Large groups acting on surfaces of given genus, and the symmetric genus of a given group

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Joint work with Tom Tucker (NY) and others

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Preamble

This lecture will concentrate on recent developments with regard to these questions:

What is the largest number of automorphisms of

- a compact orientable surface of given genus $g > 1$?
- a compact non-orientable surface of given genus $p > 2$?

Given a finite group G, what is the smallest genus of faithful actions of G on

- compact orientable surfaces?
- compact orientable surfaces, preserving orientation?
- compact non-orientable surfaces?

Background: definitions

Here, by ‘surface’ we mean a surface with local structure, or 2-manifold, which may be orientable or non-orientable.

By an automorphism of such a surface X we mean a homeomorphism that preserves the local structure; in the orientable case, this might preserve or reverse orientation.

Any group G of automorphisms of such a surface X is isomorphic to a quotient of a Fuchsian group (in the orientation-preserving case) or a non-Euclidean crystallographic group (abbreviated to NEC group, in the other cases).

Background: long-known theorems

The groups acting on the sphere and torus (orientable of genus 0 and 1) and the projective plane and Klein bottle (non-orientable of genus 1 and 2) are completely classified.

Let G be a group of automorphisms of a compact orientable surface of genus $g > 1$. Then:

- $|G| \leq 84(g-1)$ if G preserves orientation [Hurwitz 1893]
- $|G| \leq 168(g-1)$ if G has orientation-reversing elements.

Similarly, if G is a group of automorphisms of a compact non-orientable surface of genus $p > 2$, then:

- $|G| \leq 84(p-2)$ [Singerman 1971].

Also these bounds are sharp for certain values of g and p.

Maximum possible orders: Hurwitz groups

Groups meeting the Hurwitz bound $|G| \leq 84(g-1)$ are all quotients of the ordinary $(2,3,7)$ triangle group, generated by two elements x and y satisfying $x^2 = y^3 = (xy)^7 = 1$.

Among these Hurwitz groups are the alternating groups A_n for all but a few n, many families of groups of Lie type, and 12 of the 26 sporadic simple groups (including the Monster).

But: the Hurwitz bound is very rarely achieved ... in fact for genus up to 11905 the bound is met only for $g = 3$, 7, 14, 17, 118, 129, 146, 385, 411, 474, 687, 769, 1009, 1025, 1459, 1537, 2091, 2131, 2185, 2663, 3404, 4369, 4375, 5433, 5489, 6553, 7201, 8065, 8193, 8589, 11626 and 11665 [MC (1985)].

NEC groups and signatures

An non-Euclidean crystallographic group (or NEC group) Γ is a co-compact discrete subgroup of the group of orientation-preserving or -reversing isometries of the hyperbolic plane $\mathbb{H}$.

Each such group has a finite presentation in terms of various kinds of generators (elliptic elements, reflections, hyperbolic elements, glide reflections etc.), which are subject to known defining relations.

The entire presentation can be encoded by a set of data called the signature of Γ, which takes the form

$$\sigma(\Gamma) = \Big(\gamma;\ \pm;\ [m_1, ..., m_r];\ \{(n_{11}, ..., n_{1s_1}), ..., (n_{k1}, ..., n_{ks_1})\}\Big).$$

Riemann-Hurwitz formula

The area of a fundamental region for the NEC group Γ with given signature is $\mu(\Gamma) = 2\pi\xi(\Gamma)$, where

$$\xi(\Gamma) = \alpha\gamma + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right),$$

with $\alpha = 2$ if the sign is $+$ and $\alpha = 1$ otherwise.

If Λ is a subgroup of finite index in Γ, then Λ is also an NEC group, and its area is given by the Riemann-Hurwitz formula

$$\mu(\Lambda) = |\Gamma : \Lambda| \cdot \mu(\Gamma), \text{ or, equivalently, } \xi(\Lambda) = |\Gamma : \Lambda| \cdot \xi(\Gamma).$$

In particular, if $\Lambda \lhd \Gamma$ and $\Gamma/\Lambda = G$, then $\xi(\Lambda) = |G|\xi(\Gamma)$, so for given $\xi(\Lambda) > 0$ we maximise $|G|$ by minimising $\xi(\Gamma)$.

How to find bounds for 'non-Hurwitz' genera?

Need to find families of groups with presentations giving surface actions with large order to genus ratio.

Take an NEC group Γ with small hyperbolic area $2\pi\xi(\Gamma)$, and look for a suitable family of smooth finite quotients Γ/Λ_k (of Γ by torsion-free normal subgroups Λ_k), of increasing order.

Can do this using computational methods, such as the new low index normal subgroups algorithm, which finds all normal subgroups K of up to a given finite index in Γ (and hence all quotients Γ/K of up to given finite order).

Lower bounds on upper bounds

In the 1960s, Accola and Maclachlan showed that the largest number of orientation-preserving automorphisms of a compact Riemann surface of given genus $g > 1$ is at least $8(g+1)$, and that this bound is sharp for infinitely many values of g.

The corresponding surfaces are reflexible, so a lower bound for the largest number of all automorphisms of an orientable surface of given genus $g > 1$ is $16(g+1)$, and again this is sharp for infinitely many g.

For non-orientable surfaces of genus $p > 2$, the best known bound is $4p$ automorphisms [Conder et al, 2003], but it is not yet known whether this is sharp for infinitely many p.

Census of large group actions (for small genus)

Computational methods have recently enabled creation of complete lists of the largest groups of automorphisms of

- compact orientable surfaces of genus 2 to 300
- as above, preserving orientation
- compact non-orientable surfaces of genus 3 to 200.

The list for orientable surface actions corrects and extends a partial census by P.R. Hewitt (1989) for genus 2 to 26.

The non-orientable case was a little harder, involving more possibilities for the signature (of the NEC group Γ), 2010/11.

Orientable case, preserving orientation

g	Max	Signature
2	48	$[2,3,8]$
3	168	$[2,3,7]$
4	120	$[2,4,5]$
5	192	$[2,3,8]$
6	150	$[2,3,10]$
7	504	$[2,3,7]$
8	336	$[2,3,8]$
9	320	$[2,4,5]$
10	432	$[2,3,8]$

. . .

g	Max	Signature
292	6984	$[2,3,12]$
293	4672	$[2,4,8]$
294	2376	$[2,4,297]$
295	7056	$[2,3,12]$
296	2376	$[2,4,594]$
297	7104	$[2,3,12]$
298	3888	$[2,3,72]$
299	4768	$[2,4,8]$
300	4056	$[2,3,52]$

Note: In almost all small cases, the largest group action has a triangle group signature $[p,q,r] = (0;+;[p,q,r];\{-\})$, but for genus $g = 126$ it has signature $(0;+;[2,2,2,3];\{-\})$.

Orientable case, allowing orientation reversal

g	Max	Signature
2	96	$[2,3,8]^*$
3	336	$[2,3,7]^*$
4	240	$[2,4,5]^*$
5	384	$[2,3,8]^*$
6	300	$[2,3,10]^*$
7	1008	$[2,3,7]^*$
8	672	$[2,3,8]^*$
9	640	$[2,4,5]^*$
10	720	$[2,4,5]^*$

$\ldots$

g	Max	Signature
292	6984	$[2,3,12]$
293	4800	$[2,4,150]^*$
294	4752	$[2,4,297]^*$
295	14112	$[2,3,12]^*$
296	4752	$[2,4,594]^*$
297	7104	$[2,3,12]$
298	7776	$[2,3,72]^*$
299	4800	$[2,4,600]^*$
300	8112	$[2,3,52]^*$

In most cases, the maximum occurs for signature $[p,q,r]^* = (0;+;\ [-];\{(p,q,r)\})$ for some p,q,r; in some it is the same as before; and in others for signature $(0;+;\ [-];\{(2,2,2,3)\})$.

Non-orientable case [much more tricky/interesting]

p	Max	Signature
3	12	$[2,2,2,3]^*$
4	48	$[2,4,6]^*$
5	120	$[2,4,5]^*$
6	160	$[2,4,5]^*$
7	120	$[2,4,6]^*$
8	504	$[2,3,7]^*$
9	336	$[2,3,8]^*$
10	192	$[2,4,6]^*$

..

p	Max	Signature
193	1560	$[2,4,195]^*$
194	2304	$[2,6,6]^*$, $[2,2,2,3]^*$
195	1158	$(0;+;[2,3];\{(1)\})$
196	1584	$[2,4,198]^*$
197	2184	$[2,4,14]^*$
198	2352	$[2,2,2,3]^*$
199	1608	$[2,4,201]^*$
200	2640	$[2,4,10]^*$

In most cases, the maximum occurs for signature $[2,q,r]^*$, or $[2,2,2,s]^* = (0;+;[-];\{(2,2,2,s)\})$ for $s = 3$ or 4, or $(0;+;[2,3];\{(1)\})$, or $(0;+;[2];\{(2,4)\})$. In the exceptional case of genus 87, it occurs for signature $[2,2,2,87]^*$.

Symmetric genus and cross-cap number

Instead of asking for the largest group of automorphisms of a surface of given genus, once can pose the inverse problem:

What is the smallest genus of those surfaces on which a given group has a faithful action?

Formally, given a finite group G, Tucker (1980s) defined

- the symmetric genus $\sigma(G)$ as the minimum genus of all closed orientable surfaces on which G has a faithful action
- the strong symmetric genus $\sigma^{\mathrm{o}}(G)$ as the minimum genus of those in which the action preserves orientation
- the symmetric cross-cap number $\tilde{\sigma}(G)$ as the minimum genus of all closed non-orientable surfaces on which G has a faithful action.

Strong symmetric genus [concept due to Burnside]

This is now known for several families of groups, including:

- Cyclic groups: $\sigma^{\mathrm{o}}(C_n) = 0$ for all n
- Dihedral groups: $\sigma^{\mathrm{o}}(D_n) = 0$ for all n
- All finite abelian groups [Maclachlan (1965)]
- All known Hurwitz groups ($\sigma^{\mathrm{o}}(G) = \frac{|G|}{84} + 1$, optimal)
- Alternating groups A_n for all n [Conder (1984)]
- Symmetric groups S_n for all n [Conder (1984)]
- Groups $\mathrm{PSL}(2, q)$ for all q [Glover & Sjerve (1987)]
- All 26 sporadic finite simple groups [Woldar et al]
- Direct products $C_k \times D_n$ [May & Zimmerman (2003)]
- Various finite reflection groups [Jackson (2004–)]
- All finite groups of order up to 127 [Conder/MAGMA]

Strong symmetric genus spectrum

May & Zimmerman (2003) proved that for all $k, n \geq 3$,

$$\sigma^{o}(C_k \times D_n) = \begin{cases} 1 + nk\left(\frac{1}{2} - \frac{1}{2k} - \frac{1}{\mathrm{lcm}(k,n)}\right) & \text{if } k \text{ odd} \\ 1 + n(k-2) & \text{if } k \text{ even}, n \text{ even} \\ 1 + nk\left(\frac{1}{2} - \frac{1}{k} - \frac{1}{\mathrm{lcm}(k,n)}\right) & \text{if } k \text{ even}, n \text{ odd.} \end{cases}$$

The set of possible values covers almost all positive integers.

Corollary: the strong symmetric genus function is surjective

— i.e. the strong symmetric genus spectrum is complete.

Symmetric genus

This is more challenging. Nevertheless, the symmetric genus $\sigma(G)$ is known for several families of groups, including these:

- Spherical groups (groups acting on the sphere)
- Toroidal groups (groups acting on the torus)
- All finite abelian groups
- Simple groups G for which $\sigma^{\text{o}}(G)$ is known
- Symmetric groups S_n for all n
- Groups $\text{PSL}(2, q) \times C_2$ and $\text{PGL}(2, q)$
- Some other groups G for which $|G| = 168(\sigma(G) - 1)$
  [These are C_2-extensions of certain Hurwitz groups]
- All finite groups of order up to 127
- All finite groups of symmetric genus 2 to 32.

Symmetric genus spectrum

Question: Is the symmetric genus function surjective?

Partial answer (from recent work by Conder & Tucker):

There exist families of groups G with

- $|G| = 16n$ and $\sigma(G) = 4n - 1$ for all $n \geq 2$
- $|G| = 16n$ and $\sigma(G) = 4n - 3$ for all $n \geq 1$
- $|G| = 24n$ and $\sigma(G) = 3n - 3$ for all odd $n \geq 1$
- $|G| = 24n$ and $\sigma(G) = 3n + 1$ for all odd $n \geq 11$
- $|G| = 48n$ and $\sigma(G) = 9n - 7$ for all odd $n \geq 1$.

Hence the symmetric genus spectrum covers at least 8/9 of the positive integers.

Example: family with symmetric genus $4n-1$

Let $G = \langle\, x, y \,|\, x^4 = y^4 = [x^2, y] = [y^2, x] = 1,\ (xy)^{2n} = x^2 \,\rangle$.

Here $K = \langle x^2, y^2 \rangle \cong V_4$ is central, of order 4, with quotient $G/K \cong D_{2n}$, so $|V_n| = 16n$. Also $|\langle xy \rangle| = 4n$, so G is a smooth quotient of the NEC group with signature $(0; +; [4, 4, 4n]; \{-\})$, and Riemann-Hurwitz gives the genus of the action as $1 + 8n(1 - 1/4 - 1/4 - 1/(4n)) = 4n - 1$.

All involutions of G lie in K, so no action of G on an orientable surface includes reflections. This fact and similar observations make it easy to rule out actions on orientable surfaces of smaller genera. Hence $\sigma(G) = \sigma^{\text{o}}(G) = 4n - 1$.

Symmetric genus spectrum (cont.)

The main difficulty in showing this spectrum is complete lies with the integers congruent to 8 or 14 mod 18.

We have some families covering infinitely many integers in each of these classes, but not all. But ...

Conjecture: For every integer $g \geq 5$, there is a finite abelian or metabelian group G with symmetric genus $\sigma(G) = g$.

Now all we have to do is work out the symmetric genus of metabelian groups ... not so easy!!

Symmetric cross-cap number

Even less is known about the value of this parameter $\tilde{\sigma}$.

Study of $\tilde{\sigma}(G)$ involves considering epimorphisms $\theta\colon \Gamma \to G$ with the property that θ maps the orientation-preserving subgroup Γ^+ onto G (rather than a subgroup of index 2).

It is known that $\tilde{\sigma}(G) \geq \frac{|G|}{84} + 2$ whenever $\tilde{\sigma}(G) \neq 1, 2$.

Groups for which this bound is achieved are sometimes called H^*-groups [Singerman (1971)]. They include the alternating groups A_n for all but finitely many n, $\mathrm{PSL}(2, q)$ for some q, and certain other finite simple groups, as well as some groups obtainable as extensions by these.

Symmetric cross-cap number (cont.)

May (2001) found the symmetric cross-cap number of all finite abelian groups and dicyclic groups, and some others.

Etayo & Martinez (2008) showed that:

$$\tilde{\sigma}(C_m \times D_n) = \begin{cases} 2 + mn - m - n & \text{if } m \text{ and } n \text{ are odd} \\ 2 + n(m-1) & \text{if } m \text{ odd, } n \text{ even, } 2m \leq n \\ 2 + m(n-2) & \text{if } m \text{ odd, } n \text{ even, } 2m > n \\ 2 + n(m-2) & \text{if } m \equiv 0 \text{ mod } 4 \text{ and } n \text{ odd} \\ 2 + mn & \text{if } m \text{ and } n \text{ are even.} \end{cases}$$

Note: $C_m \times D_n \cong C_{\frac{m}{2}} \times D_{2n}$ when $m \equiv 2$ mod 4 and n odd.

Symmetric cross-cap number (cont.)

There is no group G with symmetric cross-cap number 3.
[This was conjectured by Tucker, and proved by May (2001)]

Recent computations [by MC (2010)] give

- $\tilde{\sigma}(G)$ for all finite groups of order up to 127
- All groups with symmetric cross-cap number 4 to 65.

It is also easy to show that for each positive integer k there is an extension of C_k by S_4 (of order $24k$, with each odd element of S_4 conjugating every element of C_k to its inverse) that has symmetric cross-cap number $3k-2$, achieved by an action with signature $(0; +; [-]; \{(2, 4, 3n)\})$.

Symmetric cross-cap number spectrum

Some of the previous families of examples give these infinite subsets of the spectrum of values of $\tilde{\sigma}(G)$:

- All $p \equiv 0, 1$ or 2 mod 4, $p > 10$, from $G = C_m \times D_n$
- All $p \equiv 11$ mod 12, from $\tilde{\sigma}(C_3 \times C_{6n}) = 12n - 1$
- All $p \equiv 1$ mod 3, from $\tilde{\sigma}(C_k \cdot S_4) = 3k - 2$.

This leaves just the class of all $p \equiv 3$ mod 12. There is no group with symmetric cross-cap number 3 (but all other values between 4 and 100 are achievable).

Conjecture: For every integer $p \geq 4$, there is a finite abelian or metabelian group with symmetric cross-cap number p.

Summary of (still) open problems

Are there infinitely many positive integers $p \equiv 3 \bmod 12$ such that the largest number of automorphisms of a compact non-orientable surface of genus p is $4p$?

For every positive integer $g \equiv 2$ mod 6, is there is a group with symmetric genus g? If so, then the symmetric genus spectrum is complete.

For every positive integer $p \equiv 3$ mod 12, is there is a group with symmetric cross-cap number p? If so, the symmetric cross-cap number can be any positive integer other than 3.

Thank You!