On the Average Distance from the Fermat-Weber Center of a Planar Convex Body

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The Fermat-Weber point for a finite point set

Given n points, $q_1, q_2, \ldots, q_n$, in the plane, the *Euclidean median*, a.k.a. *Fermat-*, *Torricelli-* or *Weber point*, is the point that minimizes the sum of distances to the n points

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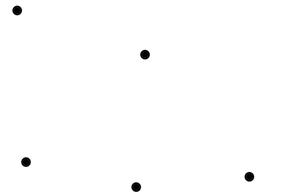

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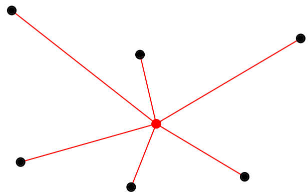

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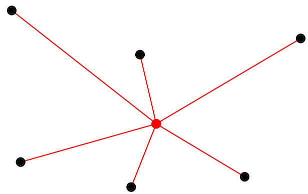
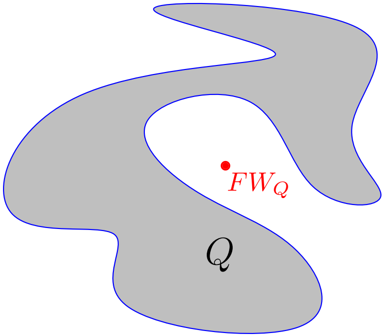


$$\min_{p \in \mathbb{R}^2} \sum_{i=1}^{n} \mathrm{dist}(p, q_i).$$

For $n = 3$ and 4, resp., Torricelli and Fagnano gave algebraic solutions for computing this point.
In general, it cannot be computed exactly for $n \geq 5$.
It can be approximated with arbitrary precision.

The study of the Weber point has a long history motivated by applications in facility location and computational statistics.

The continuous analogue of the Fermat-Weber point



Given a body Q in the plane,
the *Euclidean median* or *Fermat-Weber point*, is the point that minimizes the average distance to the points in Q.

$$\min_{p \in \mathbb{R}^2} \mu_Q(p),$$

$$\mu_Q(p) = \frac{\int_{q \in Q} \mathrm{dist}(p,q)\mathrm{d}q}{\int_{q \in Q} 1 \ \mathrm{d}q} = \frac{\int_{q \in Q} \mathrm{dist}(p,q)\mathrm{d}q}{\mathrm{area}(Q)}.$$

The Fermat-Weber point is denoted FW_Q.
The minimum average distance is $\mu_Q^* = \mu_Q(FW_Q)$.

Intuitively, the Fermat-Weber point is an ideal location for a base station serving area Q, assuming uniform density in Q.

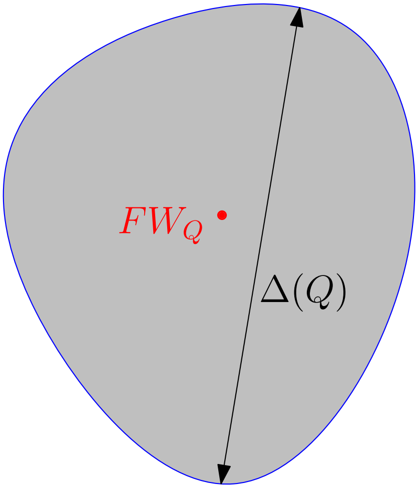
Average Distance versus Diameter

Let $\Delta(Q)$ denote the diameter of Q.

Carmi, Har-Peled, and Katz (2005) studied the ratio $\mu_Q^*/\Delta(Q)$.

Conjecture:

$$\frac{1}{6} \leq \frac{\mu_Q^*}{\Delta(Q)} \leq \frac{1}{3}.$$



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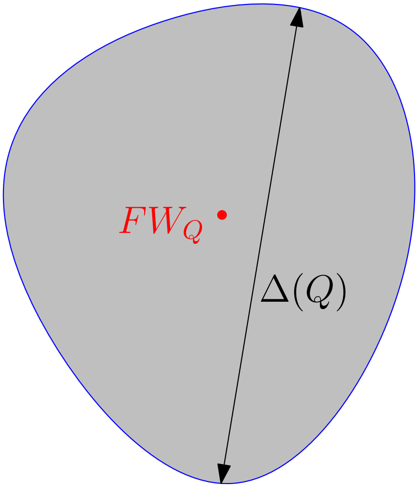

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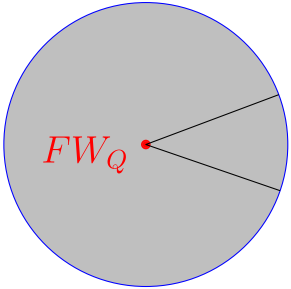

Let Q be a circular disk of radius 1, $\Delta(Q) = 2$.
By symmetry, FW_Q is the center of Q.
In every sector, the average distance is $\frac{2}{3}$.
So, we have $\mu^*_Q/\Delta(Q) = 1/3$.

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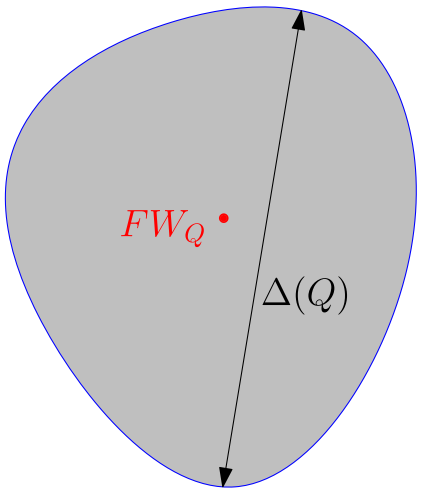

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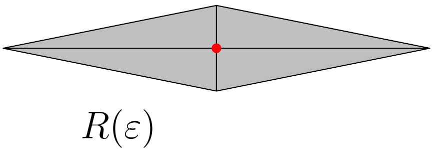


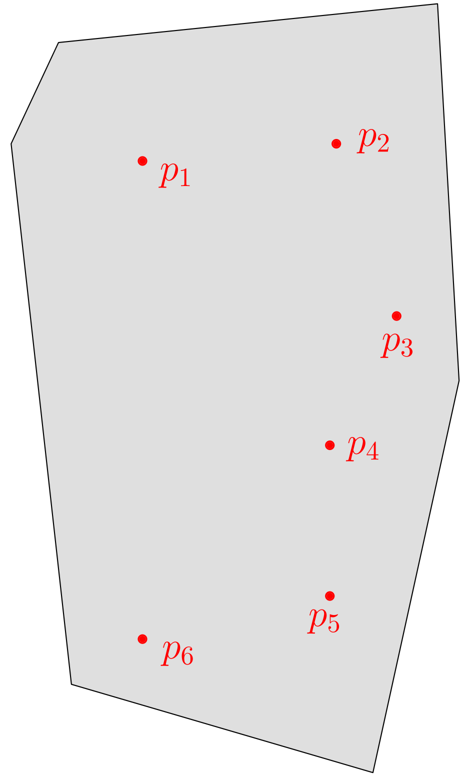
Let $R(\varepsilon)$ be a rhombus with diagonals 1 and 2ε.
By symmetry, $FW_{R(\varepsilon)}$ is the center of $R(\varepsilon)$.
In each quadrant, the average distance from $FW_{R(\varepsilon)}$ goes to $\frac{1}{3}$ as $\varepsilon \to 0$.
So, we have $\lim_{\varepsilon \to 0+} \mu_{R(\varepsilon)}^*/\Delta(R(\varepsilon)) = 1/6$.

Motivation: Geometric Load Balancing

Aronov, Carmi, and Katz (2009):

Input: a convex body D and m points $p_1, p_2, \ldots, p_m \in D$ (*facilities*).
Objective: Subdivide D into m convex regions, $R_1, R_2, \ldots, R_m$, of *equal area* such that $\sum_{i=1}^{m} \mu_{p_i}(R_i)$ is minimal, where the *cost function* $\mu_{p_i}(R_i)$ is the average travel time from facility p_i to any location in its region R_i.

Aronov, Carmi, and Katz (2009) gave an $(8 + \sqrt{2\pi})$-factor approximation if D is an $n_1 \times n_2$ rectangle for integers n_1, n_2.
This basic approximation bound is then used for several other cases, e.g. a convex fat domain D and m convex regions R_i.



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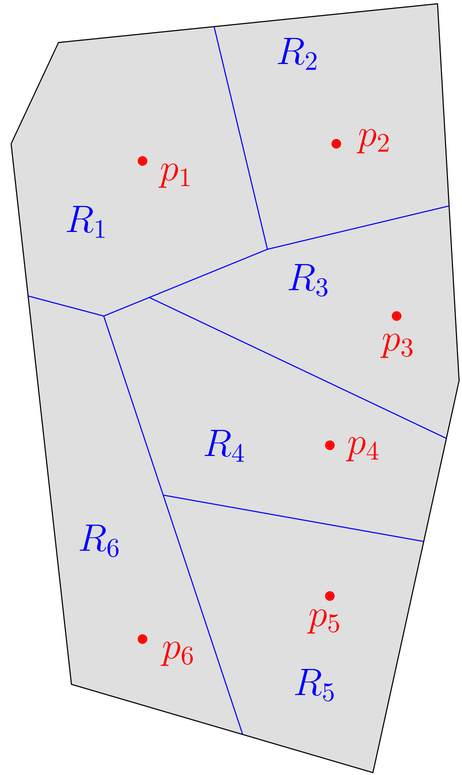

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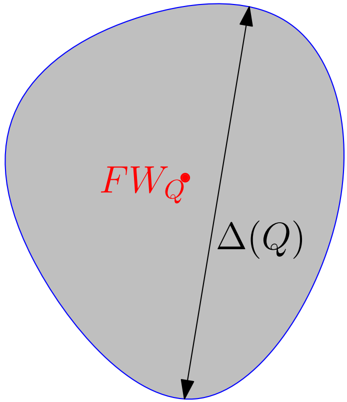
$$\frac{1}{6} \leq \frac{\mu_Q^*}{\Delta(Q)} \leq \frac{1}{3}.$$



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For every convex body $Q \subset \mathbb{R}^2$, $\frac{1}{7} \leq \frac{\mu_Q^*}{\Delta(Q)}$.

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For every convex body $Q \subset \mathbb{R}^2$, $\frac{1}{6} < \frac{\mu_Q^*}{\Delta(Q)}$.

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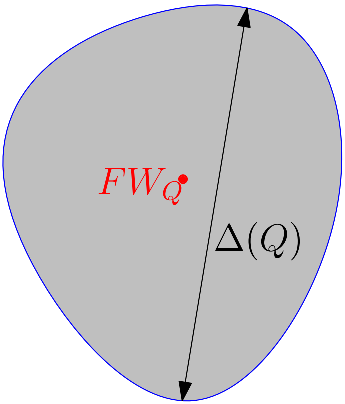

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For every convex body $Q \subset \mathbb{R}^2$, $\frac{\mu_Q^*}{\Delta(Q)} \leq \frac{2}{3\sqrt{3}} \approx 0.3849$.

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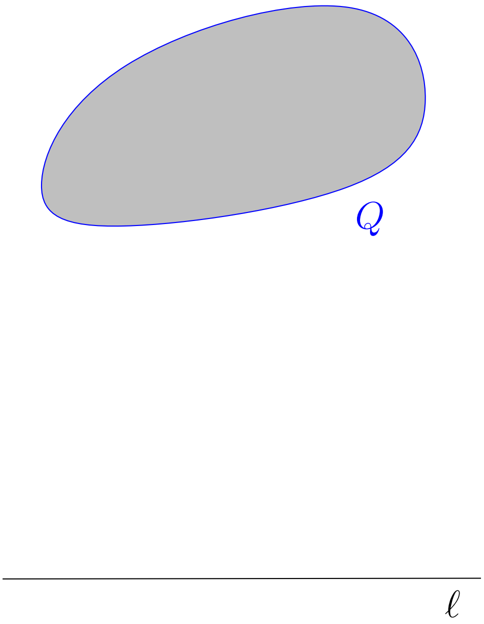
For every convex body $Q \subset \mathbb{R}^2$, $\frac{\mu_Q^*}{\Delta(Q)} \leq \frac{2(4-\sqrt{3})}{13} \approx 0.3490$.

Lower Bound for $\mu_Q^*/\Delta(Q)$

The **Steiner symmetrization** of Q with respect to an axis ℓ constructs $S(Q,\ell)$ as follows: Each chord of Q orthogonal to ℓ is displaced along its line to a new position where it is symmetric with respect to ℓ.



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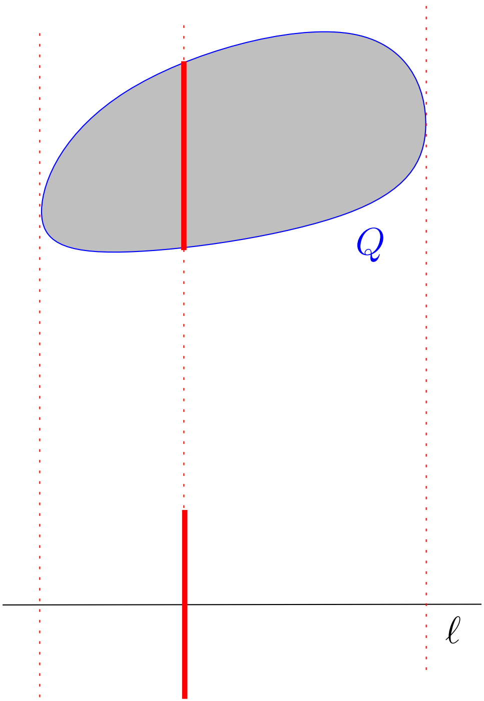

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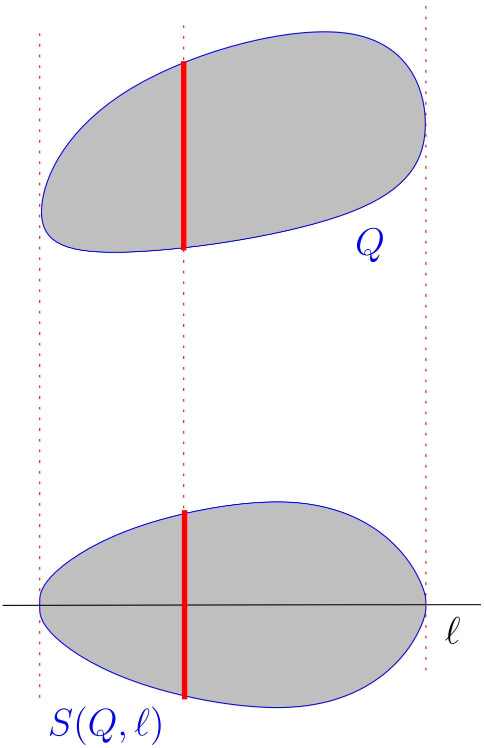

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Fact 1: Symmetrization preserves convexity.

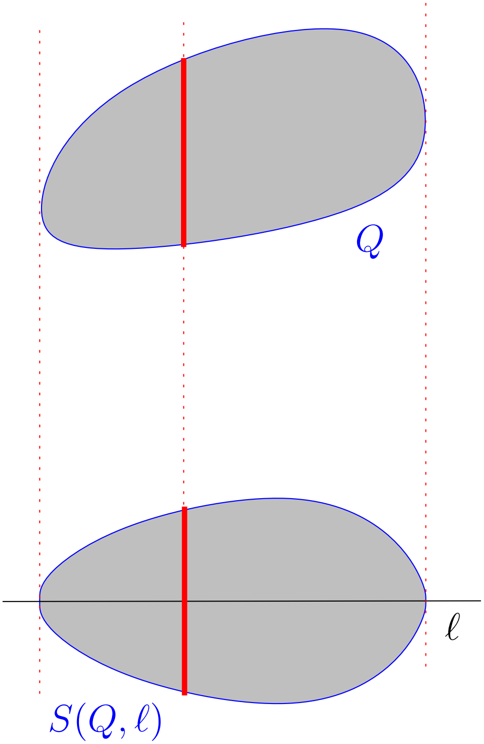
Fact 2: Symmetrization preserves area,

$$\mathrm{area}(Q) = \mathrm{area}(S(Q, \ell)).$$

Fact 3: The Fermat-Weber center of $S(Q, \ell)$ lies on the symmetry axis ℓ.

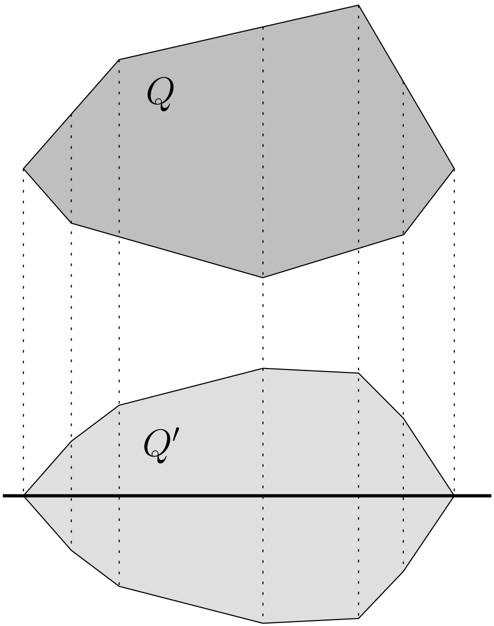
Lemma: If ℓ is parallel or orthogonal to a diagonal of Q, then the symmetrization with respect to ℓ preserves the diameter

$$\Delta(Q) = \Delta(S(Q, \ell)).$$



Lower Bound for $\mu^*_Q/\Delta(Q)$

Lemma: Let Q be a convex body, and let $Q' = S(Q, \ell)$. Then $\mu^*_{Q'} \le \mu^*_Q$.



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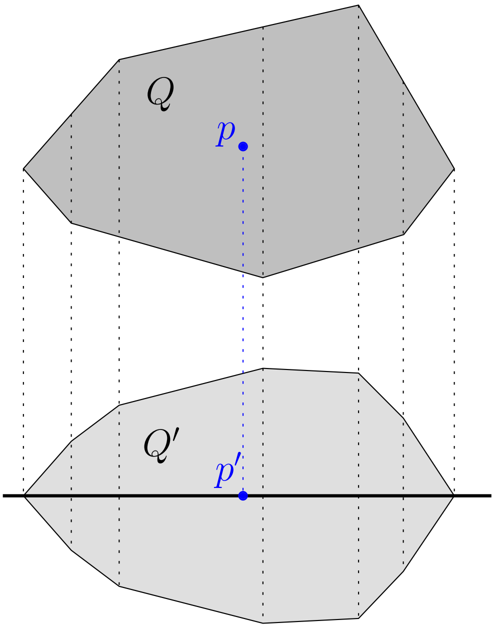

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$$\mu^*_{Q'} \le \mu_{Q'}(p') \le \mu_Q(p) = \mu^*_Q.$$

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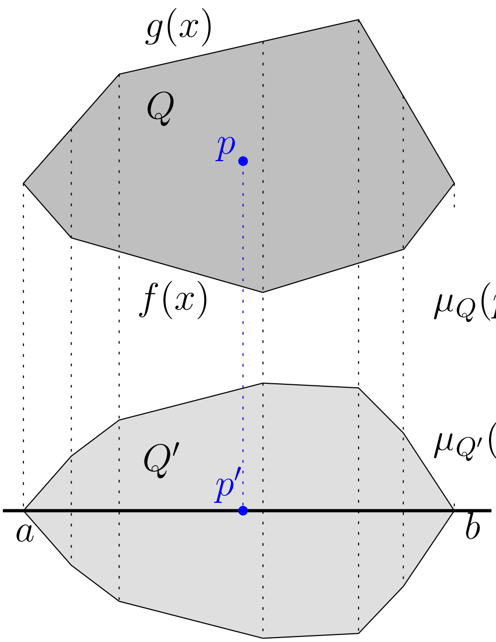

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$$\mu_Q(p) = \frac{1}{\mathrm{area}(Q)} \int_a^b \int_{f(x)}^{g(x)} \mathrm{dist}(p,(x,y))\mathrm{d}y\,\mathrm{d}x$$

$$\mu_{Q'}(p') = \frac{1}{\mathrm{area}(Q')} \int_a^b \int_{\frac{f(x)-g(x)}{2}}^{\frac{g(x)-f(x)}{2}} \mathrm{dist}(p',(x,y))\mathrm{d}y\,\mathrm{d}x$$

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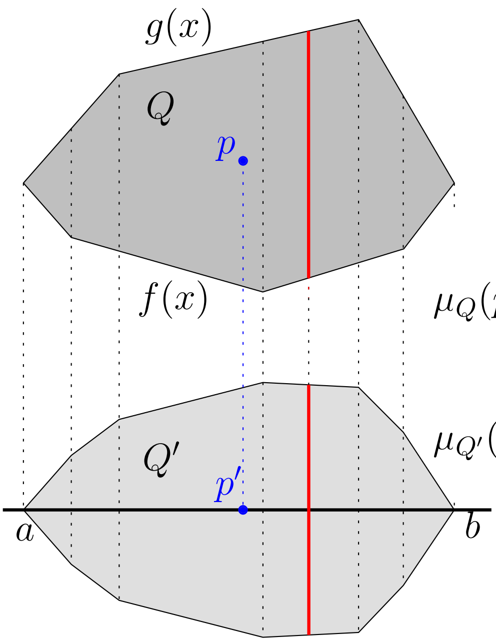
Lemma: Let Q be a convex body, and let $Q' = S(Q, \ell)$. Then $\mu_{Q'}^* \leq \mu_Q^*$.



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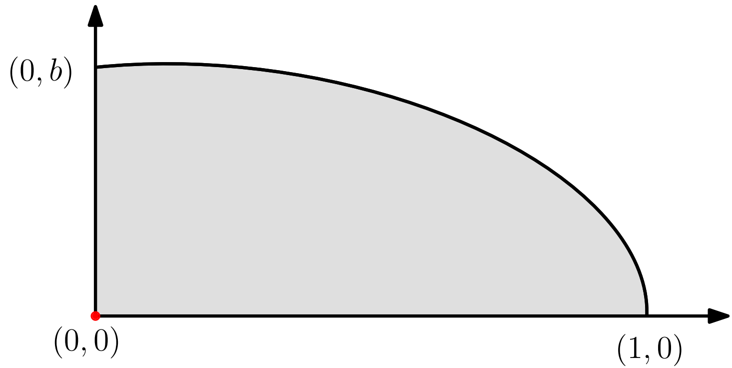
Lemma: If Q is a convex body, symmetric in both coordinate axes, then

$$\frac{1}{6} < \frac{\mu_Q^*}{\Delta(Q)}.$$

Assume $\Delta(Q) = 2$.
By symmetry, point FW_Q is the origin $(0,0)$.
The average distance from the origin is the same in each quadrant.



We partition one quadrant into pieces, and show that for some $\delta > 0$,

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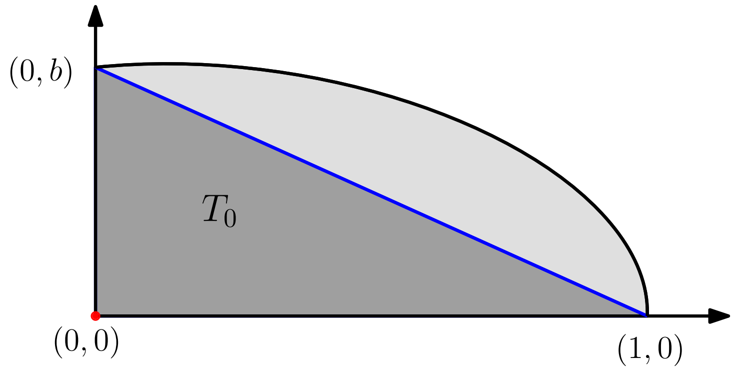

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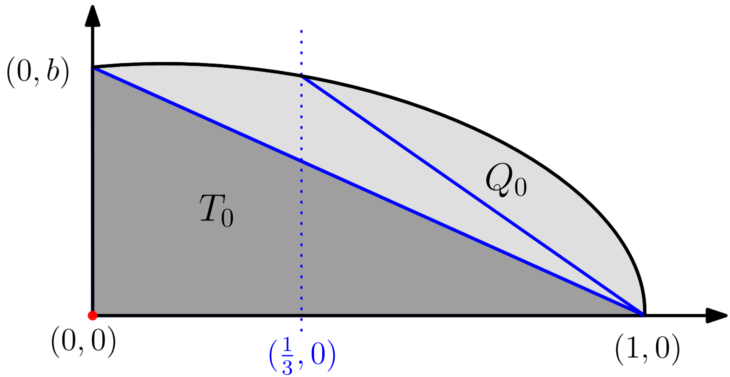

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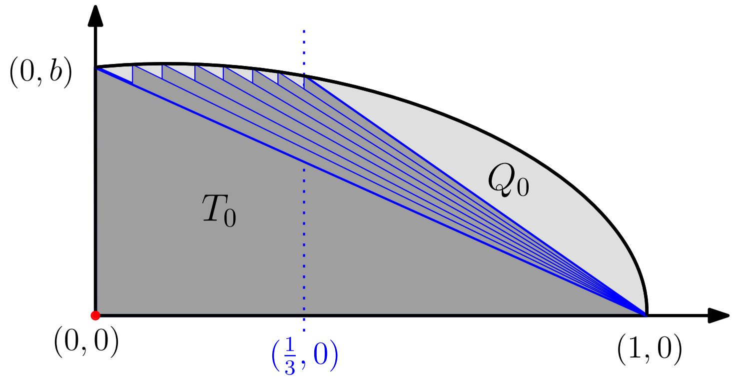

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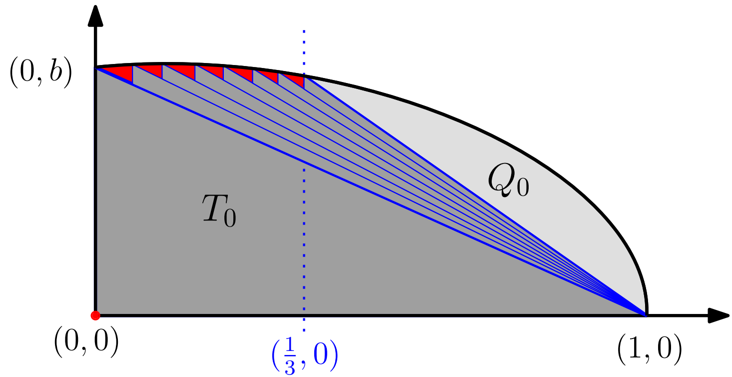

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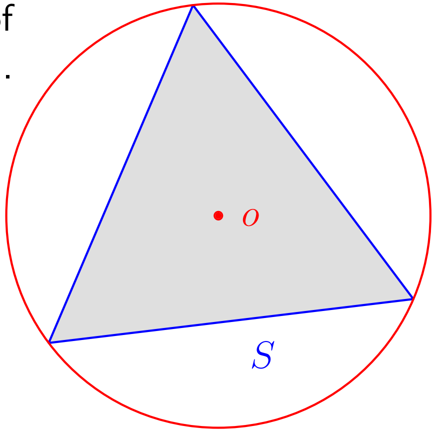
Upper Bound for $\mu_Q^*/\Delta(Q)$

Theorem (Jung, 1910): Every set $S \subset \mathbb{R}^2$ of diameter $\Delta(S)$ lies in ar disk of radius $\frac{1}{\sqrt{3}} \cdot \Delta(S)$. This bound is attained for the regular triangle.



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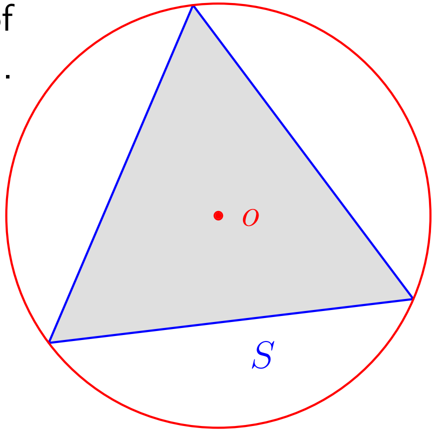

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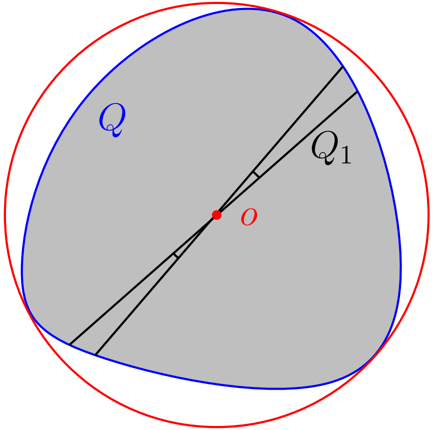
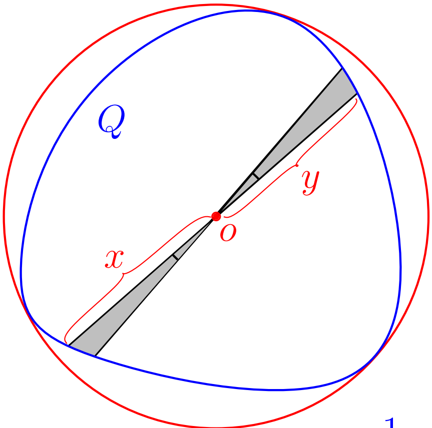


For fixed diameter $\Delta(Q)$, we have $\mu_Q^* \leq \mu_Q(o)$. We deduce an upper bound for $\mu_Q(o)$. Partition Q into n double sectors $Q_1, Q_2, \ldots, Q_n$ with a common apex o.

$$\mu_{(Q_1 \cup \ldots \cup Q_n)}(o) \leq \max(\mu_{Q_1}(o), \ldots, \mu_{Q_n}(o)).$$

Upper Bound for $\mu^*_Q/\Delta(Q)$



In a double sector Q_i, denote by x and y the maximum distance from o in two opposite directions. Assume w.l.o.g. $y \leq x$.

(By uniform continuity, x and y approximate the maximum distance from o along any two opposite directions within the sector.)

$$0 < y \leq x \leq \frac{1}{\sqrt{3}} \cdot \Delta(Q) \qquad \text{and} \qquad x + y \leq \Delta(Q).$$

$$\mu_{Q_i}(o) = \frac{\int_{q \in Q_i} \mathrm{dist}(p,q)\mathrm{d}q}{\mathrm{area}(Q)} \approx \frac{2}{3} \cdot \frac{x^3 + y^3}{x^2 + y^2}.$$

Under these constraints, $\mu_{Q_i}(o)$ is maximized for
$x_0 = \Delta(Q)/\sqrt{3}$ and $y_0 = \left(1 - \frac{1}{\sqrt{3}}\right)\Delta(Q)$. With $n \to \infty$, we get

$$\mu_Q(o) \leq \max_i \mu_{Q_i}(o) \leq \frac{2}{3} \cdot \frac{x_0^3 + y_0^3}{x_0^2 + y_0^2} \frac{2(4 - \sqrt{3})}{13} \cdot \Delta(Q),$$

Q.E.D.

Further Directions

In some applications, the cost of serving a location q from a facility at point p is $\mathrm{dist}^\kappa(p,q)$ for some exponent $\kappa \geq 1$, rather than $\mathrm{dist}(p,q)$.

For a convex body $Q \subset \mathbb{R}^2$, let

$$\mu_Q^\kappa(p) = \frac{\int_{q\in Q} \mathrm{dist}^\kappa(p,q)\ \mathrm{d}q}{\mathrm{area}(Q)} \quad \text{and} \quad \mu_Q^{\kappa *} = \inf\{\mu_Q^\kappa(p) : p \in \mathbb{R}^2\}.$$

The proof of our lower bound carries over for this variant and shows that $\mu_Q^{\kappa *}/\Delta^\kappa(Q) > \frac{1}{(\kappa+2)2^\kappa}$ for any convex body $Q \subset \mathbb{R}^2$, and this bound is the best possible.

For the upper bound, the picture is not so clear:
$\mu_Q^*/\Delta(Q)$ is conjectured to be maximal for the circular disk.
However, there is a $\kappa \geq 1$ such that $\mu_Q^{\kappa *}/\Delta^\kappa(Q)$ cannot be maximal for the disk (e.g., a regular or a Reuleaux triangle).