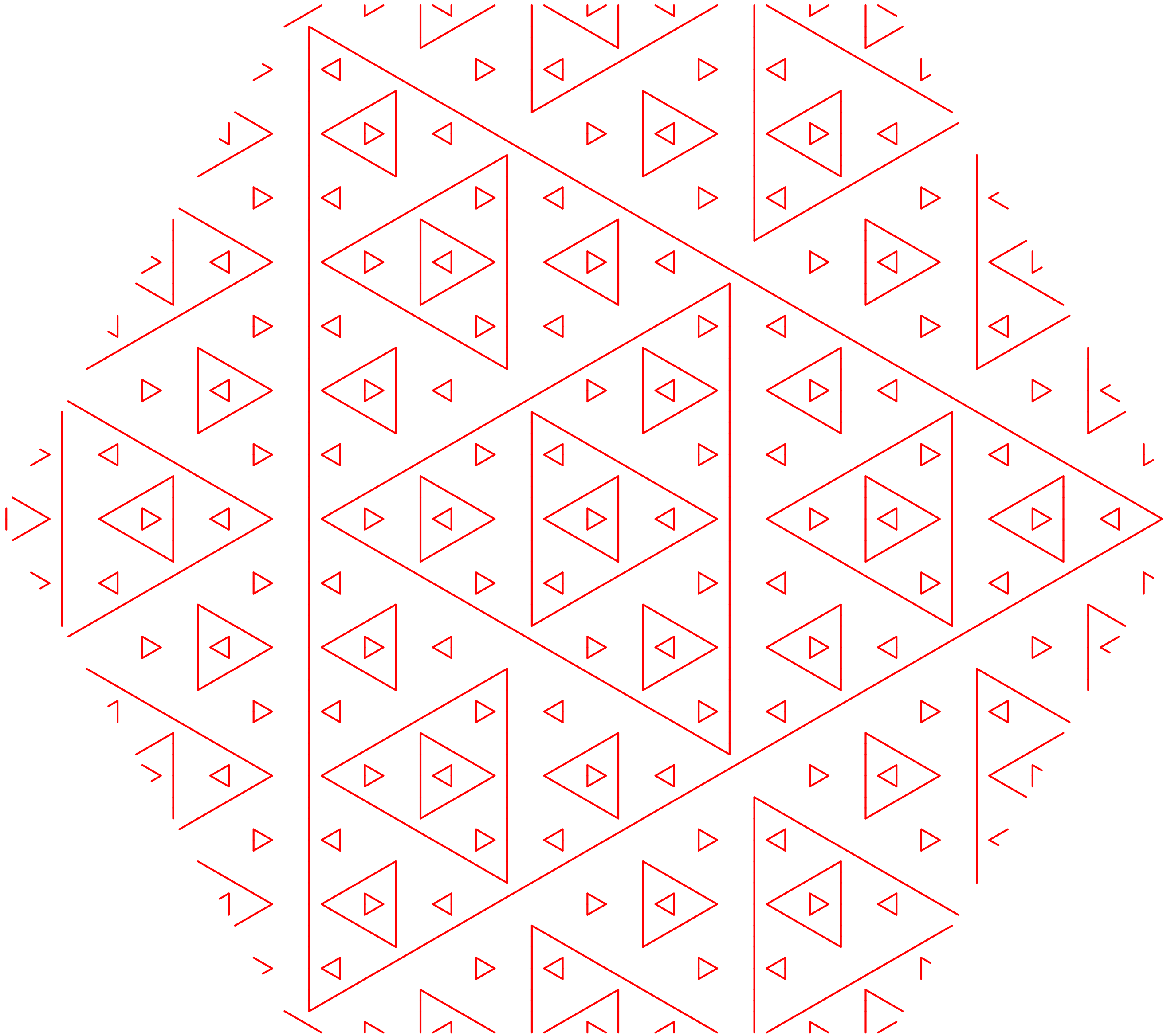
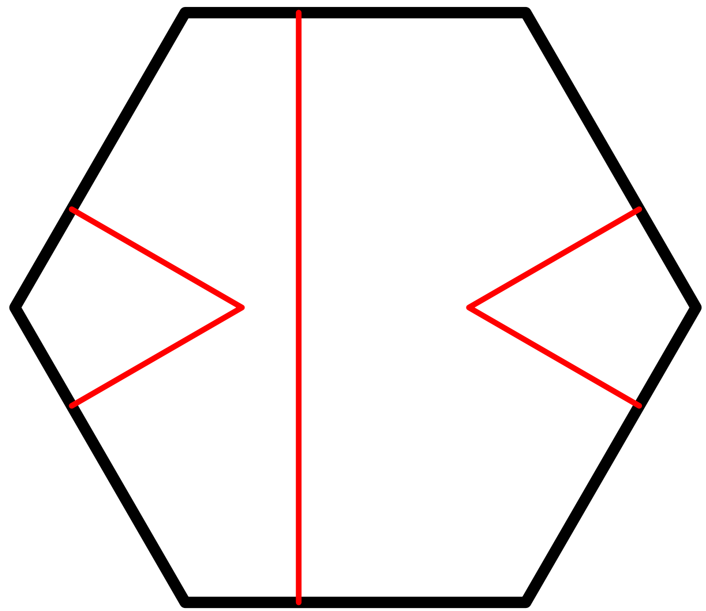


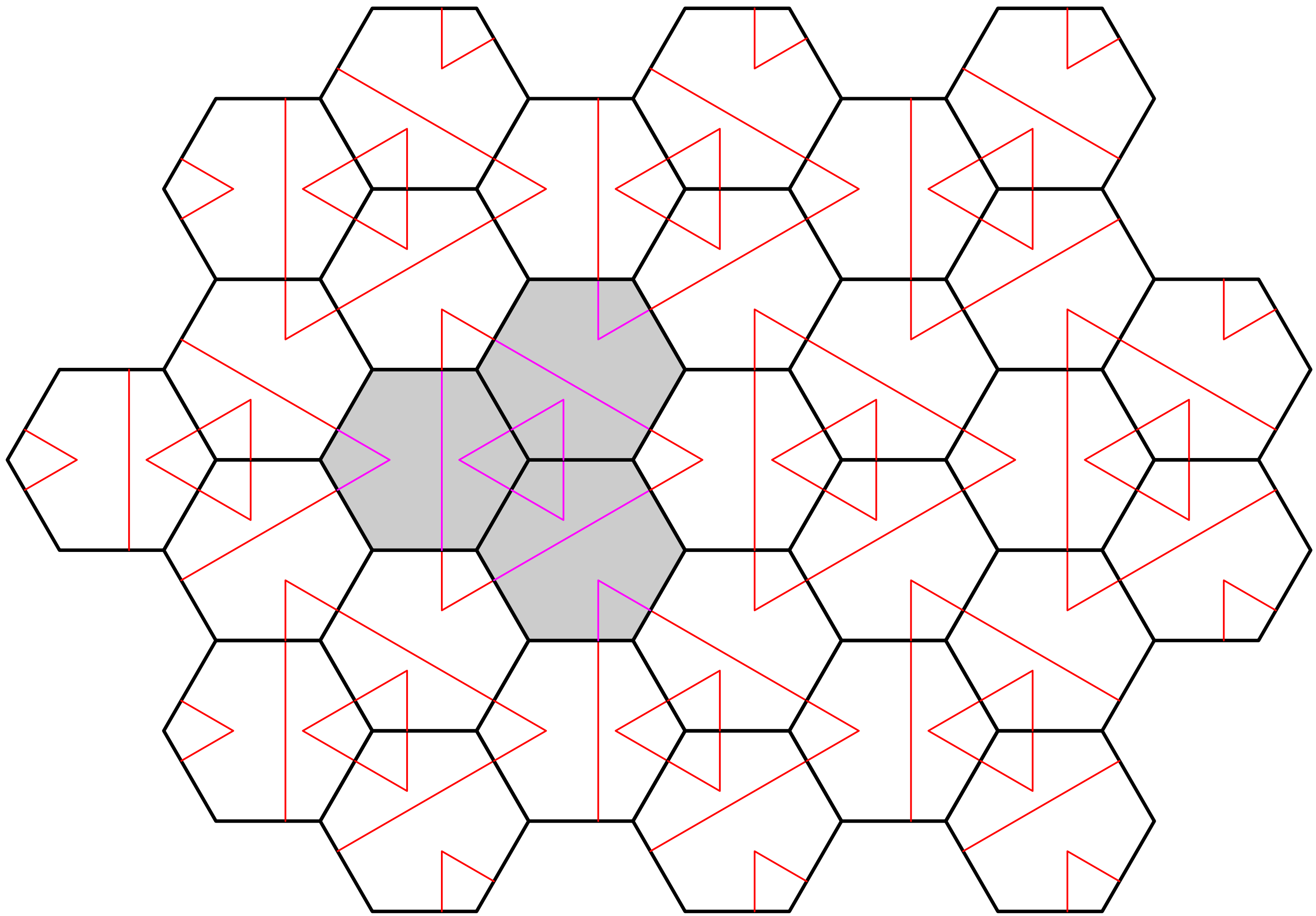
Half-hex tiling

hexagonal tile

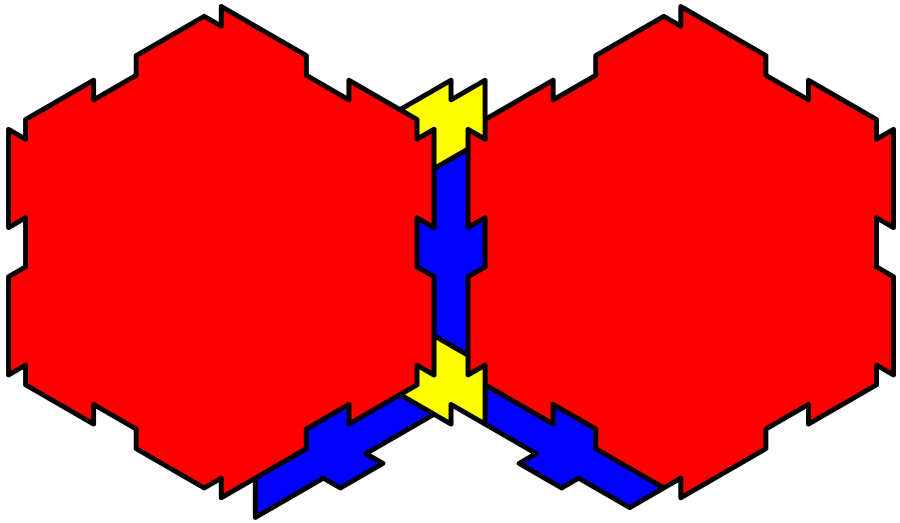


still admits periodic tilings of the plane

Half-hex tiling



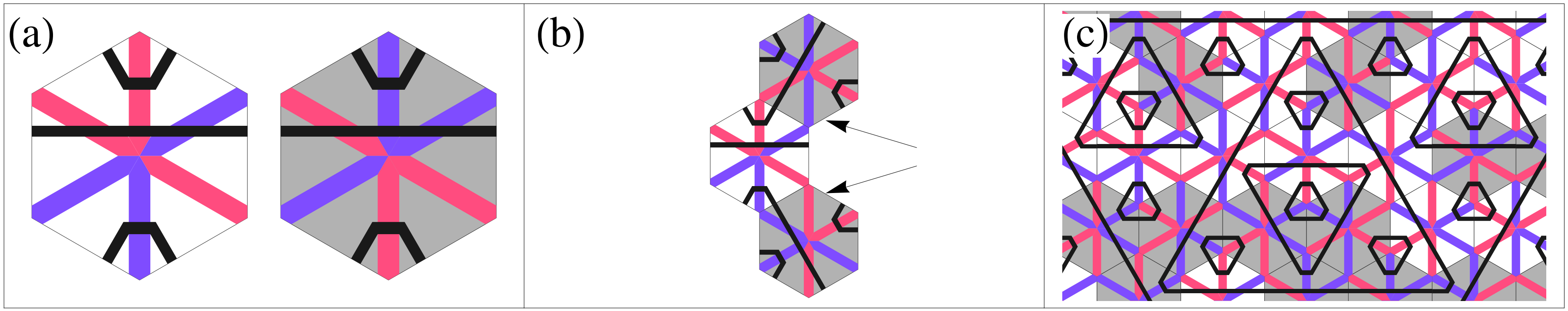
Penrose's $1 + \varepsilon + \varepsilon^2$ tiling

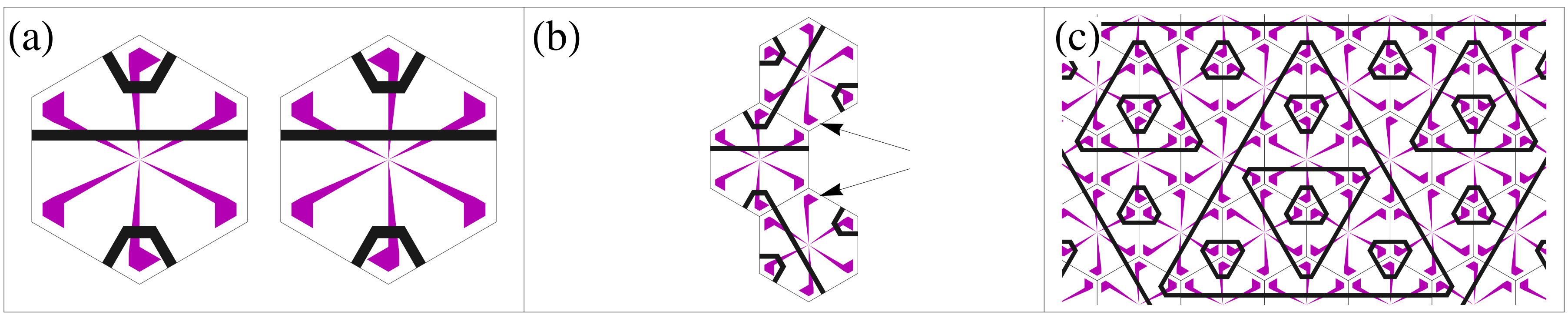


- 3 tiles: $1 + \varepsilon + \varepsilon^2$
- ‘key tiles’ encode matching rule information
- proof of aperiodicity (Penrose)
- the ε tile transmits information along edge

The monotile

(figures from Socolar & Taylor *An aperiodic hexagonal monotile*)





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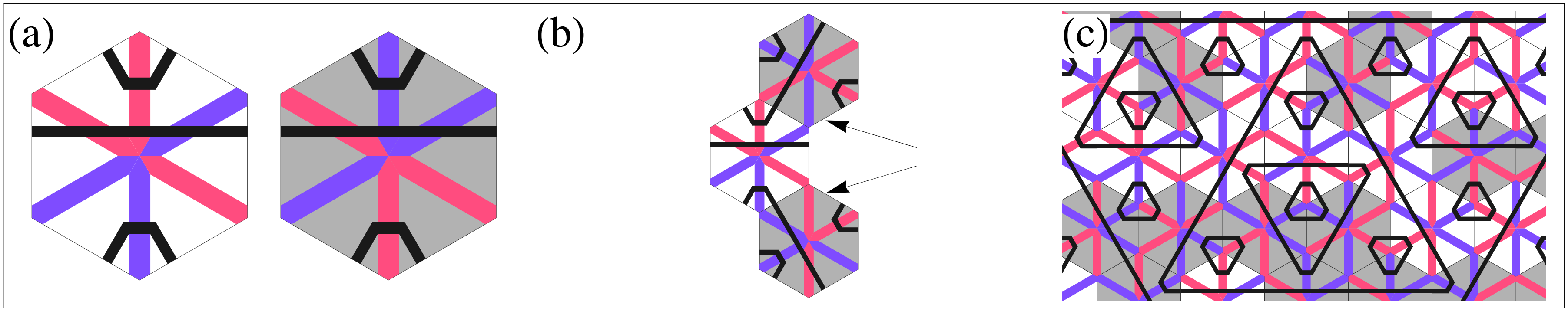
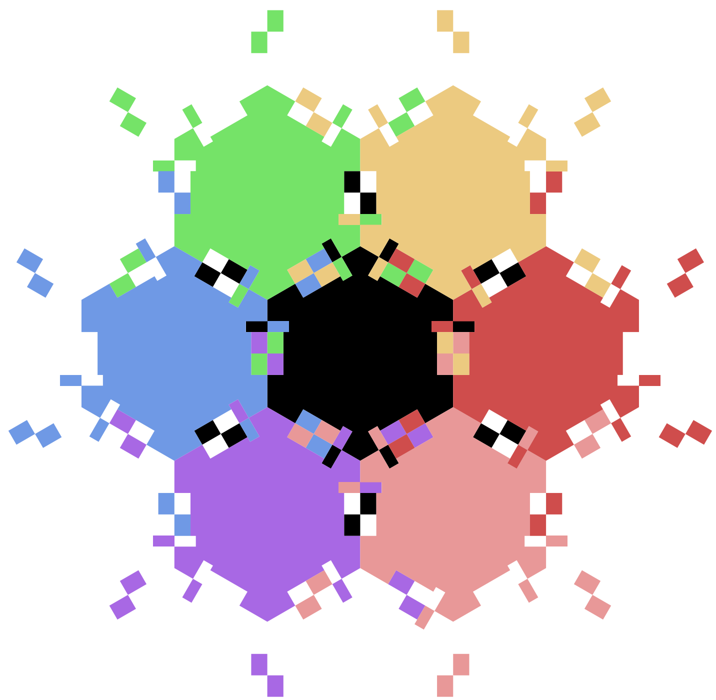
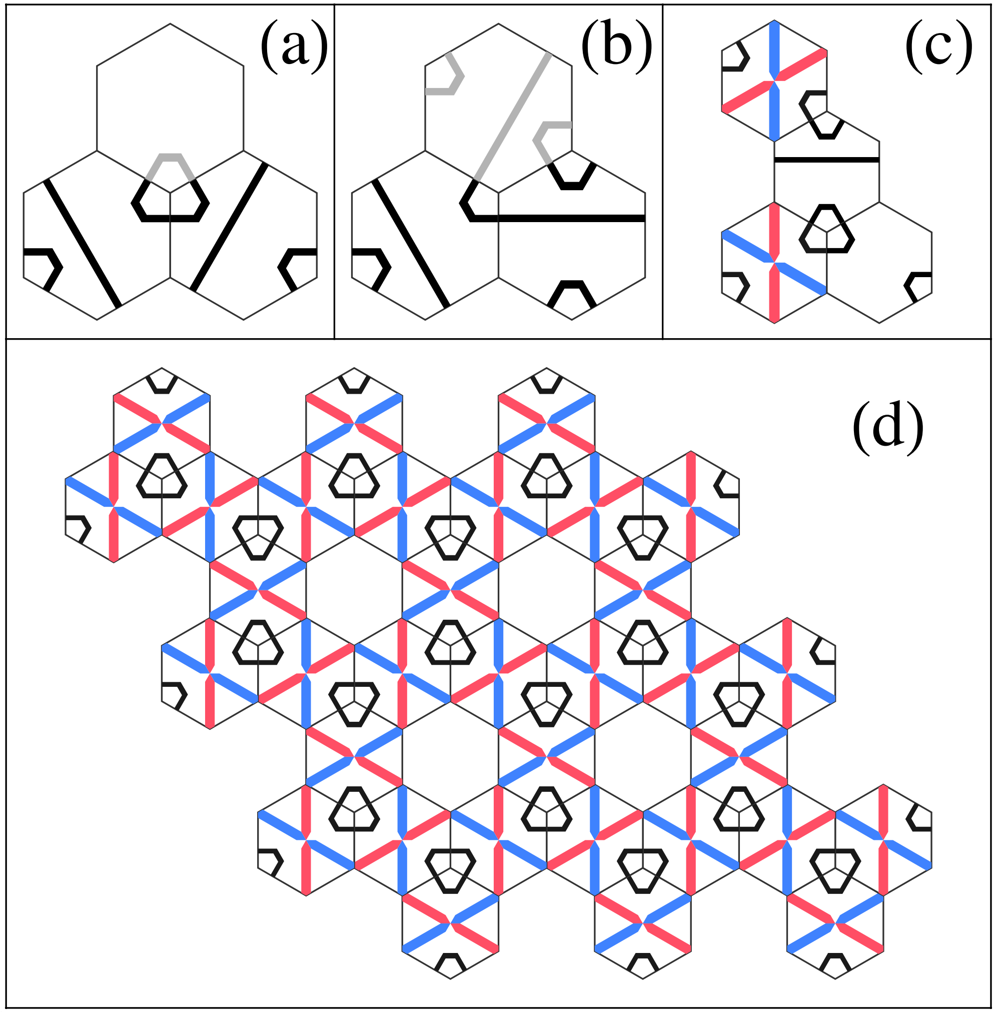




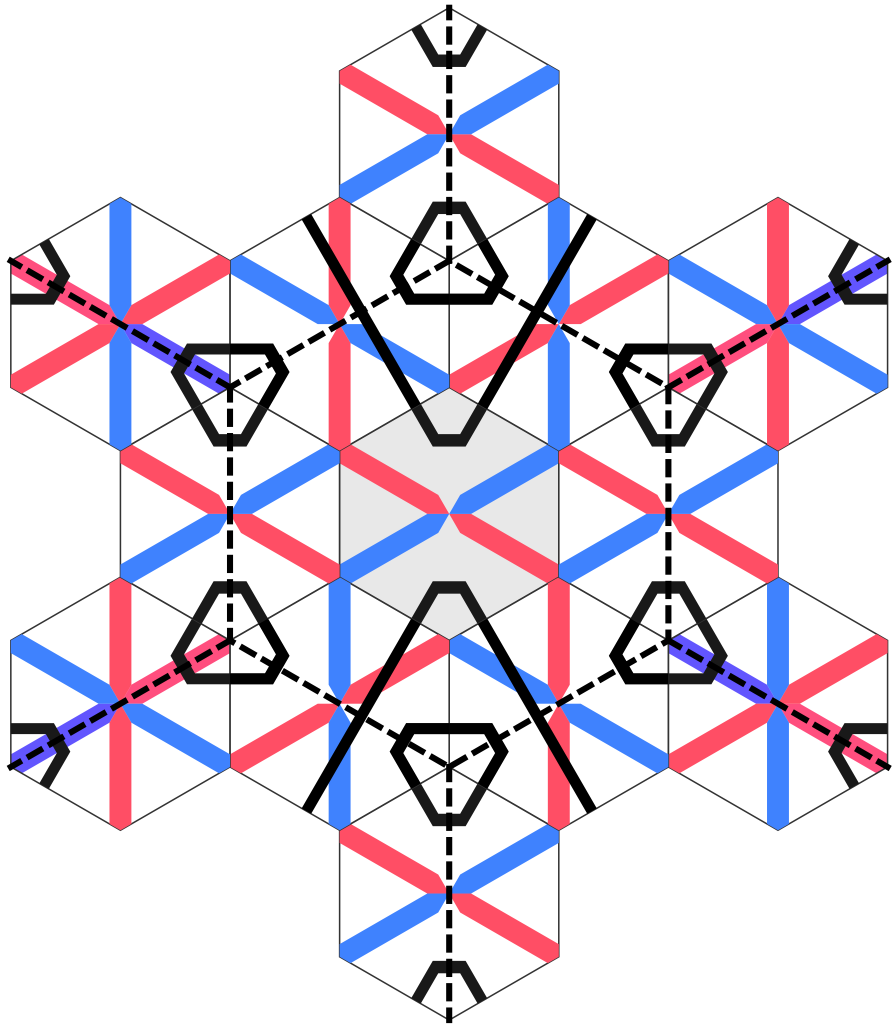
Forced patterns

(figures from Socolar & Taylor *An aperiodic hexagonal monotile*)



Filling the gaps

(figures from Socolar & Taylor *An aperiodic hexagonal monotile*)



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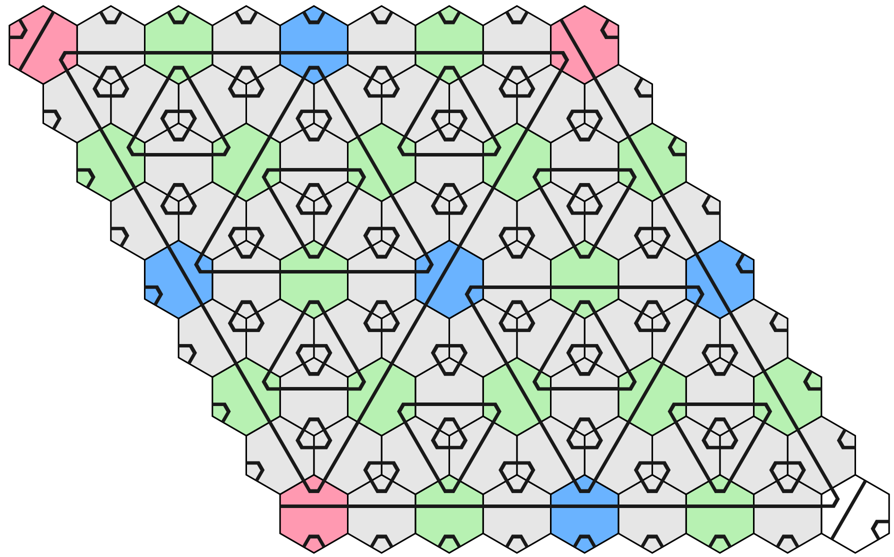

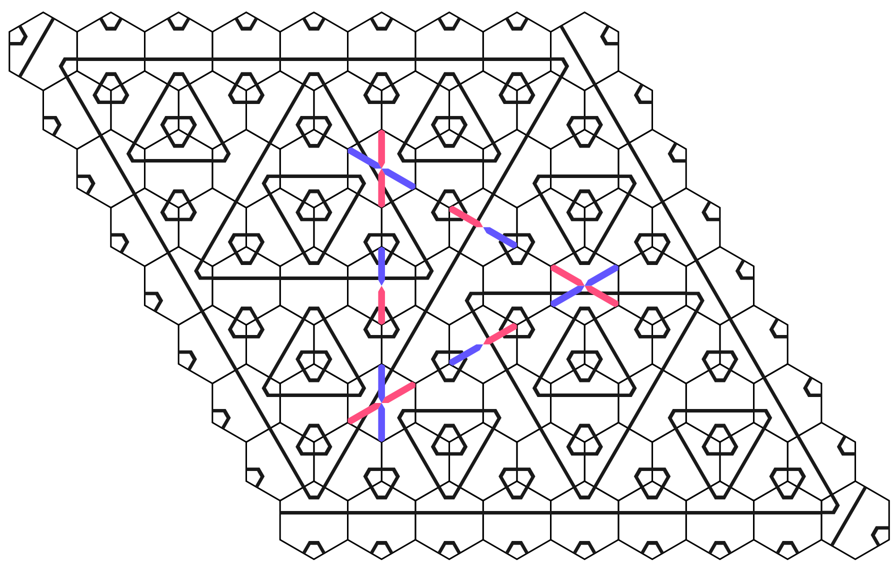

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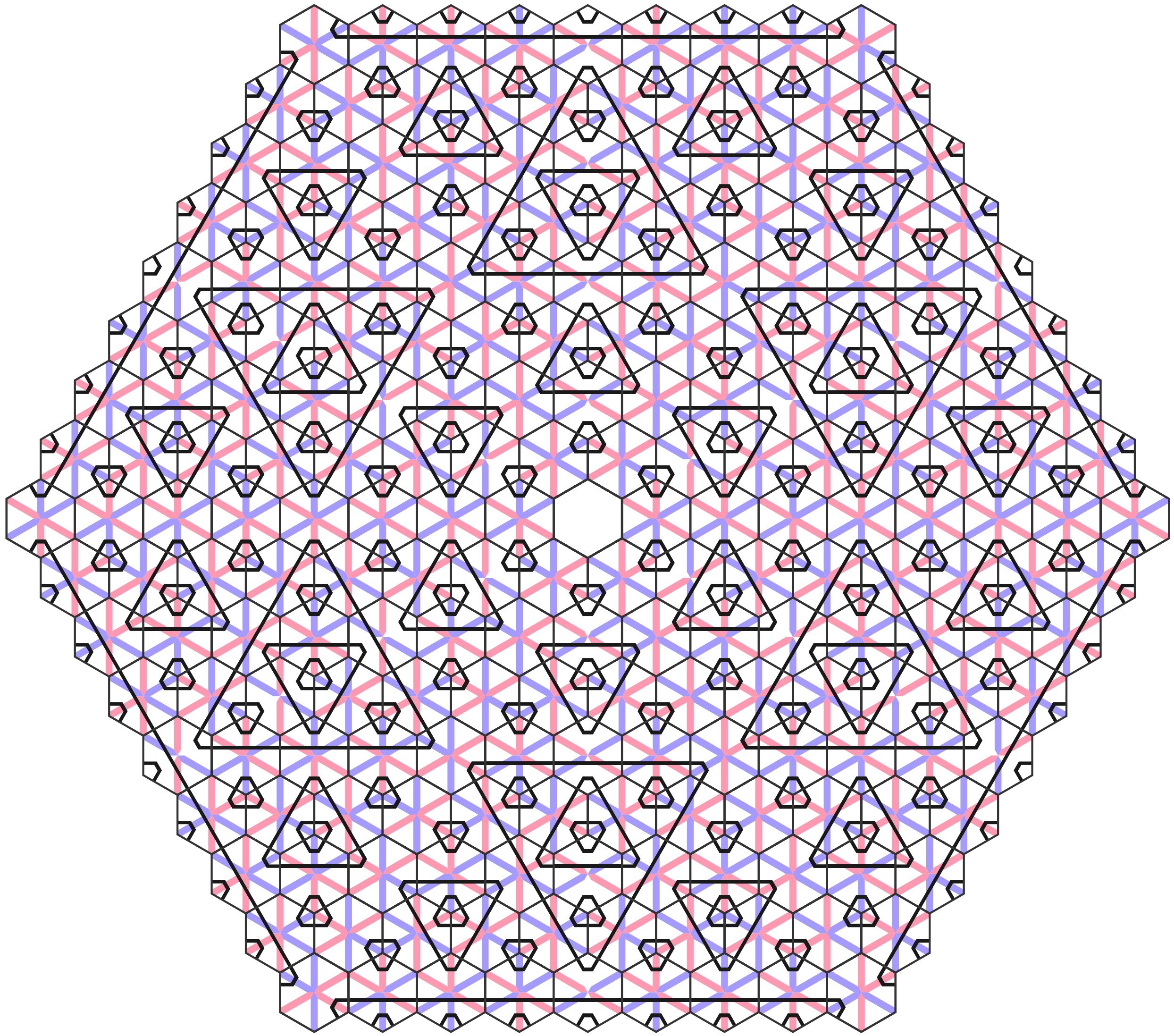


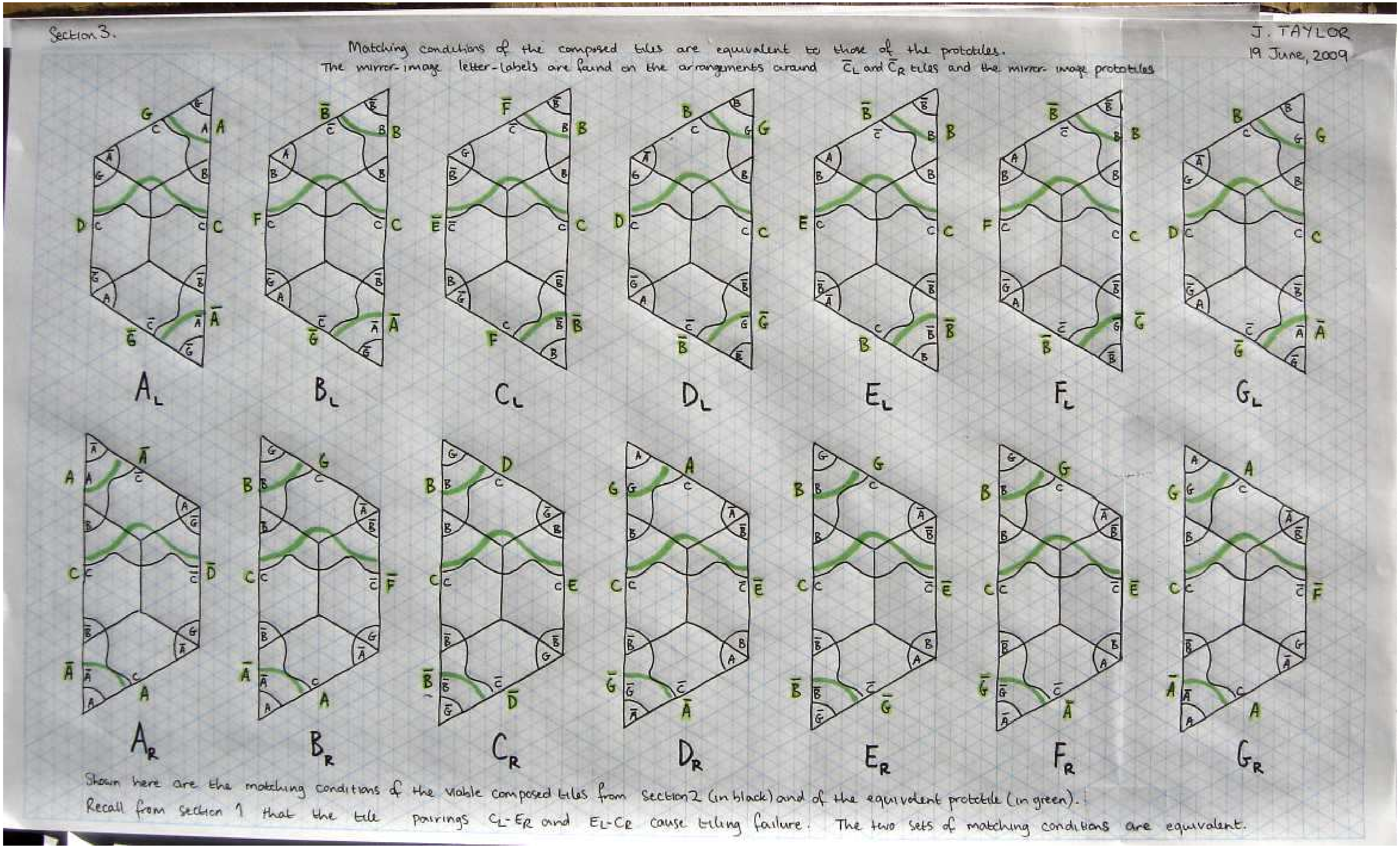
Composition-decomposition method

(Franz Gähler 1993)

- method to show that matching rules (local rules) enforce non-periodicity
- based on inflation (self-similarity)
- requirements:
  - Inflation rule has to respect matching rules: Tiles that match must have decompositions that match
  - In any admitted tiling, each tile can be composed, together with part of its neighbours, to a unique *supertile*
  - The *supertiles* inherit markings that enforce equivalent matching rules

Taylor's substitution

(figures from Taylor's manuscript *Aperiodicity of a functional monotile*)



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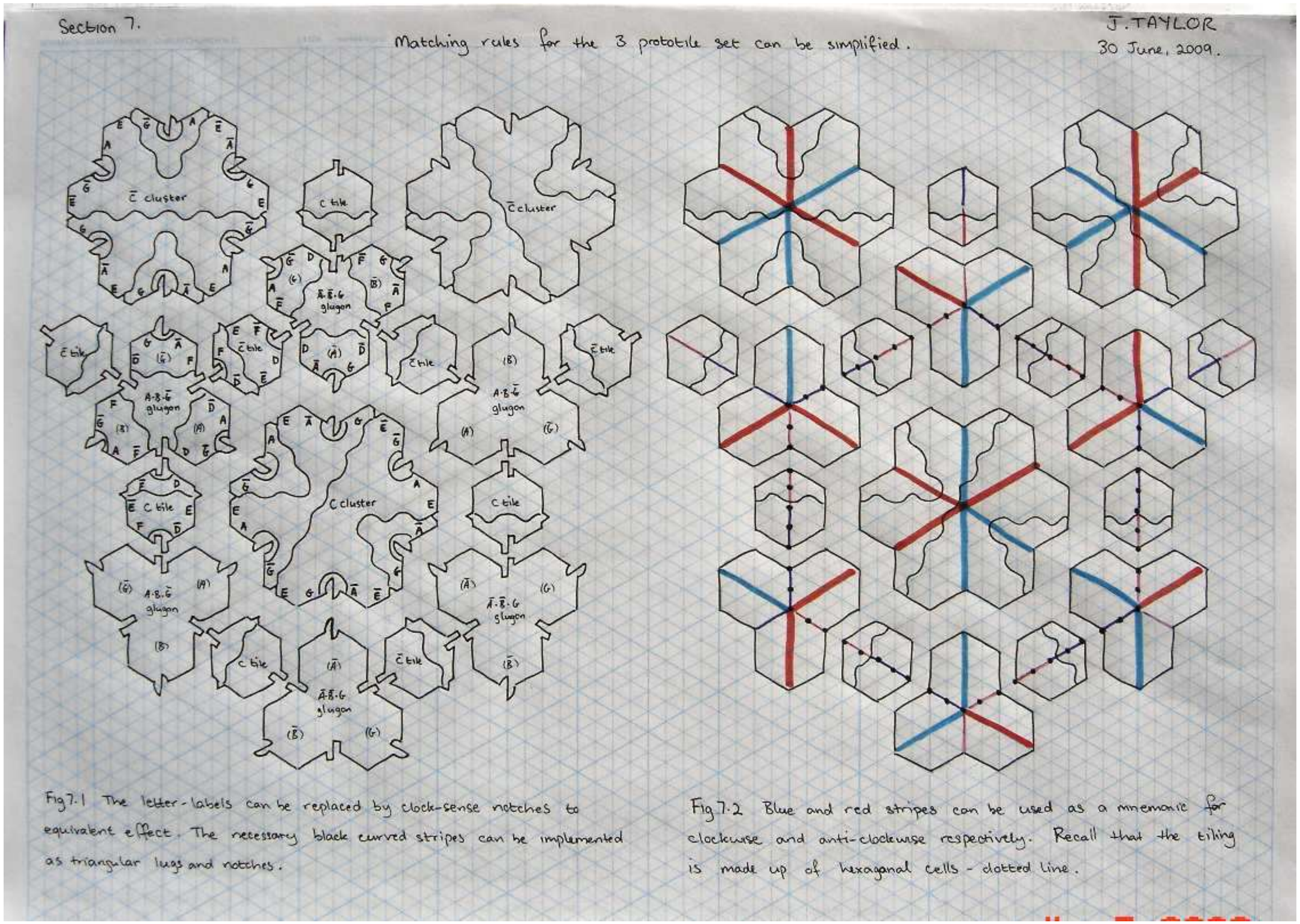
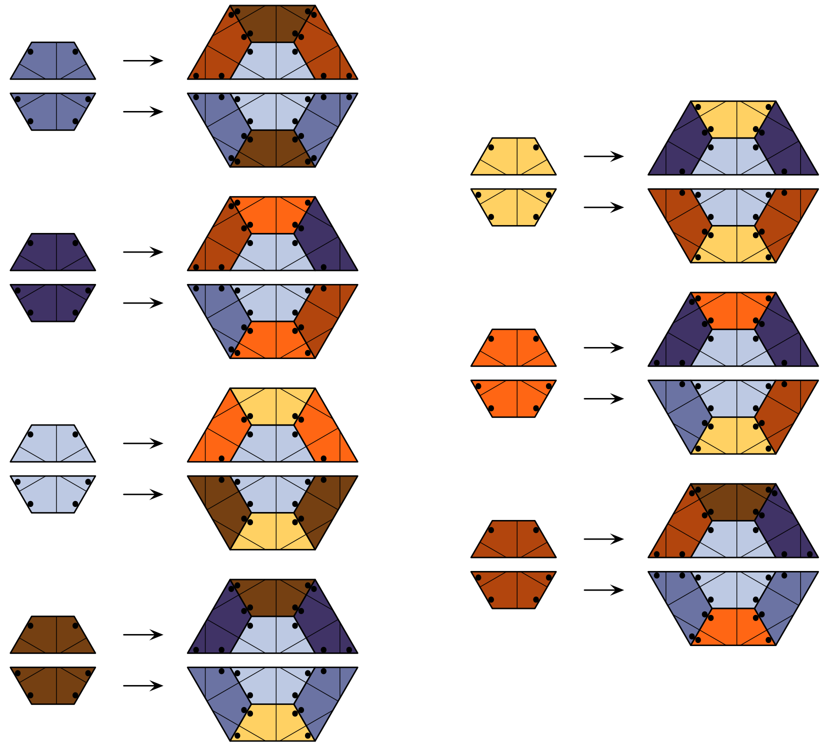
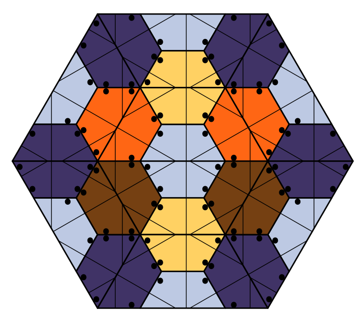


Inflation tiling





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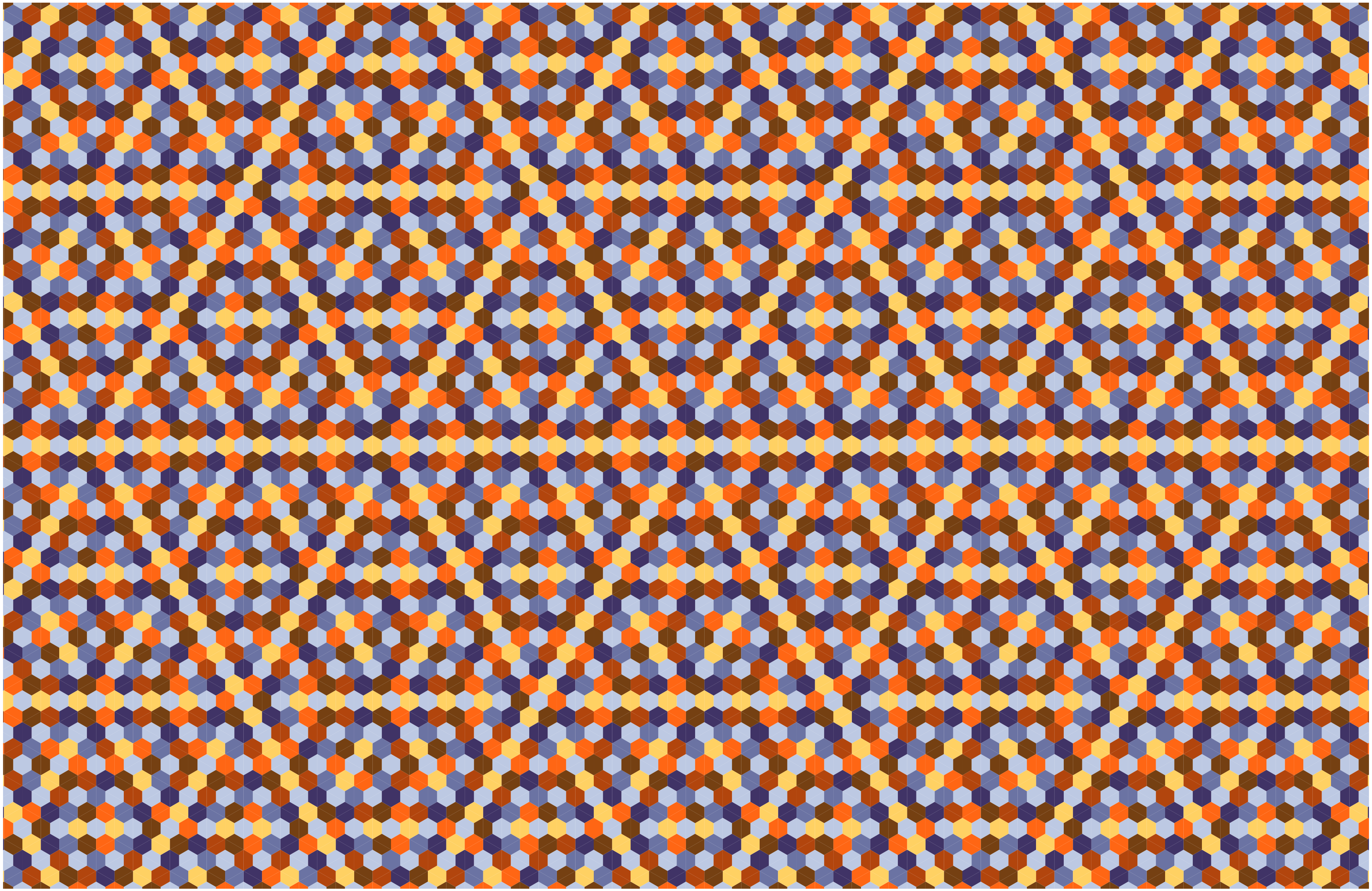
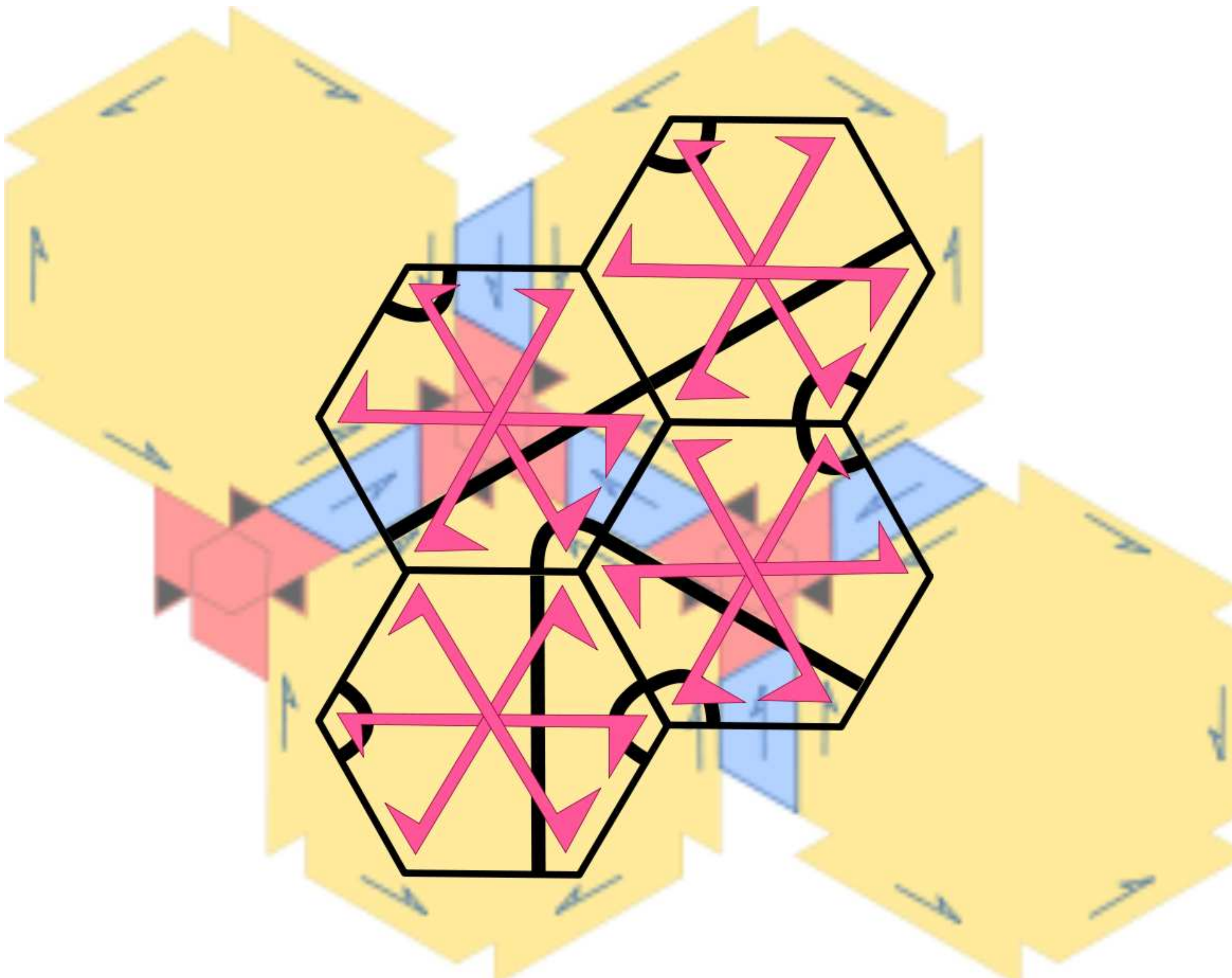


Inflation tiling

Relation to Penrose's $1+\varepsilon+\varepsilon^2$ tiling:



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