Shortest Path Avoiding Balls

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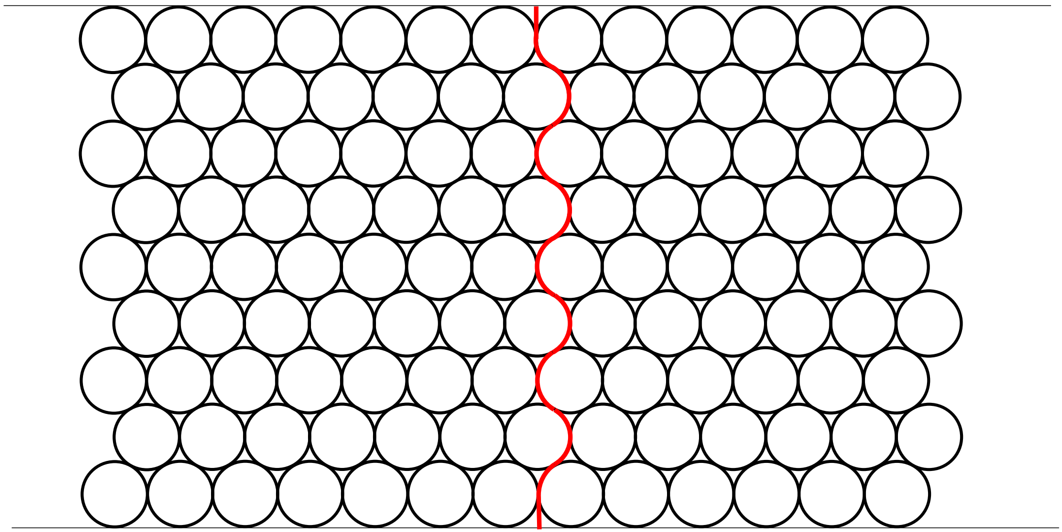
A system of non-overlapping, nowhere accumulating open convex discs contained in a parallel strip is said to form a *layer*. The *permeability* p of the layer is defined by the quotient

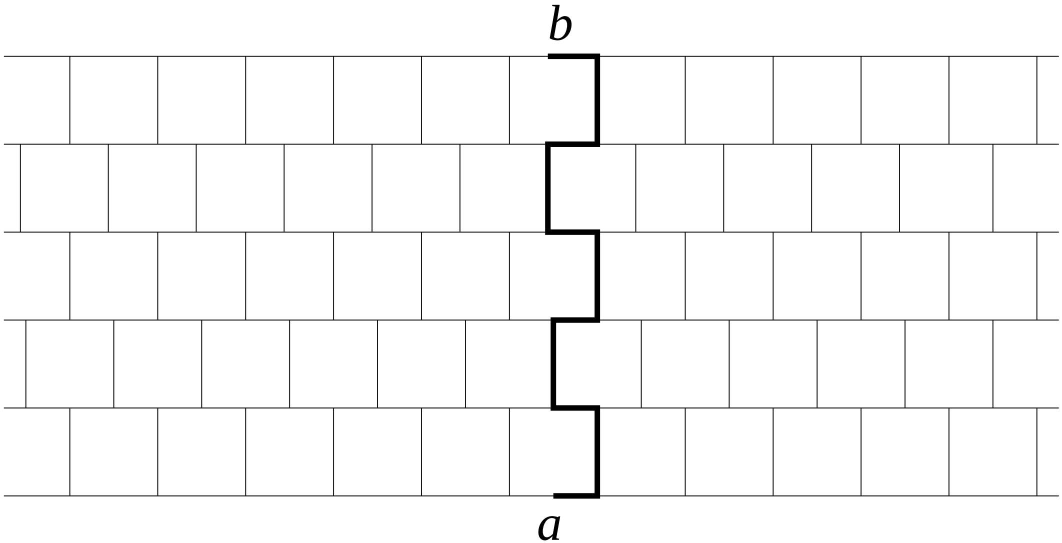
$$p = w / \inf l,$$

where w is the width of the strip, l is the length of a path connecting the two edges of the strip avoiding all members of the layer, and the infimum extends over all paths of this kind.

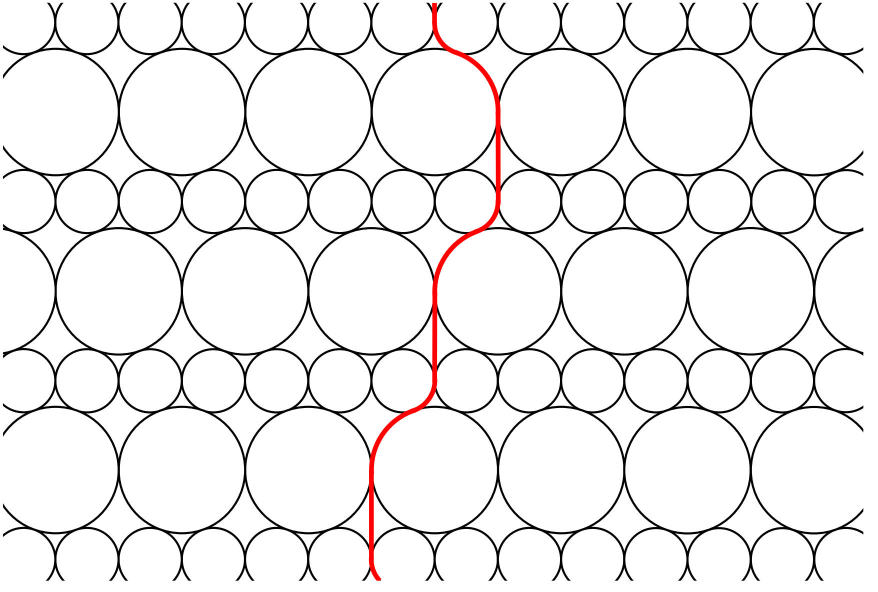
Theorem (LFT). *The permeability of a layer of congruent circles is at least* $\sqrt{27}/2\pi = 0.82699\ldots$, *and the permeability of a layer of squares is at least* $2/3$.

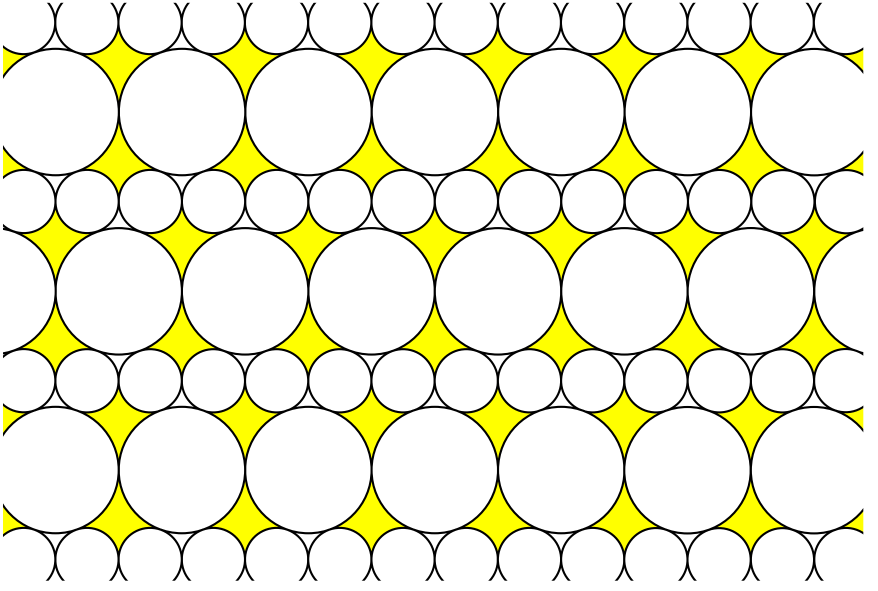
Both bounds are sharp.




b
a

Given a convex disc D, let $p_s(D)$ be the infimum of the permeabilities by similar copies of D and let $p_c(D)$ be the infimum of the permeabilities by congruent copies of D. It is an interesting question which convex discs D share the property with the parallelograms that $p_s(D) = p_c(D)$. The circle is not among such discs.





Placing in the gaps bounded by four circles very small circles arranged in a hexagonal lattice we can enforce a greater detour, obtaining a layer of circles with permeability

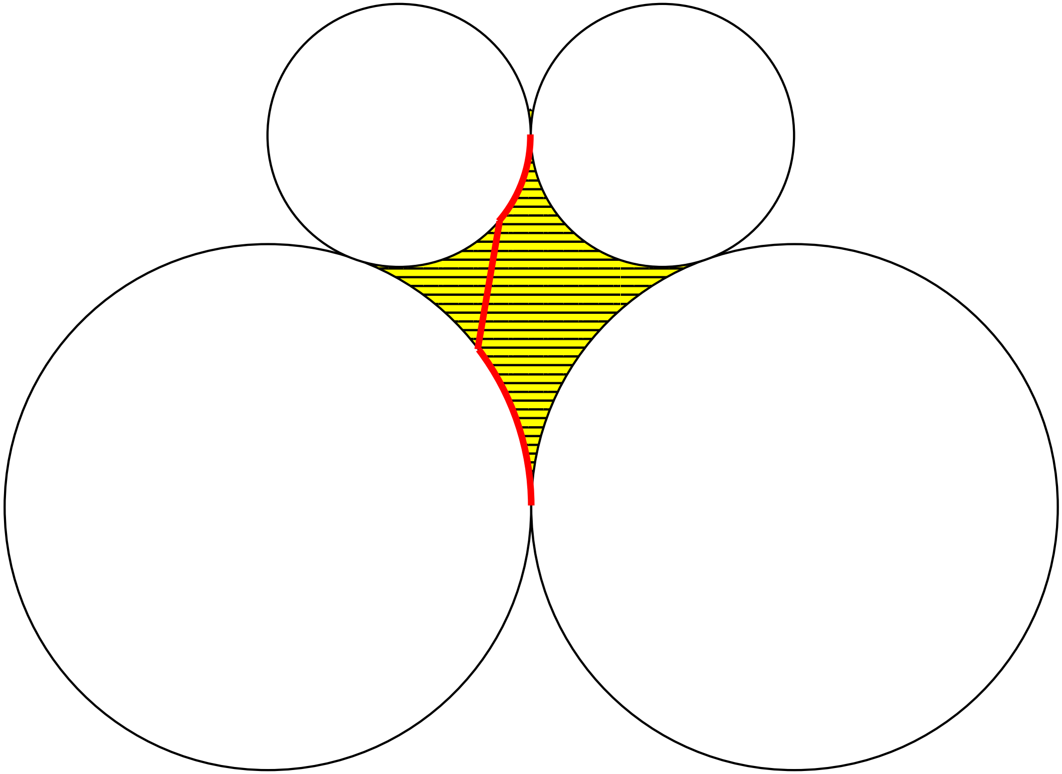
$$q < \frac{\sqrt{27}}{2\pi}\,.$$

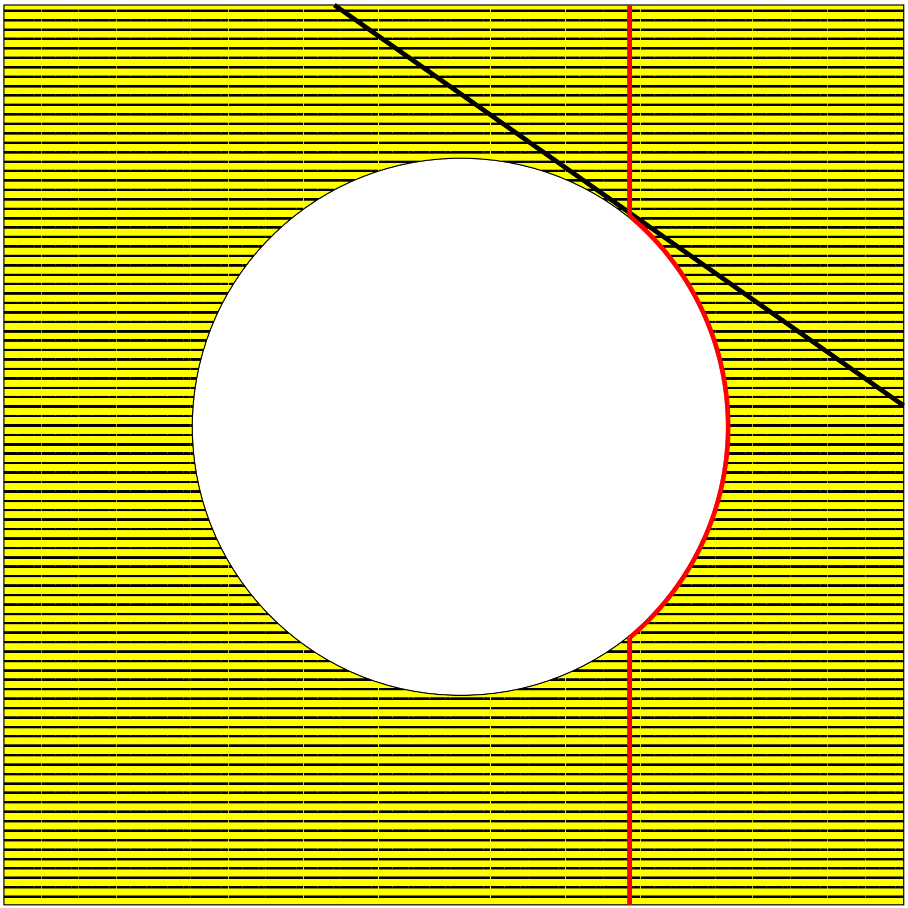
We can enforce even a greater detour if we replace the small circles in the gaps by circles arranged in this new arrangement. Iterating this process we get a layer of circles with permeability

$$0.82349\,.$$

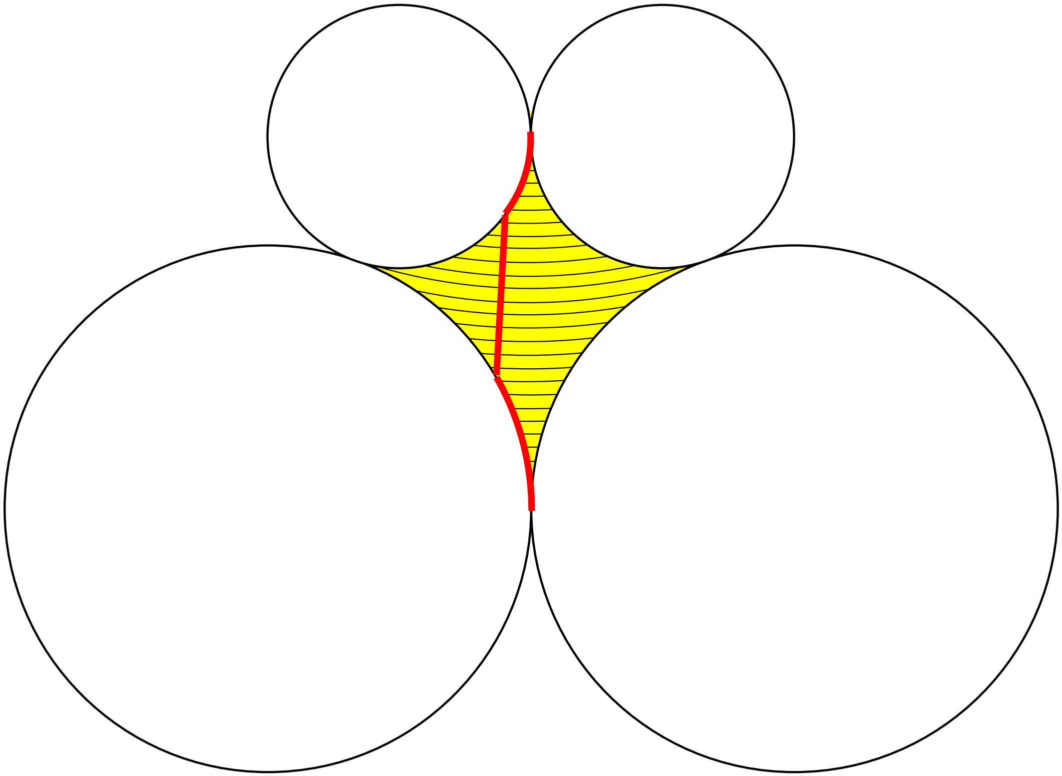
By a modification of this construction Danzer improved this bound to

$$0,82231\,.$$





At a point where we first or last meet a circle the sin of the angle of the tangent of the circle with a horizontal line is equal to the permeability of the small circles.



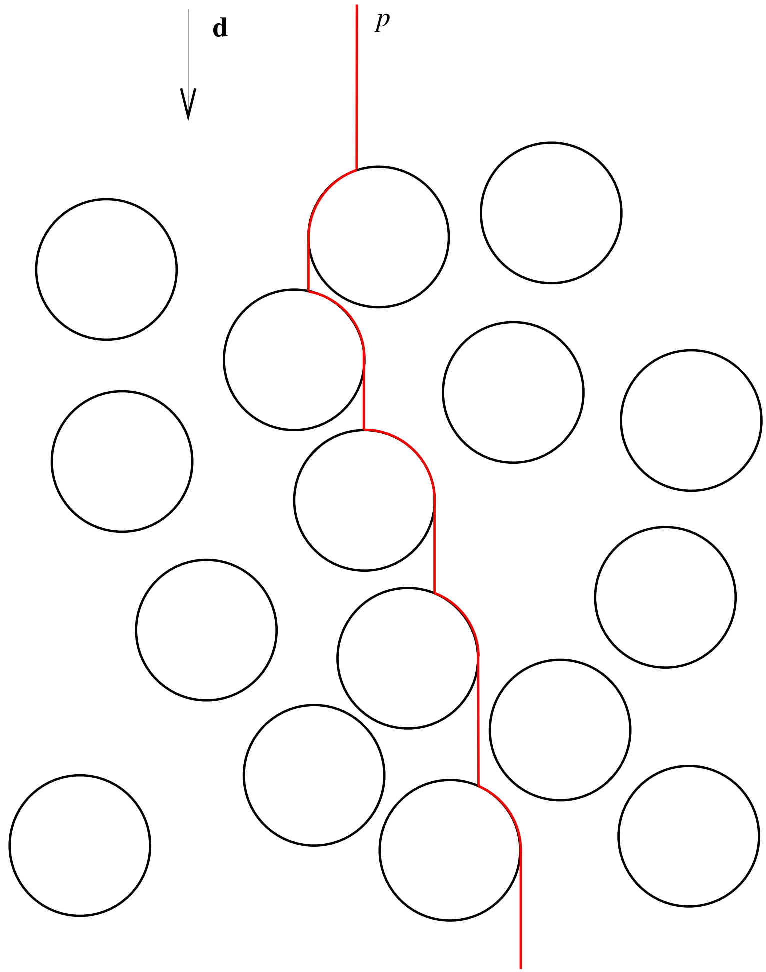
Given a packing of uniformly bounded congruent (similar) copies of a convex disc D and two points outside the discs at distance d from one another, is it true that the two points can be joined by a path traveling outside the discs of the packing and having length $\frac{d}{p_c(D)} + o(d)$ ($\frac{d}{p_s(D)} + o(d)$)?

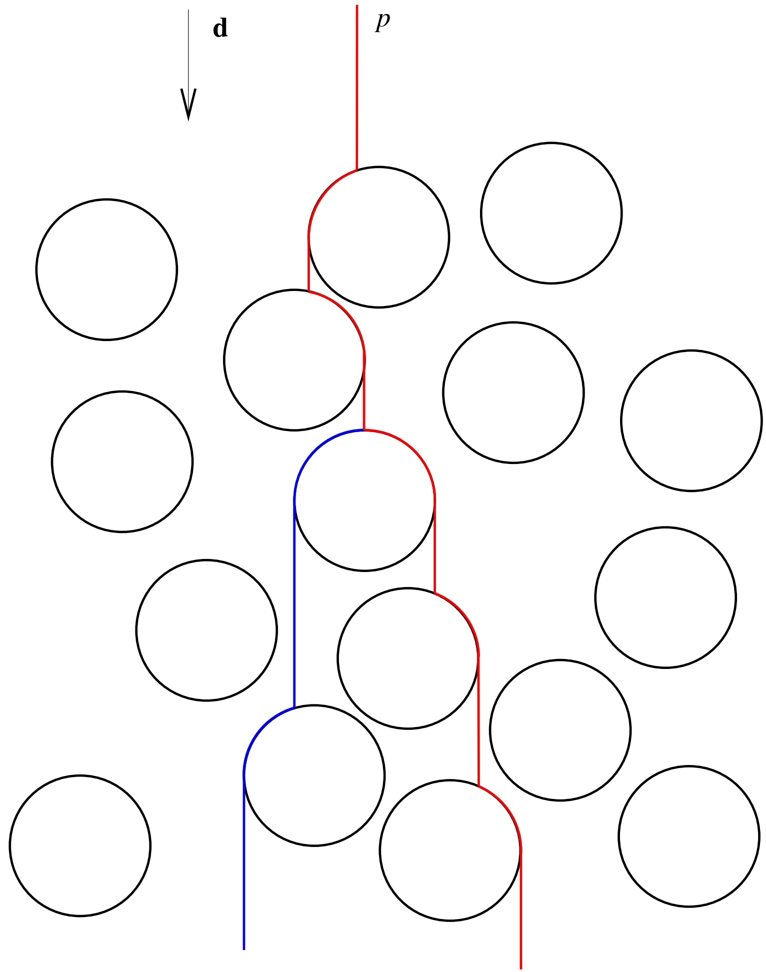
The first result in this direction was achieved by János Pach who used a mean-value argument to show that in a packing of open squares with side-lengths not exceeding 1, any two points lying outside the squares at distance d from one another can be connected by a path avoiding the squares and having length at most

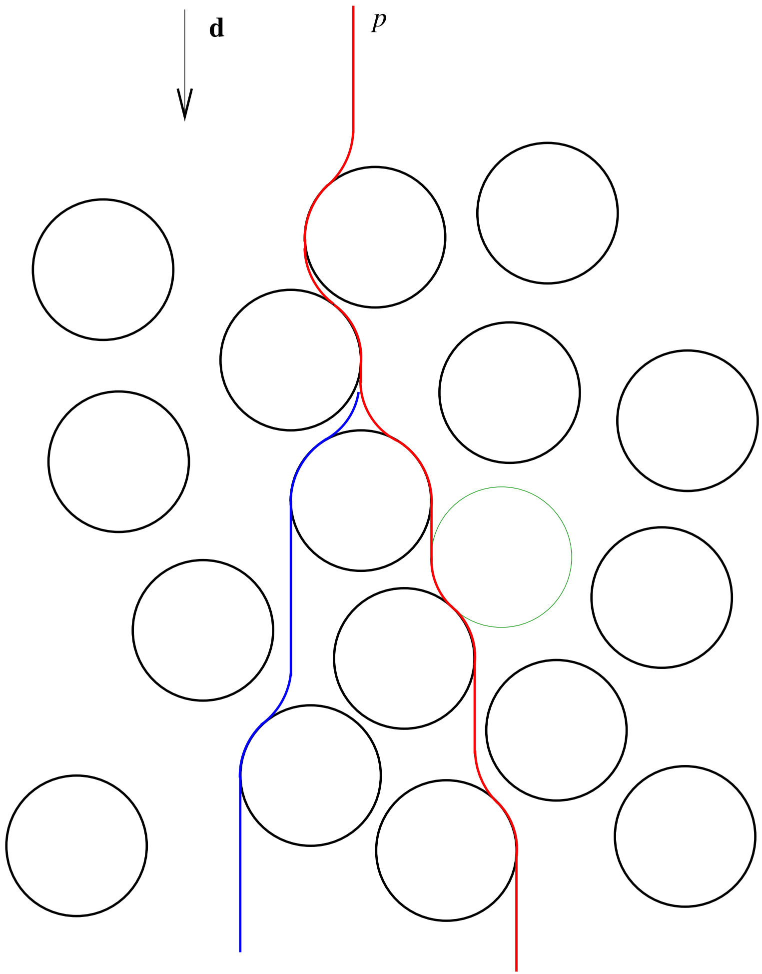
$$\frac{3}{2}d + 4\sqrt{d} + 1.$$

Later I improved the bound for the length of the path to $\frac{3d+1}{2}$ which cannot be improved for odd values of d. Moreover, in a packing of open unit circles, any two points lying outside the circles at distance d from one another can be connected by a path avoiding the circles and having length at most $\frac{2\pi}{\sqrt{27}}(d-2)+\pi$.

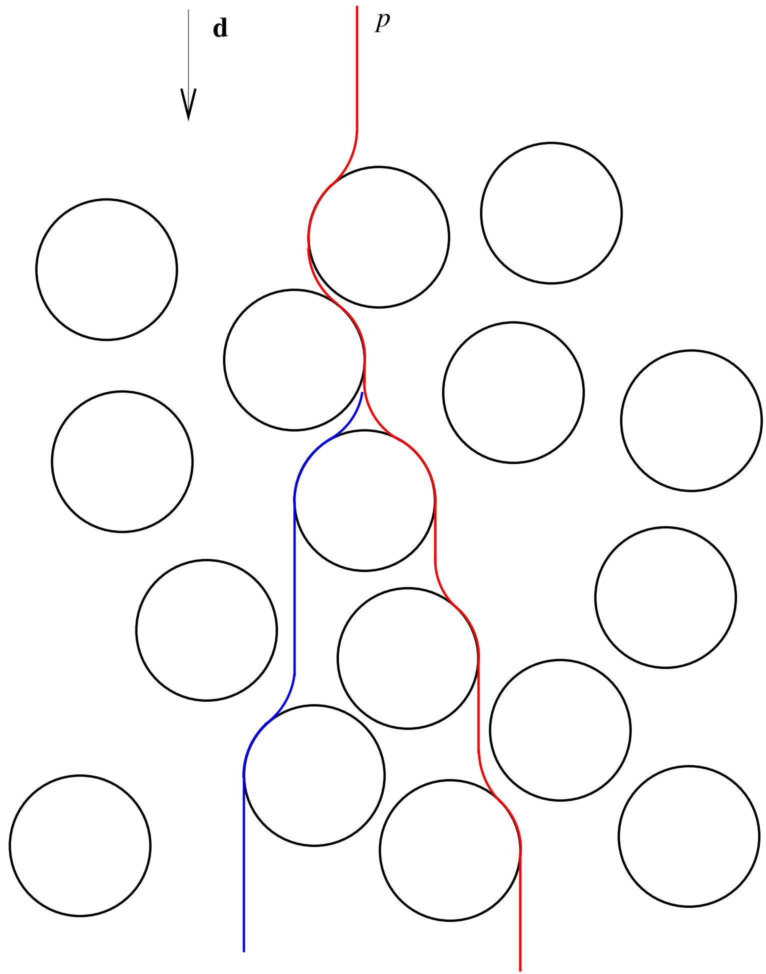
Consider a packing $\mathcal{P}$ of unit circles. Let p be a point outside the circles and let $\mathbf{d}$ be a unit vector. We construct an infinite *path emanating from p in the direction* $\mathbf{d}$.


d
p


d
p



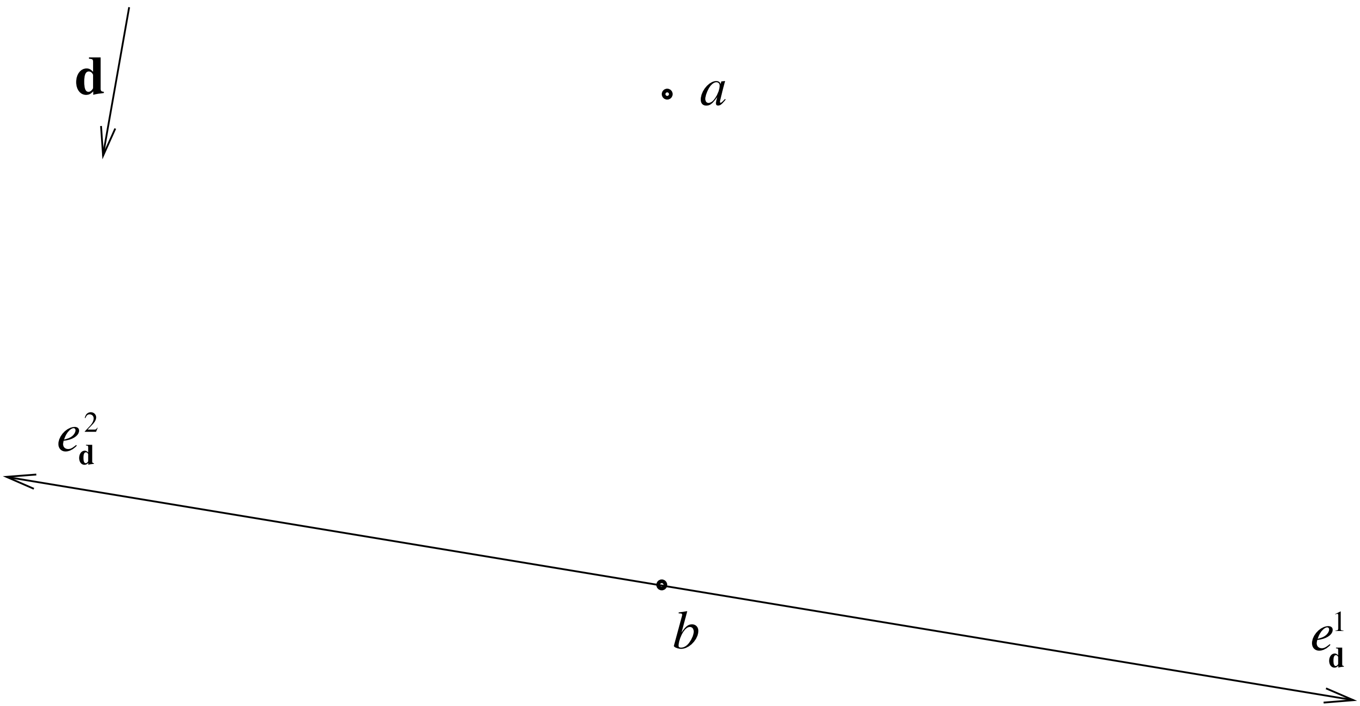
d
p

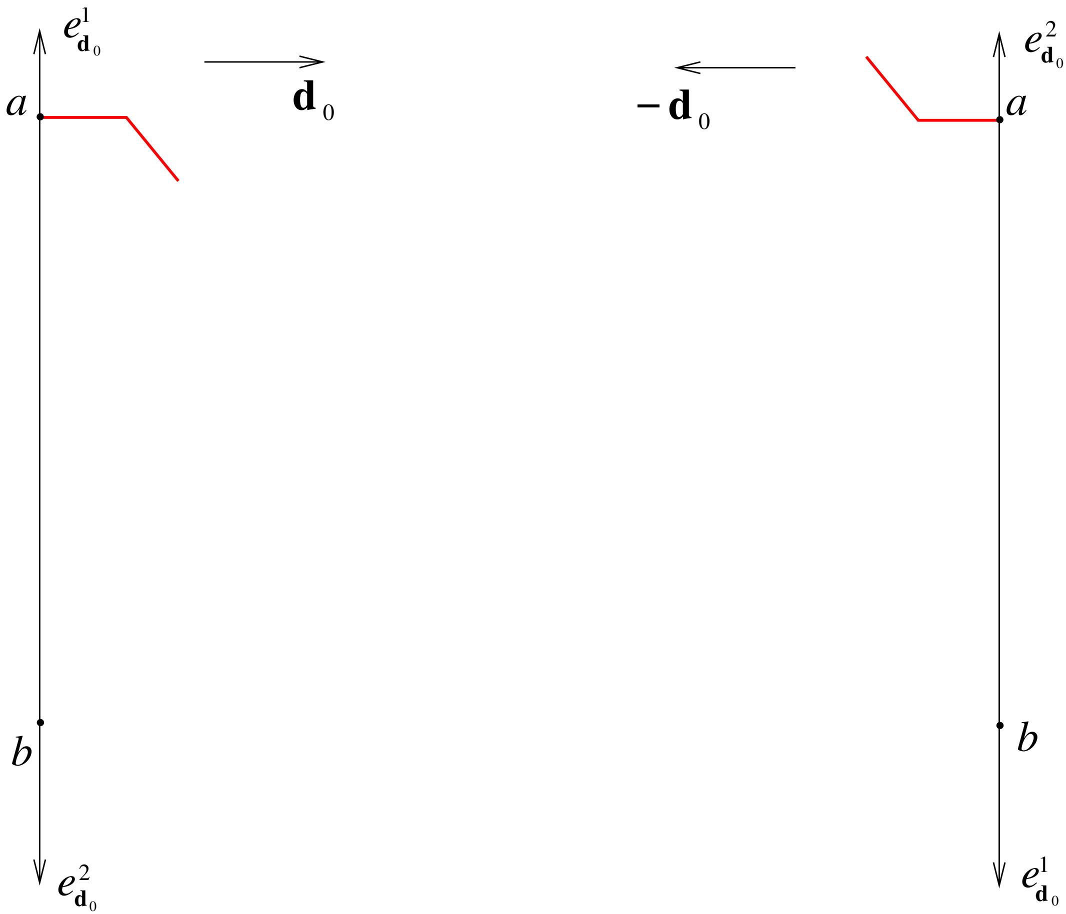

d
p

Lemma (LFT). *Consider a path emanating from p in the direction $\mathbf{d}$. For a point q on the path let $l(q)$ be the length of the arc of the path between p and q and let $\delta(q)$ be the inner product of $\mathbf{d}$ and the vector $\overrightarrow{pq}$. If q is an endpoint of a diameter of a circle perpendicular to $\mathbf{d}$ or lies on a straight segment that is part of the path, then*

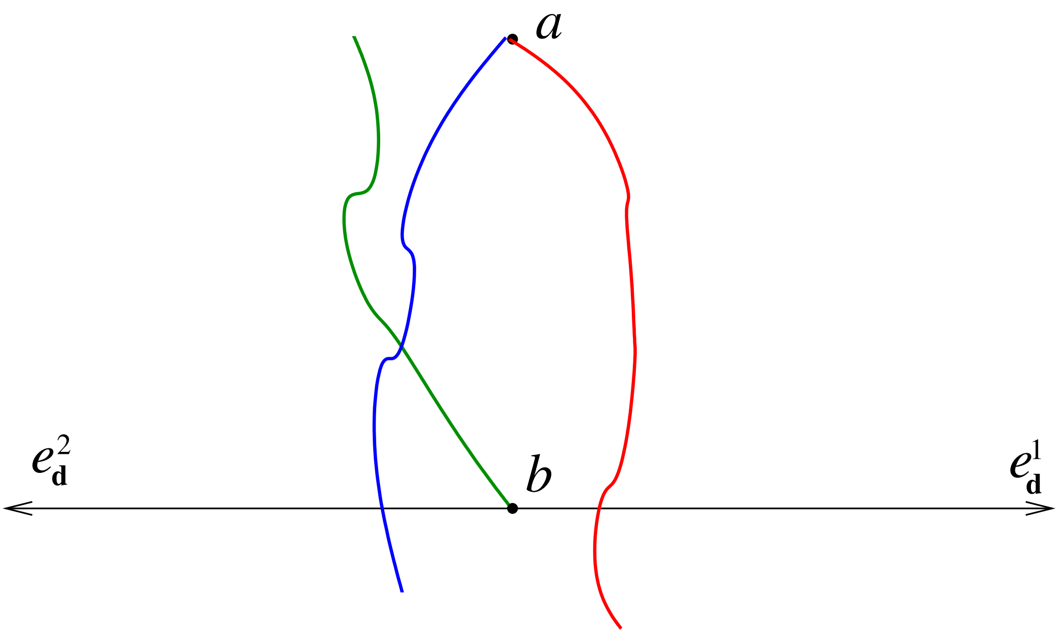
$$l(q) \leq \frac{2\pi}{\sqrt{27}}(\delta(q) - 1) + \frac{\pi}{2}.$$

Let $e_{\mathbf{d}}$ be the line through b perpendicular to the direction $\mathbf{d}$. b divides $e_{\mathbf{d}}$ into two half-lines $e_{\mathbf{d}}^1$ and $e_{\mathbf{d}}^2$. We choose the notation so that the direction of $e_{\mathbf{d}}^1$ is obtained from $\mathbf{d}$ by a rotation of 90° in the counter-clockwise direction.





$e^1_{\mathbf{d}_0}$
$\mathbf{d}_0$
$-\mathbf{d}_0$
$e^2_{\mathbf{d}_0}$
a
a
b
b
$e^2_{\mathbf{d}_0}$
$e^1_{\mathbf{d}_0}$


a
b
$e_{\mathbf{d}}^2$
$e_{\mathbf{d}}^1$

Observation. *Let a and b be two points outside the circles. If there is a direction* $\mathbf{d}$ *and two paths, one emanating from a in the direction* $\mathbf{d}$, *and another emanating from b in the direction* $-\mathbf{d}$ *which intersect, then it is possible to compose from them a path from a to b whose length is at most*

$$\frac{2\pi}{\sqrt{27}}(d-2)+\pi\,.$$

In E^n we use coordinates $x_1, \ldots, x_{n-1}, y$, or $(\mathbf{x}, y)$, for short. Let H denote the coordinate hyperplane $y = 0$. For a ball B in R^n we define the function $l_B(\mathbf{x})$, $\mathbf{x} \in H$ as follows. If the line

$$l(\mathbf{x}) = \{(\mathbf{x}, y), -\infty < y < \infty\}$$

intersects the boundary of B in two points, let $l_B(\mathbf{x})$ be the length of the shortest arc on the boundary of B between the two points of intersection. Let $l_B(\mathbf{x}) = 0$ otherwise. Define λ_n as

$$\lambda_n = \int_H l_{B^n}(\mathbf{x})\, d\mathbf{x},$$

where B^n denotes the unit ball in E^n.

We write $V(B^n) = \kappa_n$ for the volume of the unit ball and σ_n for its surface area.

Given an arrangement $\mathcal{A}$ of convex bodies and a domain D in E^n, the *inner density* and the *outer density of $\mathcal{A}$ relative to D* is defined as

$$d_{\mathrm{inn}}(\mathcal{A}, D) = \frac{1}{V(D)} \sum_{A \in \mathcal{A}, A \subset D} V(A)$$

and

$$d_{\mathrm{out}}(\mathcal{A}, D) = \frac{1}{V(D)} \sum_{A \in \mathcal{A}, A \cap D \neq \emptyset} V(A)\,,$$

respectively.

The *upper density* of the arrangement $\mathcal{A}$ is defined as

$$d_+(\mathcal{A}) = \limsup_{r\to\infty} d_{\mathrm{out}}(\mathcal{A}, rB^n).$$

With these definitions we can state our main result as follows.

Theorem. *Let $\mathcal{P}$ be a packing of open balls in E^n with radii not exceeding* 1 *and upper density δ. Then any two points lying outside the balls at distance d from one another can be connected by a path avoiding the balls and of length at most*

$$(1 - \delta + \tfrac{\lambda_n \delta}{\kappa_n})d + O(\sqrt{d}).$$

Recall that

$$\sigma_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \quad \text{and} \quad \kappa_n = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}.$$

We have

$$\begin{aligned}\lambda_n &= 2\sigma_{n-1} \int_0^1 t^{n-2} \arccos t \, dt = \\ &= \sigma_{n-1} \frac{\Gamma(\frac{n}{2})\pi^{\frac{1}{2}}}{(n-1)\Gamma(\frac{n+1}{2})} = \\ &= \frac{2\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})}{(n-1)\Gamma(\frac{n-1}{2})\Gamma(\frac{n+1}{2})}.\end{aligned}$$

We obtain that

$$\frac{\lambda_n}{\kappa_n} = \frac{n\Gamma^2(\frac{n}{2})}{(n-1)\Gamma(\frac{n-1}{2})\Gamma(\frac{n+1}{2})}.$$

In particular, we have

$$\frac{\lambda_2}{\kappa_2} = \frac{4}{\pi} \quad \text{and} \quad \frac{\lambda_3}{\kappa_3} = \frac{3\pi}{8}.$$

In view of this and the celebrated result of Hales that the density of congruent balls in E^3 is at most $\frac{\pi}{\sqrt{18}}$ we get the following corollaries.

Corollary 1. *Two points at distance d from one another lying outside the members of a packing of open circles with radii at most 1 can be connected by a path avoiding the circles and having length at most*

$$\frac{4}{\pi} d + O(\sqrt{d}) < 1.27324\, d + O(\sqrt{d})\,.$$

Corollary 2. *In E^3 two points at distance d from one another lying outside the members of a packing of open balls with radii at most 1 can be connected by a path avoiding the balls and of length at most*

$$\frac{3\pi}{8}d + O(\sqrt{d}) < 1.1781\, d + O(\sqrt{d}).$$

Corollary 3. *In E^3 two points at distance d from one another lying outside the members of a packing of open unit balls can be connected by a path avoiding the balls and of length at most*

$$\left(1 - \frac{\pi}{\sqrt{18}} + \frac{\pi^2\sqrt{2}}{16}\right) d + O(\sqrt{d}) < 1.1319\, d + O(\sqrt{d}).$$

Remembering that $\Gamma(z) = \sqrt{\frac{2\pi}{z}}(\frac{z}{e})^z(1 + O(\frac{1}{z}))$ it follows that

$$\frac{\lambda_n}{\kappa_n} = 1 + O(\tfrac{1}{n}).$$

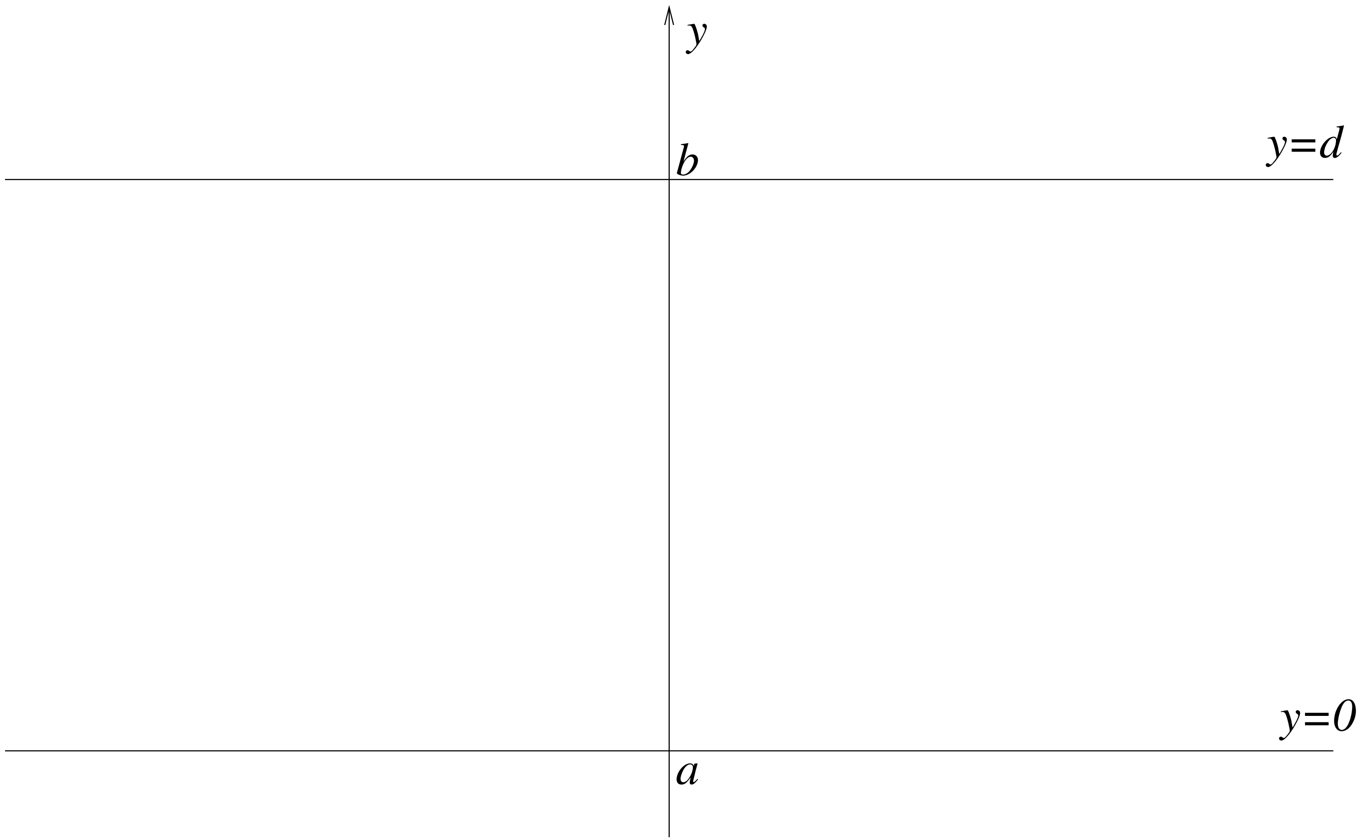
This means that in E^n, even for a packing of balls with arbitrary but bounded radii, where the density might be 1, we need not make a detour greater than $O(d/n)$ in order to connect two points lying at distance d outside the balls by a path avoiding the balls.

Given any packing of open balls and two points u and v outside the balls there is a natural way to connect the two points by a path avoiding all balls. If a ball B of the packing is intersected by the segment uv in two boundary points $u(B)$ and $v(B)$, let $a(B)$ be the shorter arc (one of the shortest arcs) of the boundary of B between $u(B)$ and $v(B)$. The path to which we shall refer as the *natural path connecting u and v* is the union of all arcs $a(B)$ for the balls of the packing intersecting uv and the segments of the line uv between u and v disjoint from the balls.

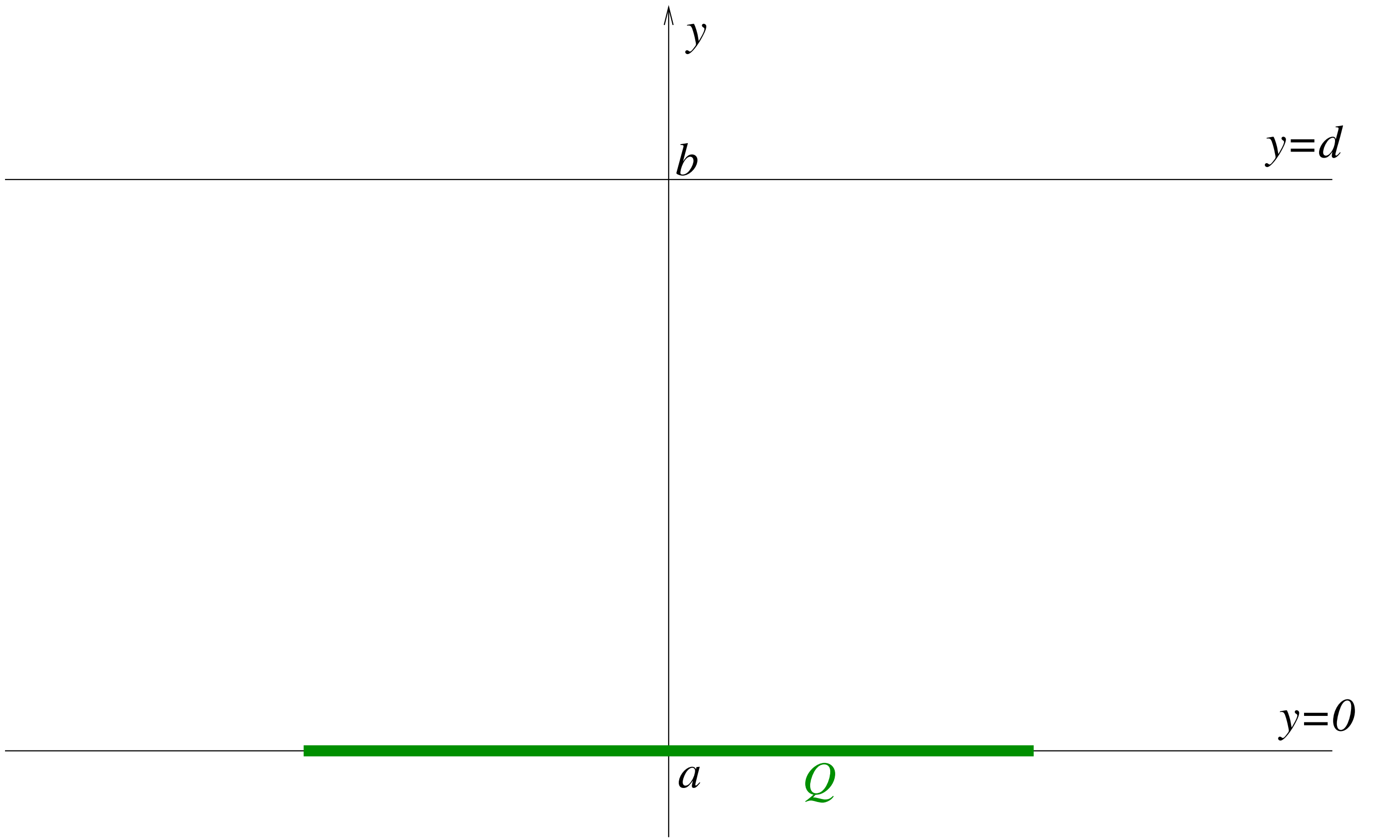
We shall use the following obvious fact

Observation. *The length of the natural path connecting two points at distance d outside the members of a ball packing is at most* $\frac{\pi}{2}d$.

Let $\mathcal{P}$ be a packing of balls with radii at most 1, and let a and b two points outside the balls. We choose the coordinate system so that a is the origin and $b = (\mathbf{0}, d)$.



y
b
y=d
y=0
a



y
b
y=d
y=0
a
Q

Q is the cube of side length

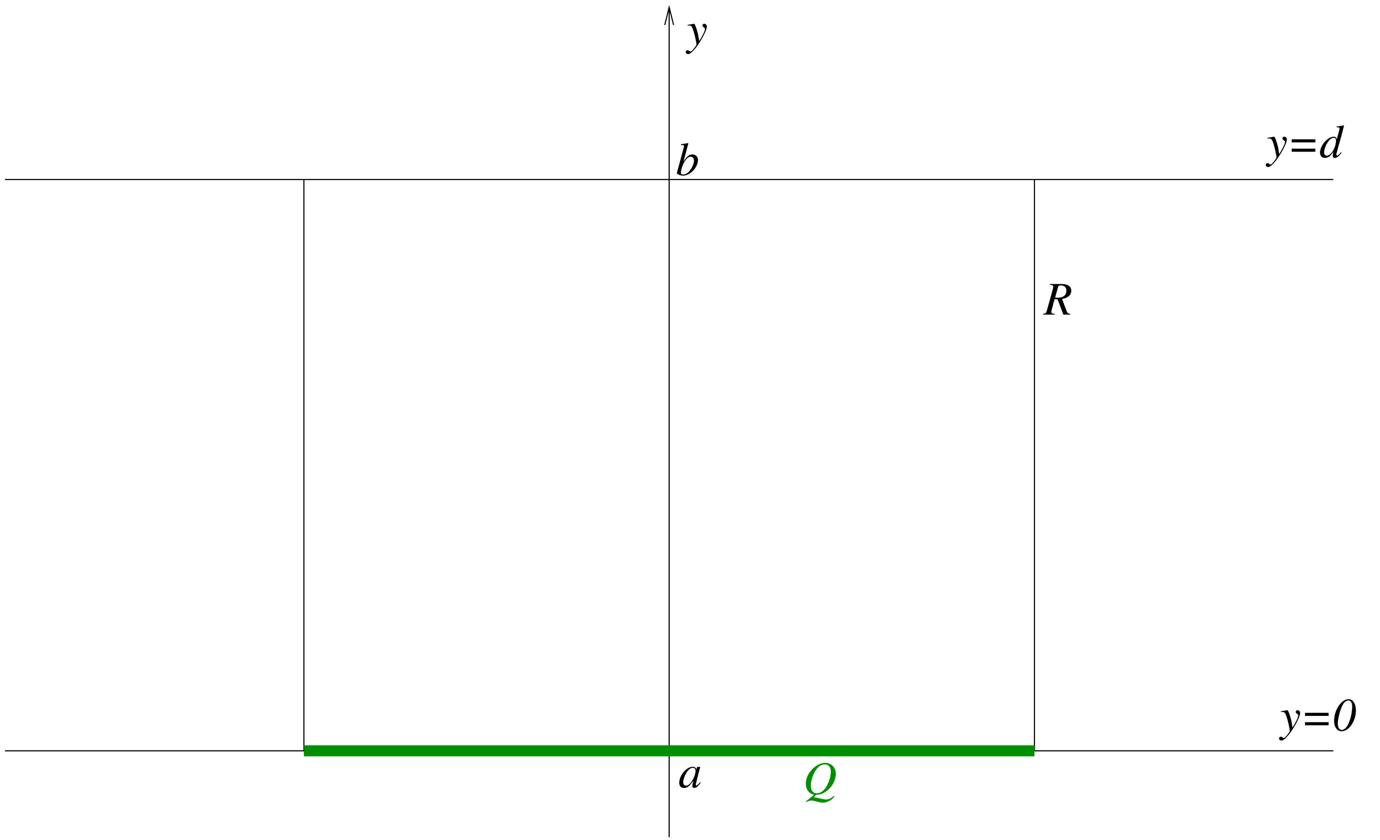
$$s = \sqrt{d}$$

in H centered at the origin and

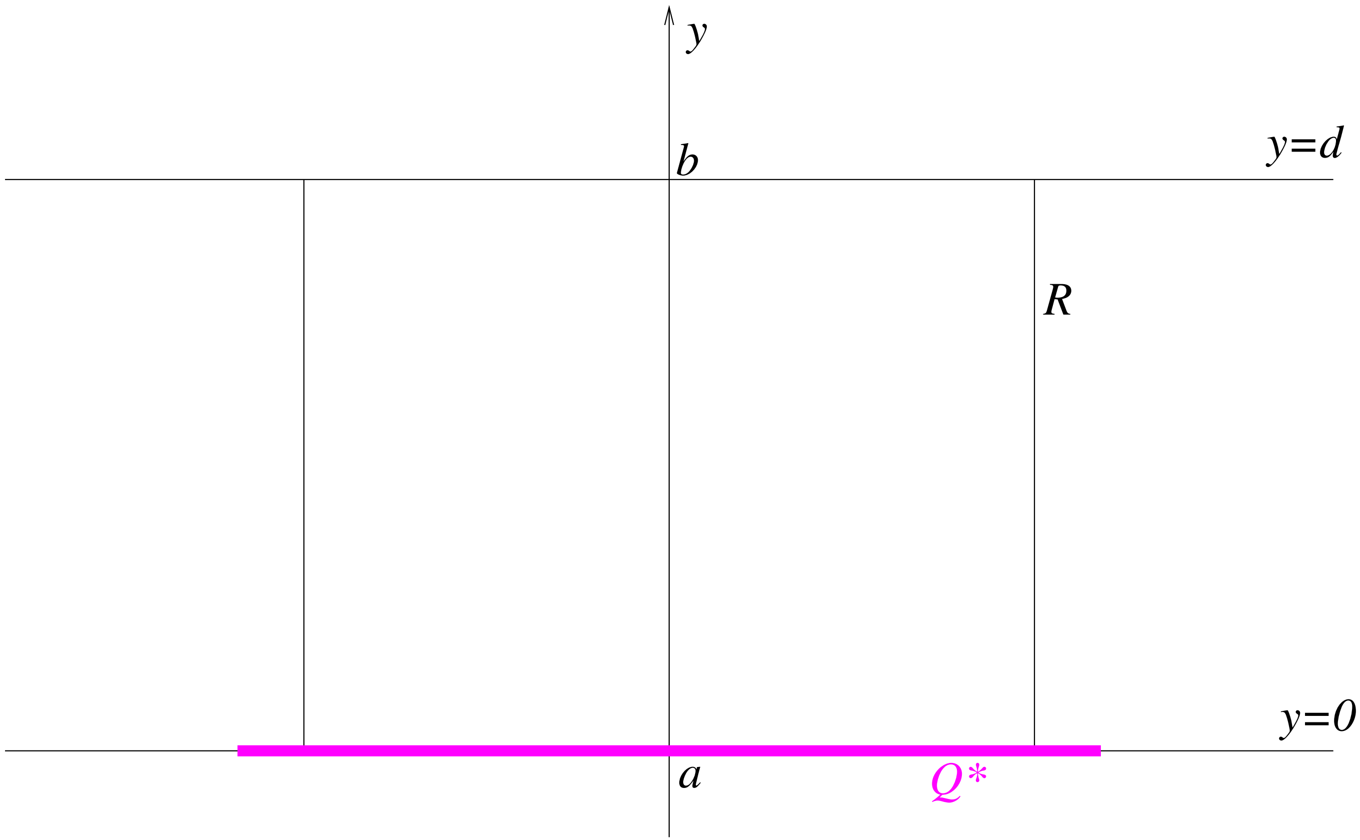
$$R = \{(\mathbf{x}, y) | \mathbf{x} \in Q, \, 0 < y < d\} \, .$$

Q^* is the cube of side length $s + 4$ concentric and homothetic with Q and

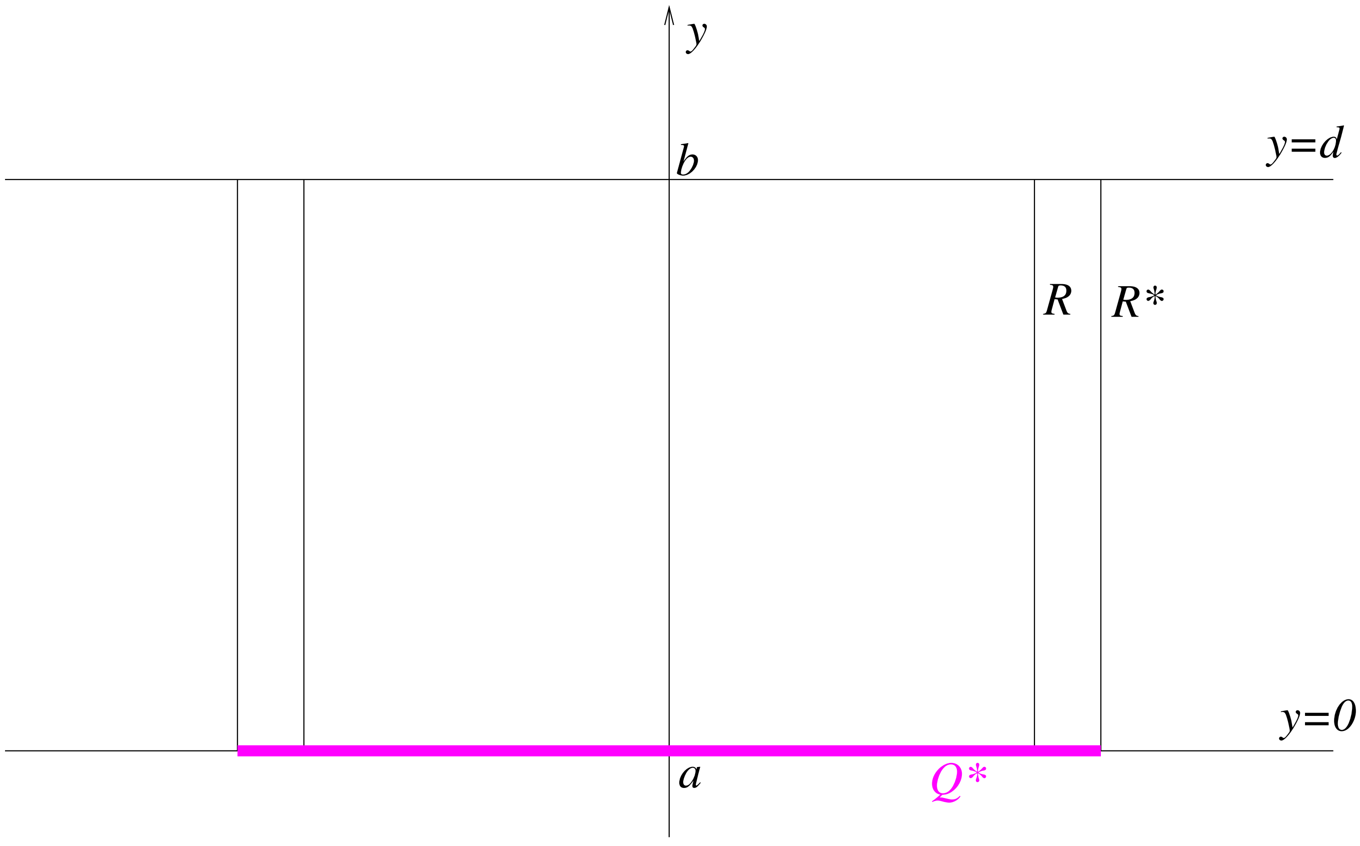
$$R^* = \{(\mathbf{x}, y) | \mathbf{x} \in Q^*, \, 0 < y < d\} \, ,$$



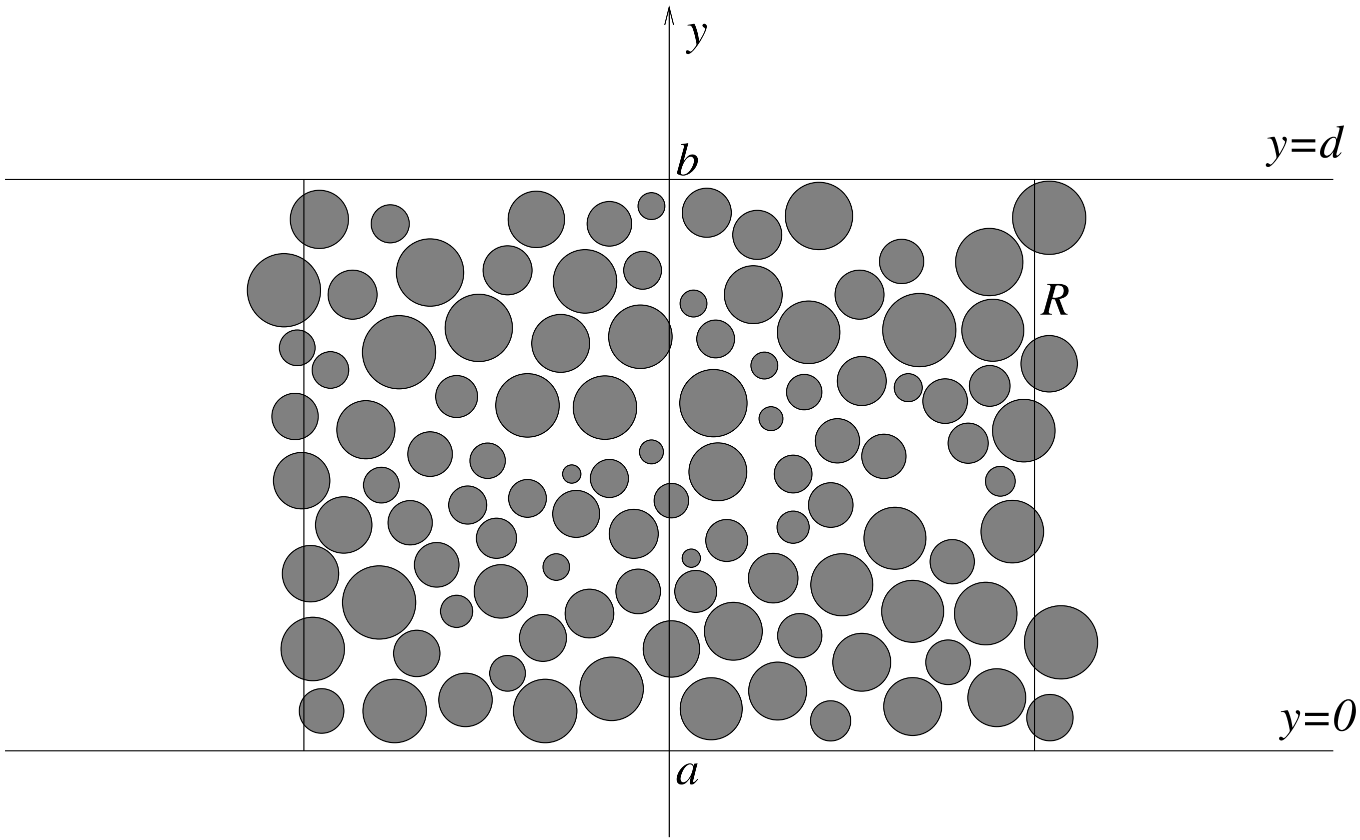
y
b
y=d
R
y=0
a
Q

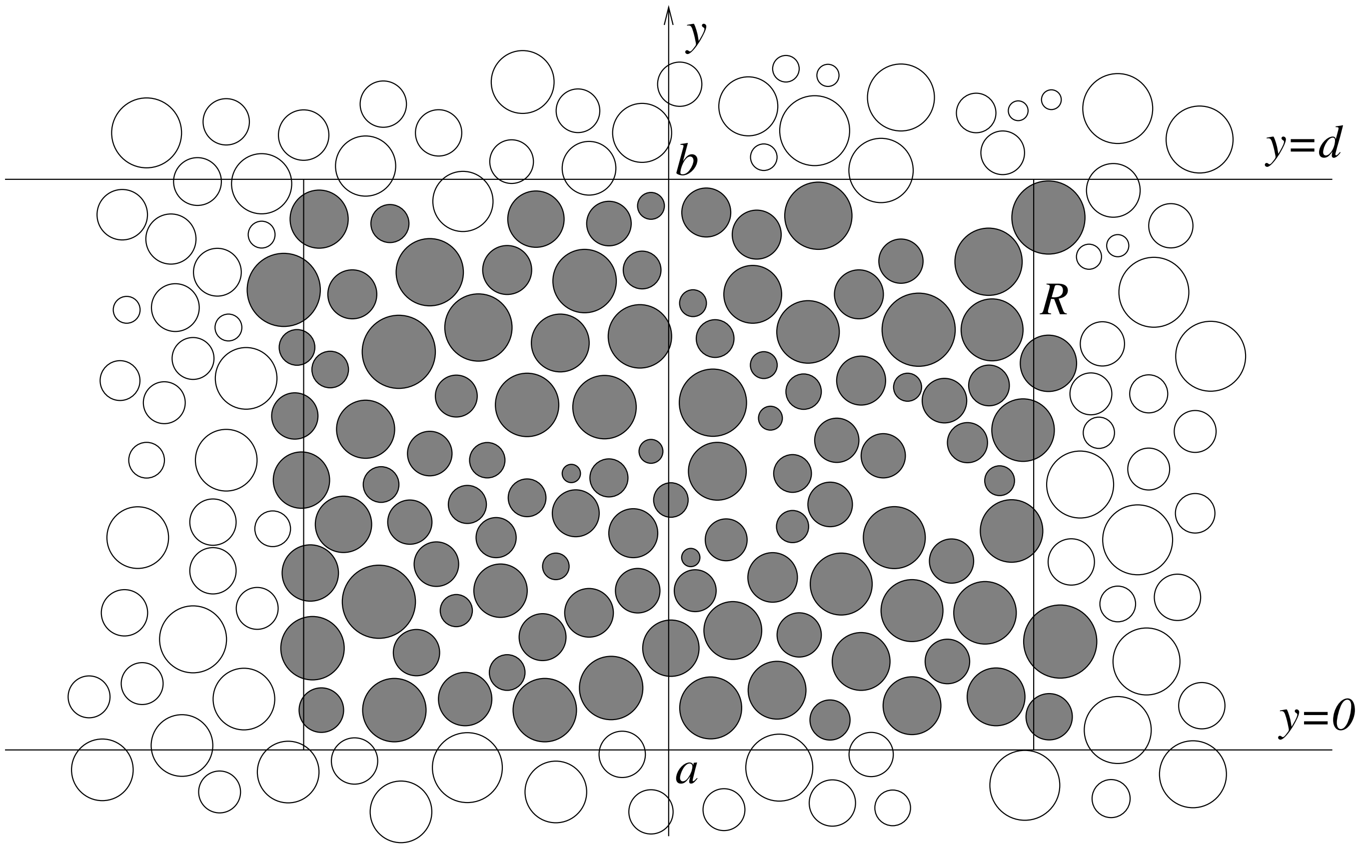


y
b
y=d
R
y=0
a
Q*

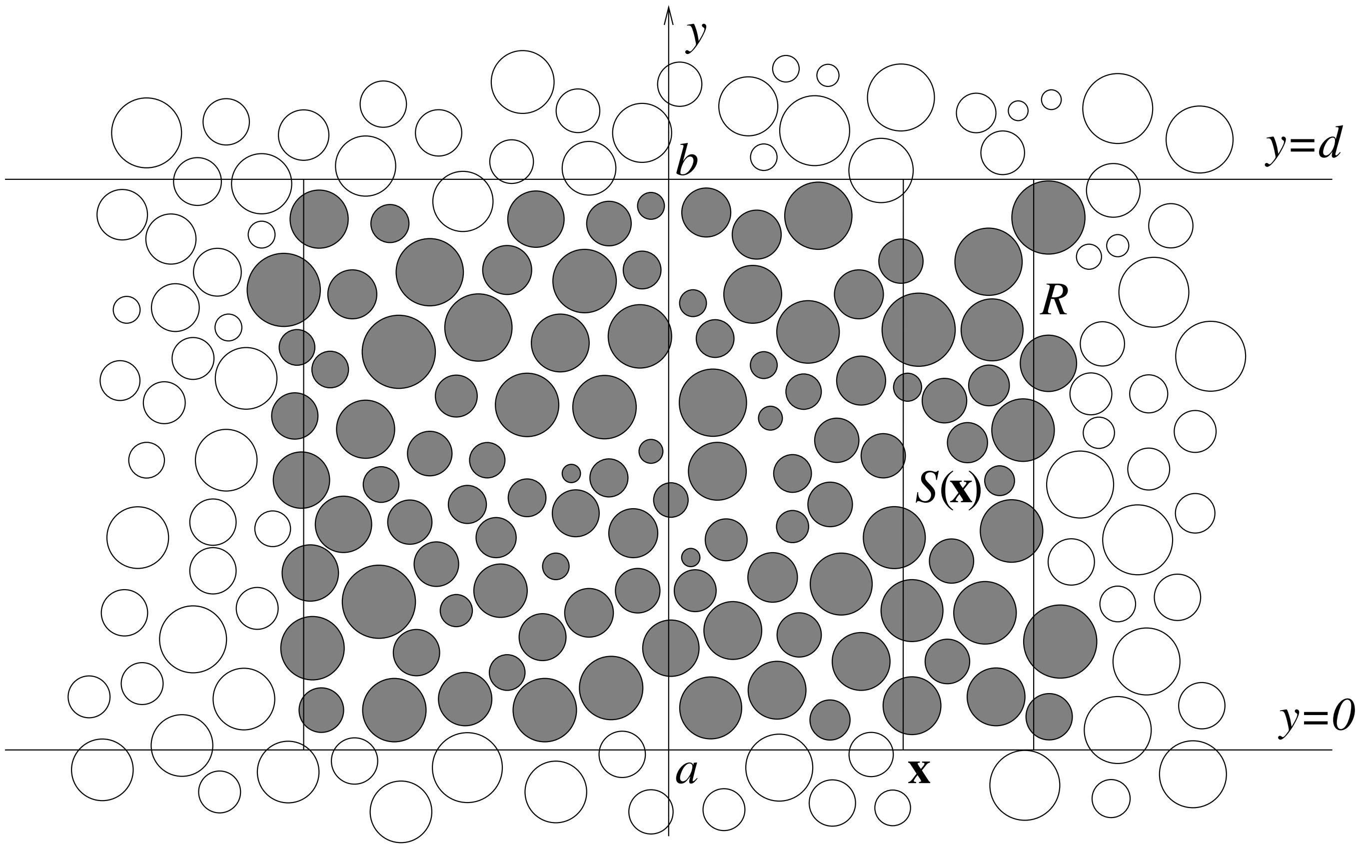

y
b
y=d
R
R*
y=0
a
Q*

$\tilde{\mathcal{P}}$ consist of those balls of $\mathcal{P}$ that are contained in the parallel strip bounded by the hyper planes $y = 0$ and $y = d$ and which intersect the box R.


y
b
y=d
R
y=0
a

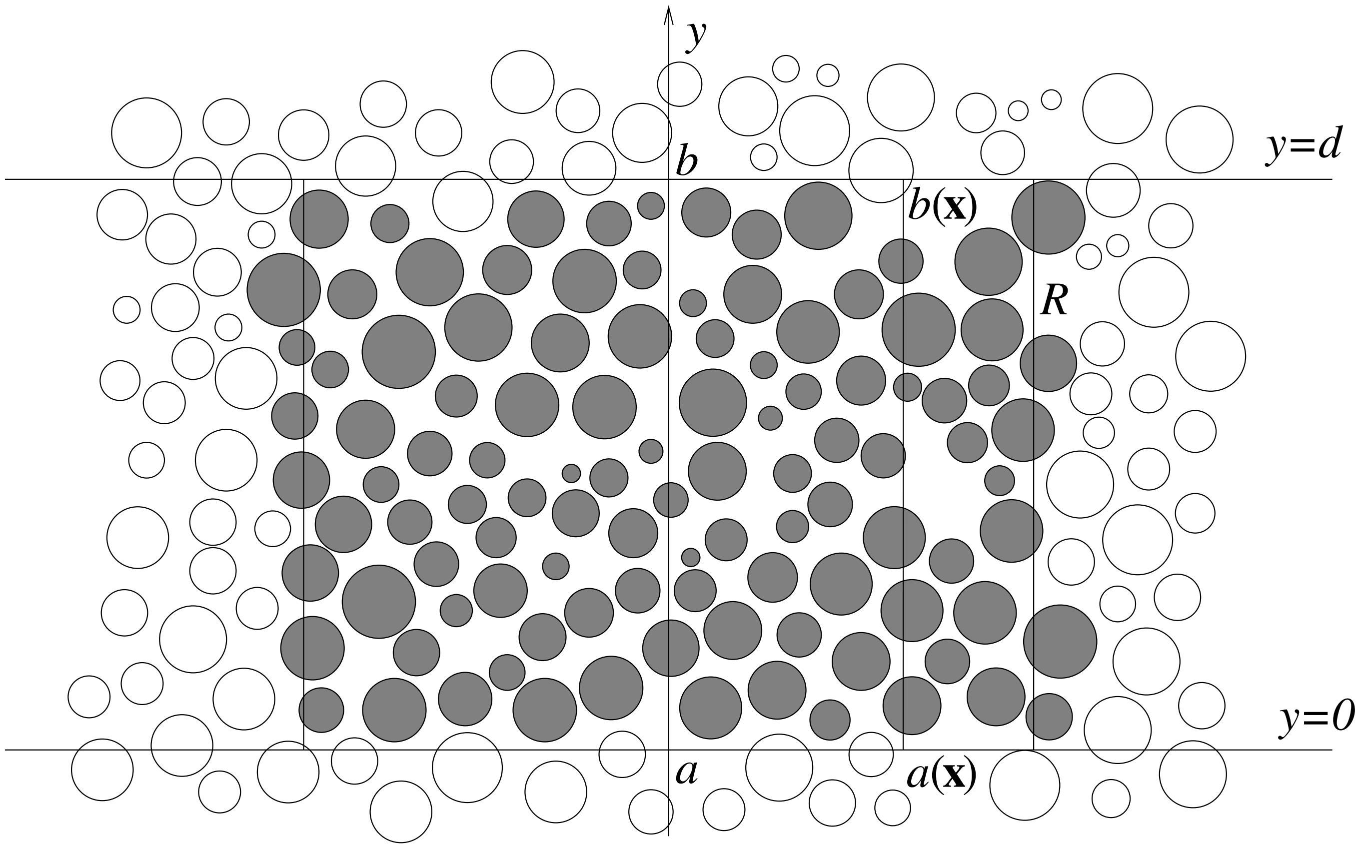

y
y=d
b
R
y=0
a

$S(\mathbf{x})$ is the segment with the endpoints $(\mathbf{x}, 0)$ and $(\mathbf{x}, d)$.

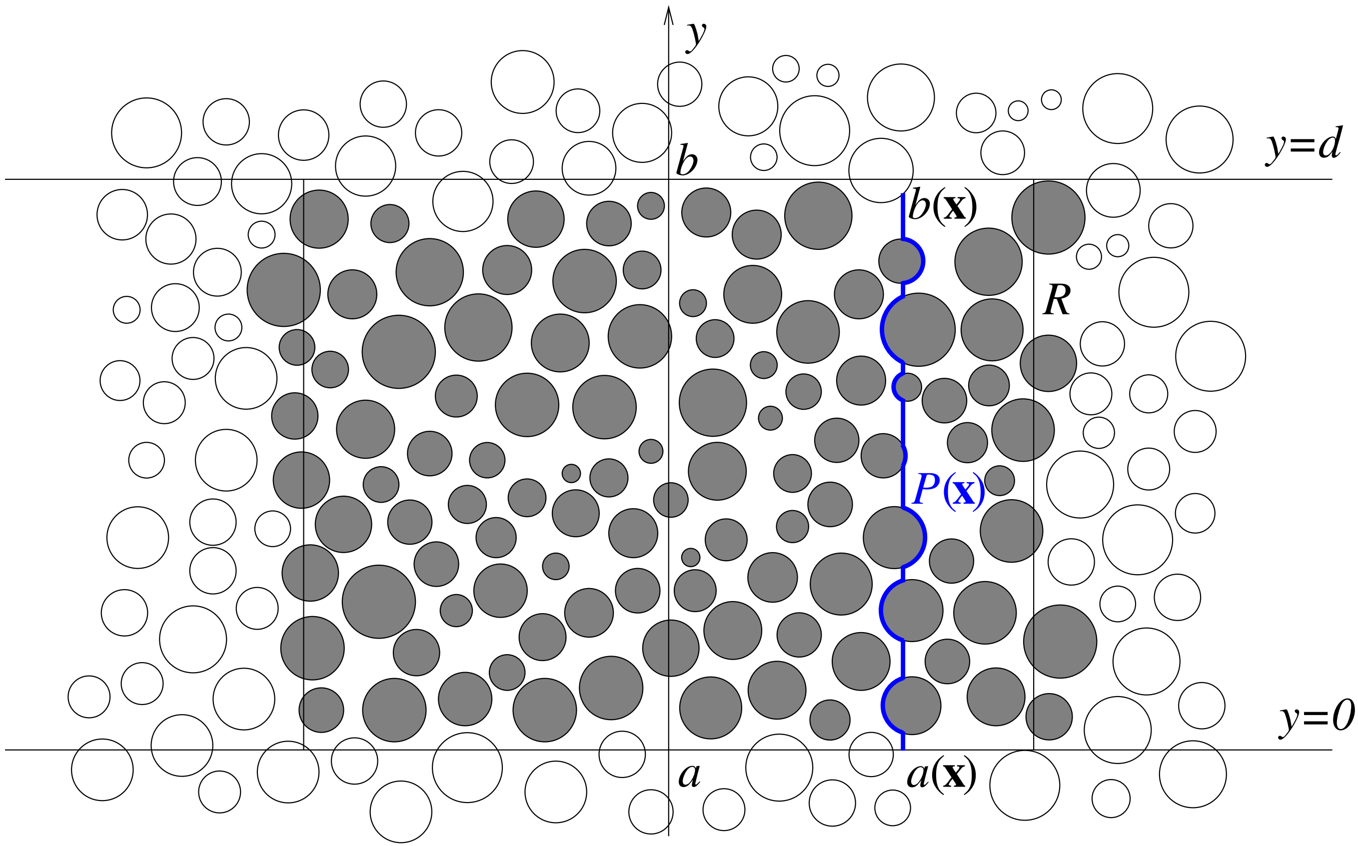


y
b
y=d
R
S(x)
y=0
a
x

Let $a(\mathbf{x})$ and $b(\mathbf{x})$ be the points of $S(\mathbf{x}) \setminus (\cup_{B \in \mathcal{P}} B)$ closest to the hyper plane H and to the hyper plane $y = d$, respectively.


y
y=d
b
b(x)
R
y=0
a
a(x)

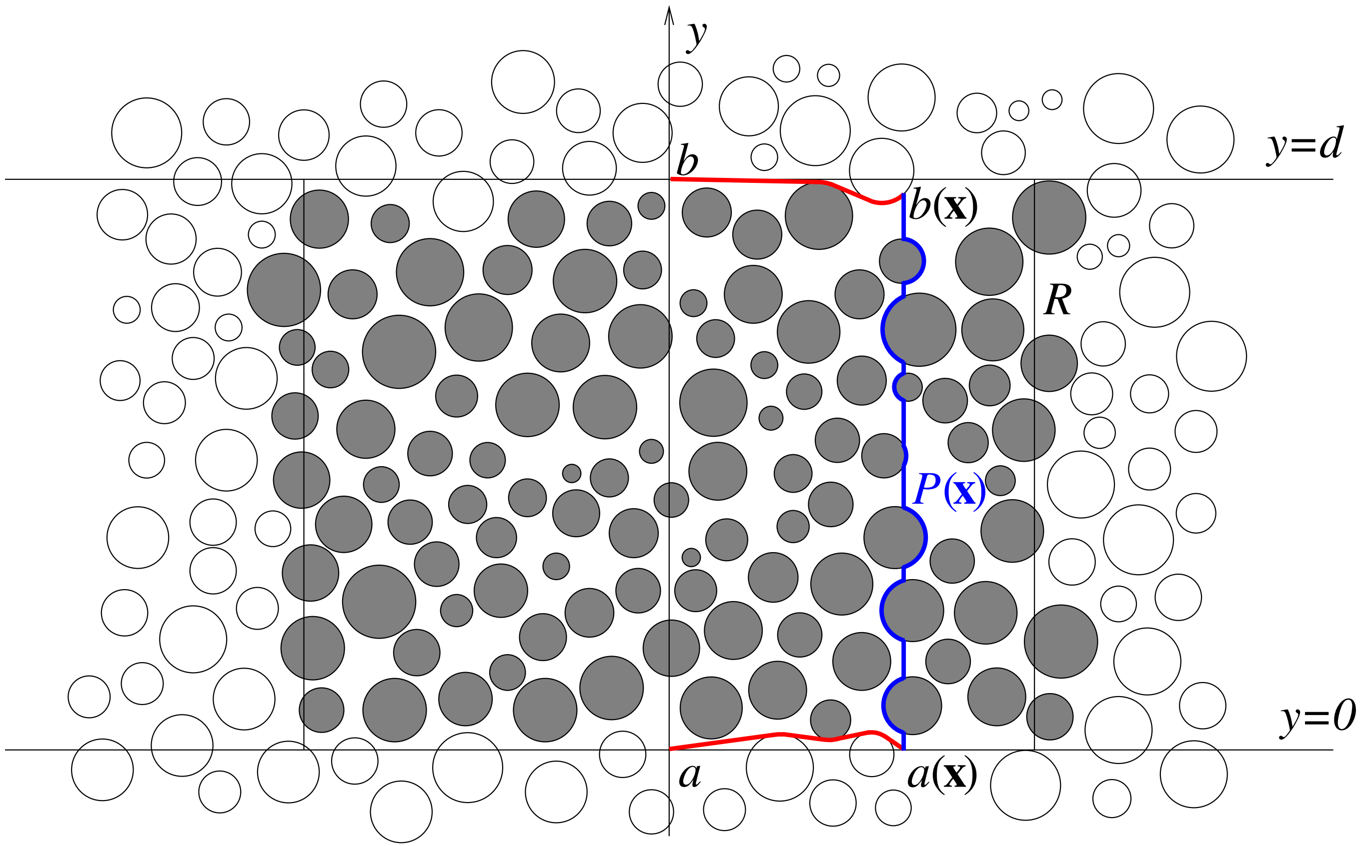
Let $P(\mathbf{x})$ denote the natural path between $a(\mathbf{x})$ and $b(\mathbf{x})$ and let $L(\mathbf{x})$ denote its length. Further, let $L_1(\mathbf{x})$ and $L_2(\mathbf{x})$ be the total length of straight segments and circular arcs on $P(\mathbf{x})$, respectively.



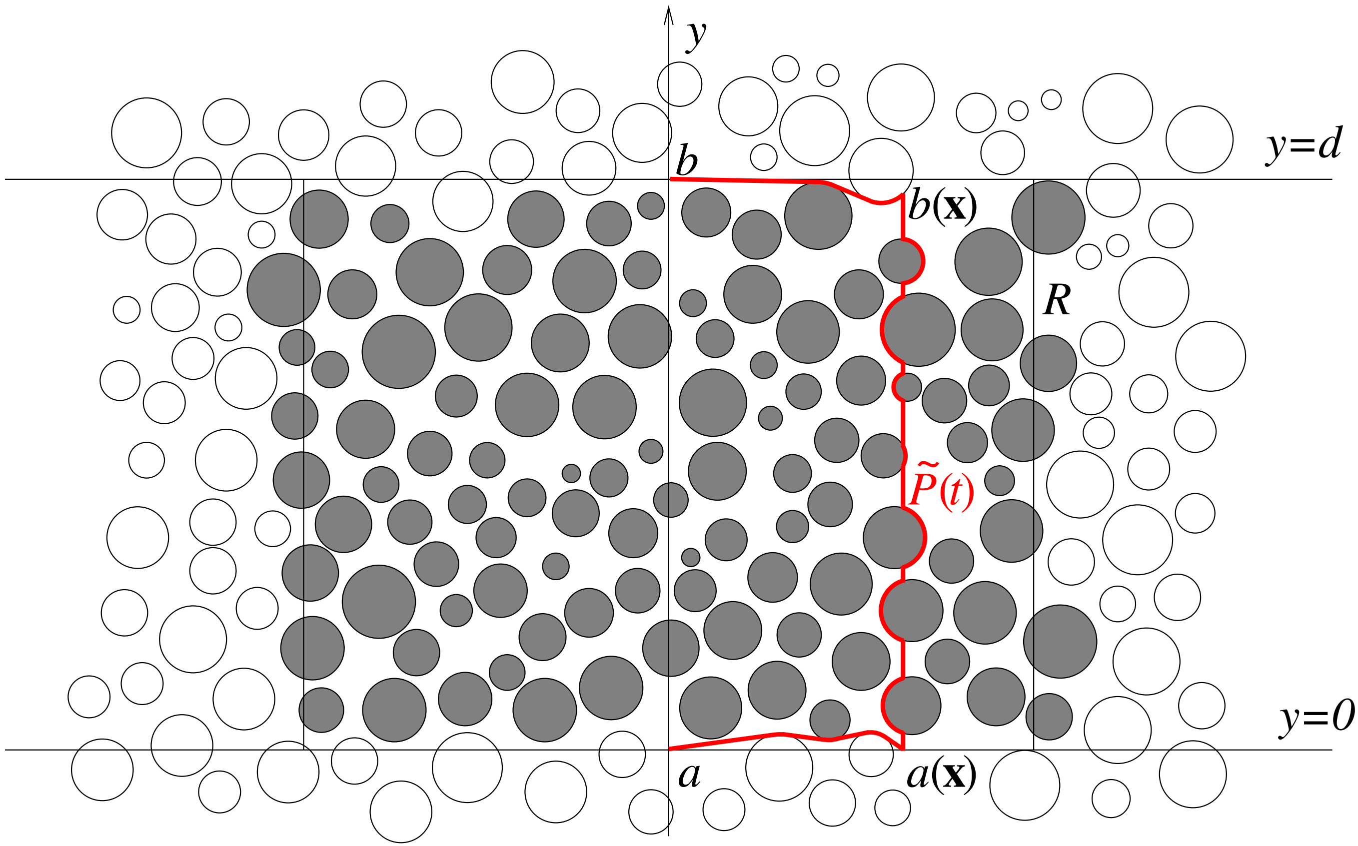
y
y=d
b
b(x)
R
P(x)
y=0
a
a(x)

For each $\mathbf{x} \in Q^*$ we construct a path joining a to b as follows.

Connect a and $a(\mathbf{x})$, as well as b and $b(\mathbf{x})$ by a shortest path avoiding all members of $\mathcal{P}$. Let $\tilde{P}(\mathbf{x})$ be the union of these two paths and $P(\mathbf{x})$.



y
y=d
b
b(x)
R
P(x)
y=0
a
a(x)



y
y=d
b
b(x)
R
P̃(t)
y=0
a
a(x)

The distance from a to $a(\mathbf{x})$, as well as the distance from b to $b(\mathbf{x})$ is at most $|\mathbf{x}| + 2$. Using Observation 1 we conclude that the length of $\tilde{P}(\mathbf{x})$ is at most

$$\pi(|\mathbf{x}| + 2) + L(\mathbf{x}),$$

hence the length $l(a, b)$ of the shortest path from a to b avoiding the balls of $\mathcal{P}$ satisfies the inequality

$$l(a, b) \leq L(\mathbf{x}) + \pi(|\mathbf{x}| + 2) \,.$$

We have

$$\begin{aligned} s^{n-1} l(a,b) &= \int_Q l(a,b)\, d\mathbf{x} \leq \\ &\leq \int_Q \left(L(\mathbf{x}) + \pi(|\mathbf{x}| + 2)\right) d\mathbf{x} \leq \\ &\leq \int_Q L(\mathbf{x})\, d\mathbf{x} + \frac{\pi}{2} s^n + 2\pi s^{n-1}. \end{aligned} \tag{1}$$

Let

$$F = \bigcup_{B \in \tilde{\mathcal{P}}} B \cap R, \quad G = R \setminus F,$$

$$F^* = \bigcup_{B \in \tilde{\mathcal{P}}} B, \quad \text{and} \quad G^* = R^* \setminus F^*.$$

Then we have

$$\begin{aligned}
\int_Q L(\mathbf{x})\,d\mathbf{x} \le \int_{Q^*} L(\mathbf{x})\,d\mathbf{x} = \\
&= \int_{Q^*} L_1(\mathbf{x})\,d\mathbf{x} + \int_{Q^*} L_2(\mathbf{x})\,d\mathbf{x} \le \\
&\le V(G^*) + \sum_{B\in\tilde{\mathcal{P}}} \int_{Q^*} l_B(\mathbf{x})\,d\mathbf{x} = \\
&= V(G^*) + \frac{\lambda_n}{\kappa_n} \sum_{B\in\tilde{\mathcal{P}}} V(B) = \\
&= V(G^*) + \frac{\lambda_n}{\kappa_n} V(F^*)\,.
\end{aligned}$$

Next we make the following

Observation 2. *The inner density of an arrangement relative to a space-filling domain cannot exceed the upper density of the arrangement.*

It follows that $V(F^*) \le \delta V(R^*)$, and since $\frac{\lambda_n}{\kappa_n} \le 1$ we have

$$\begin{aligned}
\int_Q L(\mathbf{x})\, d\mathbf{x} &\le V(G^*) + \tfrac{\lambda_n}{\kappa_n} V(F^*) \le \\
&\le (1 - \delta + \tfrac{\lambda_n \delta}{\kappa_n}) V(R^*) = \\
&= (1 - \delta + \tfrac{\lambda_n \delta}{\kappa_n}) d(s+4)^{n-1} = \\
&= (1 - \delta + \tfrac{\lambda_n \delta}{\kappa_n}) ds^{n-1} + O(ds^{n-2}).
\end{aligned}$$

Substituting this bound for $\int_Q L(\mathbf{x})\,d\mathbf{x}$ into (1) we get

$$s^{n-1} l(a,b) \leq (1-\delta+\tfrac{\lambda_n \delta}{\kappa_n}) d s^{n-1} + O(d s^{n-2})\,.$$

Remembering that $s=\sqrt{d}$ we get

$$l(a,b) \leq (1-\delta+\tfrac{\lambda_n \delta}{\kappa_n}) d + O(\sqrt{d})\,.$$

Let K be a convex body and $\mathbf{u}$ a unit vector in E^n. We choose the coordinate system so that $\mathbf{u}$ points in the direction of the y axis. As before, Let H denote the coordinate hyperplane $y = 0$.

We define the function $l_K(\mathbf{x}, \mathbf{u})$, $\mathbf{x} \in H$ as follows. If the line $l(\mathbf{x}) = \{(\mathbf{x}, y), -\infty < y < \infty\}$ intersects the boundary of K in two points, let $l_K(\mathbf{x}, \mathbf{u})$ be the length of the shortest arc on the boundary of B between the two points of intersection. Let $l_K(\mathbf{x}, \mathbf{u}) = 0$ otherwise. Define $\lambda_K(\mathbf{u})$ as

$$\lambda_K(\mathbf{u}) = \int_H l_K(\mathbf{x}, \mathbf{u})\, d\mathbf{x}$$

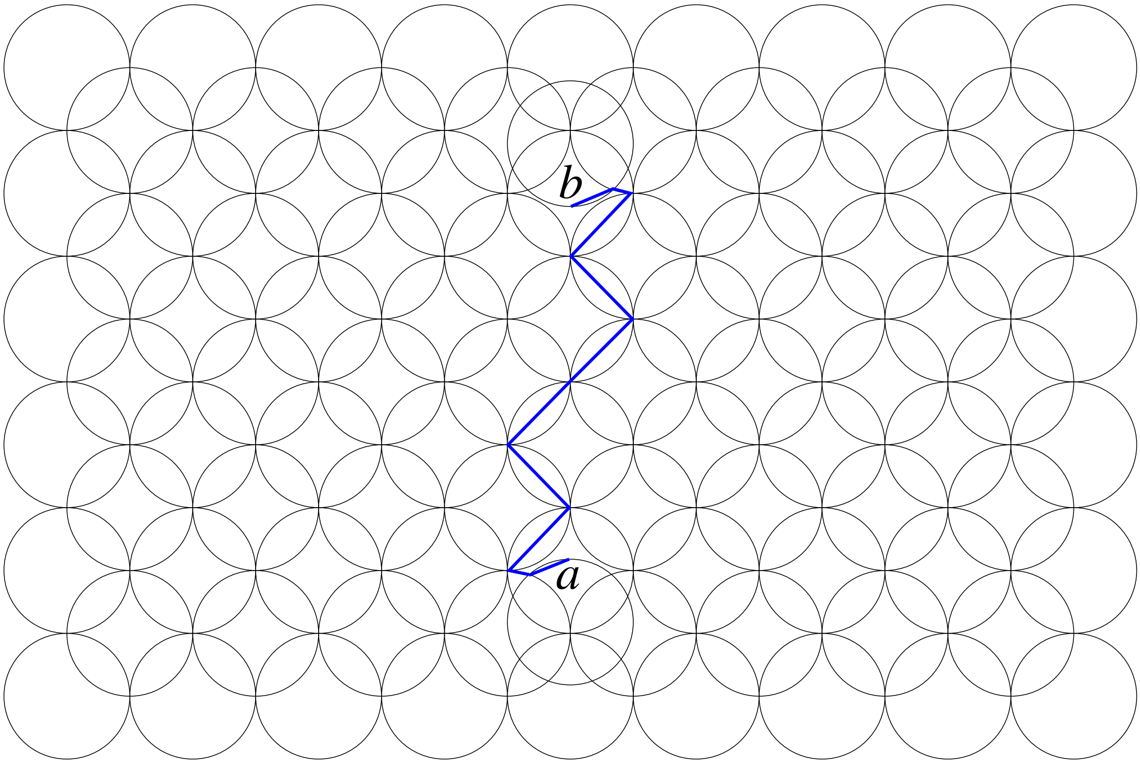
and let

$$\lambda_K = \min_{\mathbf{u}} \lambda_K(\mathbf{u}).$$

The argument we used for packings of balls can be repeated for packings of uniformly bounded similar copies of K with upper density δ showing that in such a packing any two points at distance d from one another lying outside the members of a packing can be connected by a path remaining in uncovered part of the space and of length at most

$$(1-\delta+\tfrac{\lambda_K}{V(K)}\delta)d+O(\sqrt{d}).$$

If compact sets cover the space, then the set of points that are covered at least twice is connected. Thus the shortest path problems we considered for packings have their natural dual counterparts concerning shortest paths within the part of the space covered at least wtice by the members of a covering. We emphasize the problem about the paths within the region covered at least twice by the members of a covering by unit circles.


b
a

The figure shows that for values of d close to $2(\sqrt{3}+i)$, $i=-1, 0, 1\ldots$, the length of the shortest path between two points at distance d can be close to $\sqrt{2}d+2.889\ldots$. Is this the worth case? Does there exist a constant c such that for all coverings of the plane by closed unit circles any two points situated at distance d from one another and covered by at least two circles can be connected by a path of length $\sqrt{2}d+c$ traveling within at least doubly covered part of the plane? Can a covering by incongruent circles force us to a greater detour than a covering by congruent circles?

There are two special types of paths in the region covered at least twice by circles: one that uses only boundary arcs of the circles, and another type that travels along the sides of the Dirichlet cells. Prove or disprove the following conjectures.

There is a constat c_1 such that if closed unit circles cover the plane and a and b are two points at distance d apart, both lying on the boundary of some of the circles, then a and b can be connected by a path whose length is at most $\pi d/2 + c_1$ and which uses only boundary arcs of the circles.

There is a constat c_2 such that if closed unit circles cover the plane and a and b are two points at distance d apart, both lying on the boundary of the Dirichlet cell of some of the circles, then a and b can be connected by a path whose length is at most $\sqrt{2}d + c_1$ and which uses only boundary arcs of the Dirichlet cells.

D.R. Baggett and András Bezdek confirmed the second conjecture for lattice-coverings. For arbitrary coverings with unit circles Edgardo Roldán-Pensado showed that two points at distance d apart lying in at least doubly covered part of the plane can be connected by a path that remains in the part of the plane covered at least twice and whose length is at most $(\pi/3 + \sqrt(3))d + c$ for some constant $c < 17$.