

# Circle Packings and Circle Arrangements: Searching for Common Ground

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Workshop on Sphere Arrangements  
Fields Institute

November 2011



- Circle packing:

- Circle packing: *bringing geometry to combinatorics*



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- A brief topical tour

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- Summary





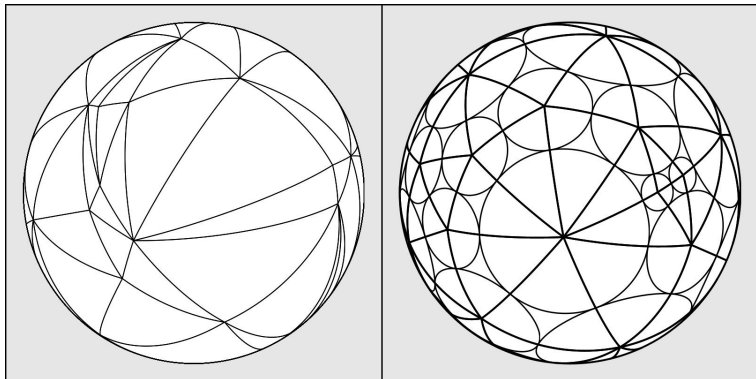
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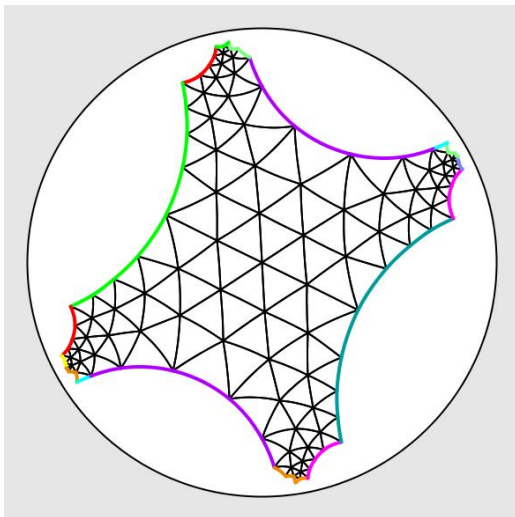
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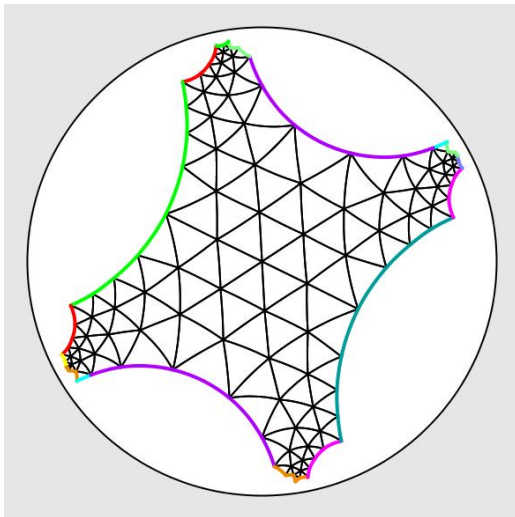
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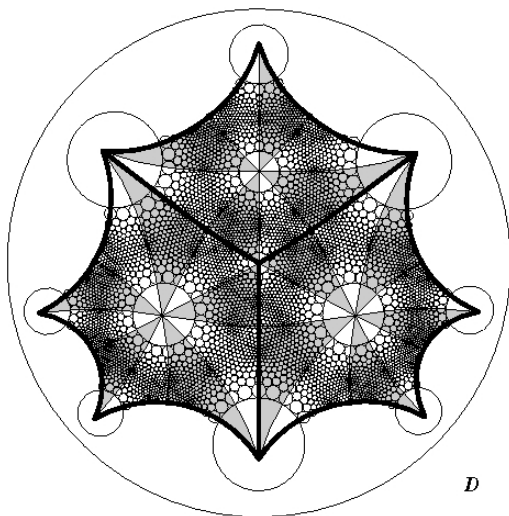
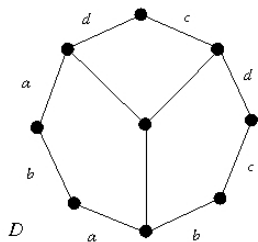
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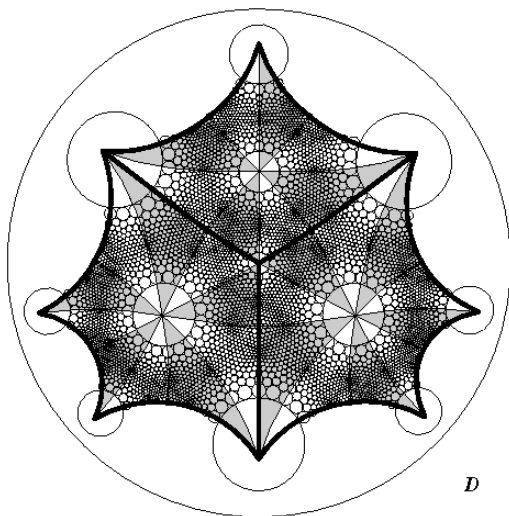
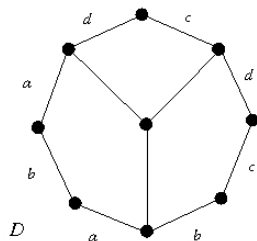




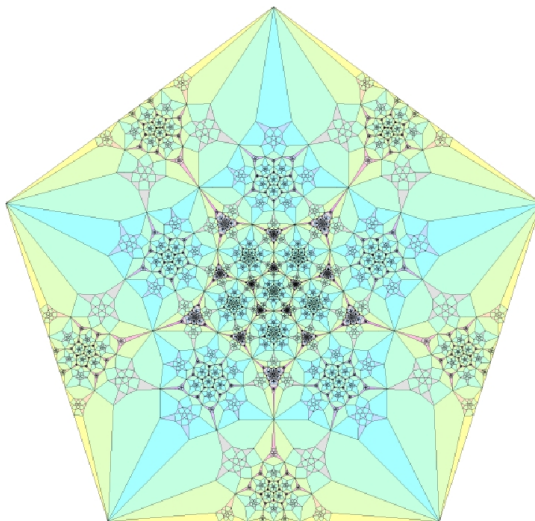


Pairs of pants

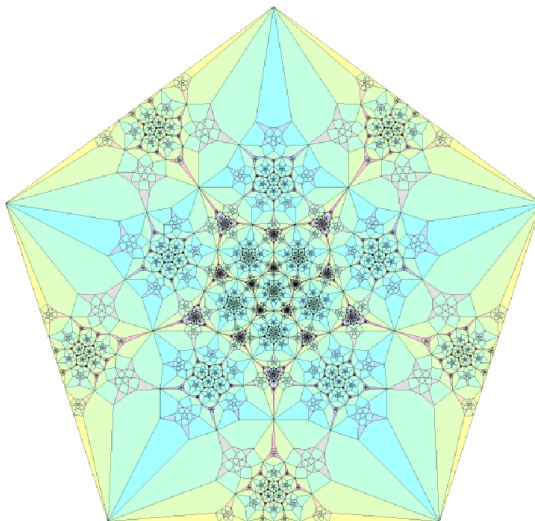




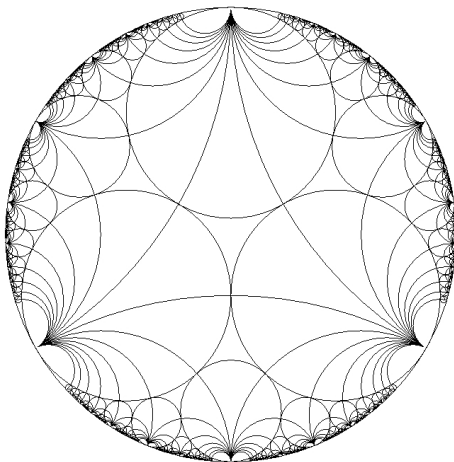
Discrete Gothendieck *Dessin d'Enfants*, Bowers and S.

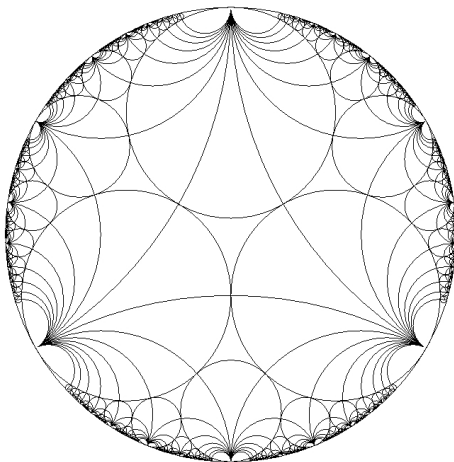




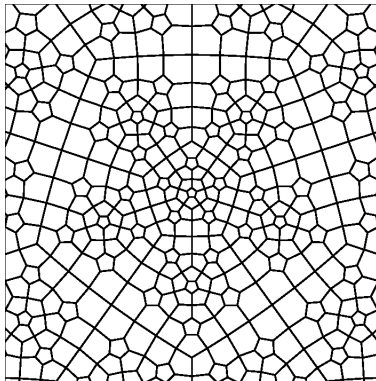


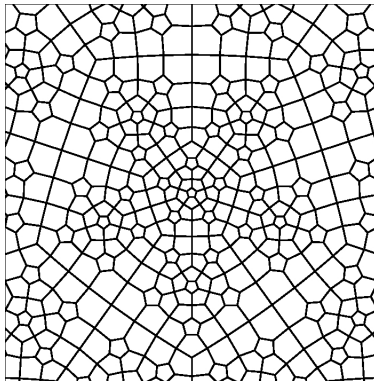
Knot complements, Brian Rushton



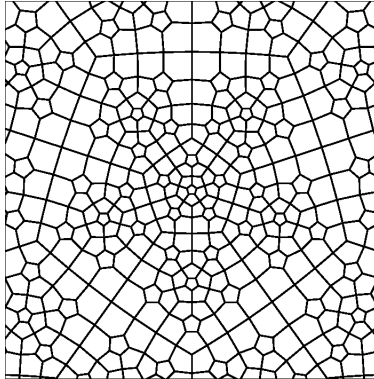


Farey triangulation

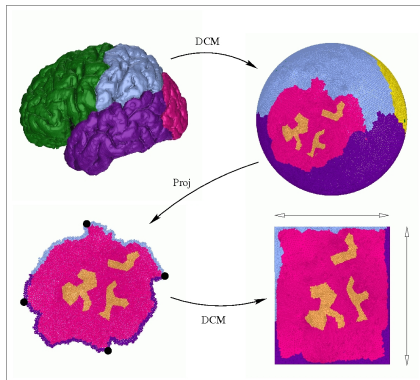


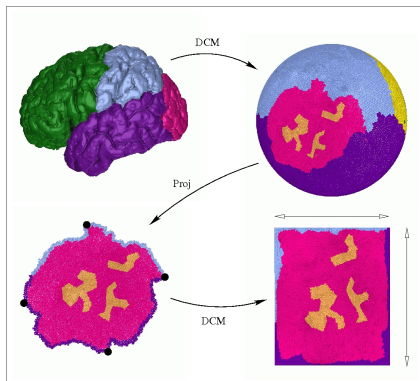


Subdivision rules, Cannon/Floyd/Parry



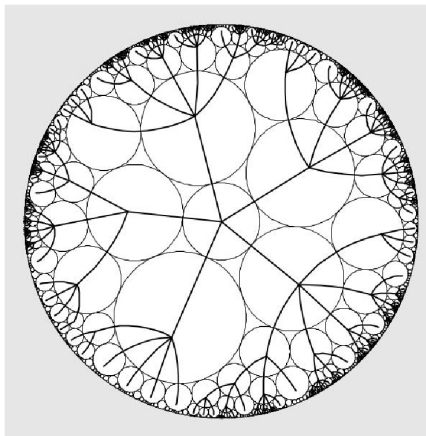
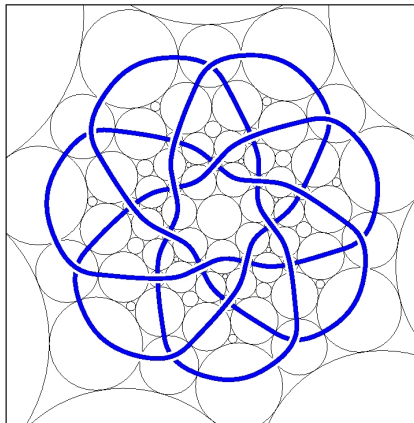
Subdivision rules, Cannon/Floyd/Parry  
Conformal tilings, Bowers and S.

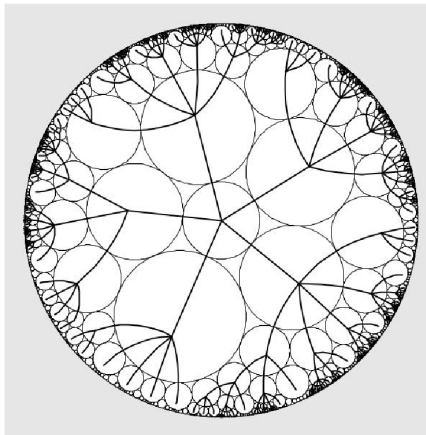
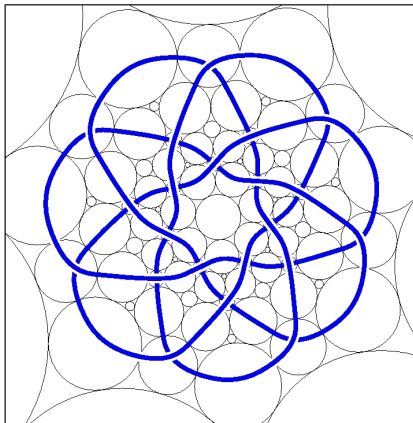




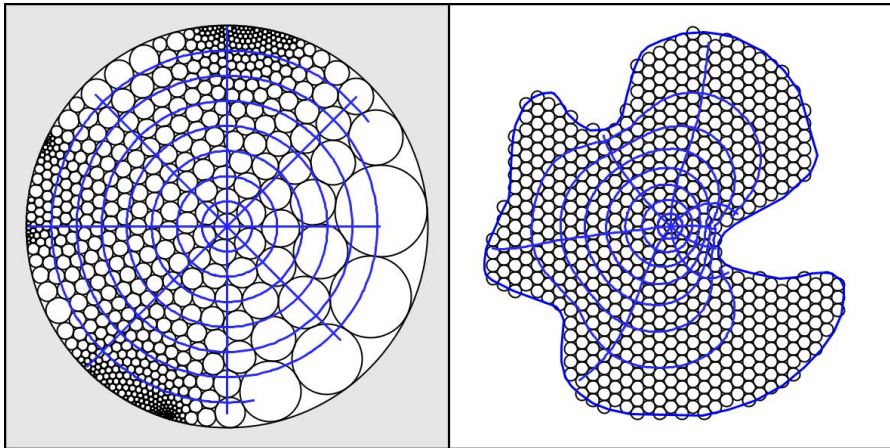
Brain imaging: Sumners, Rothenberg, Hurdal, Bowers, S., et al

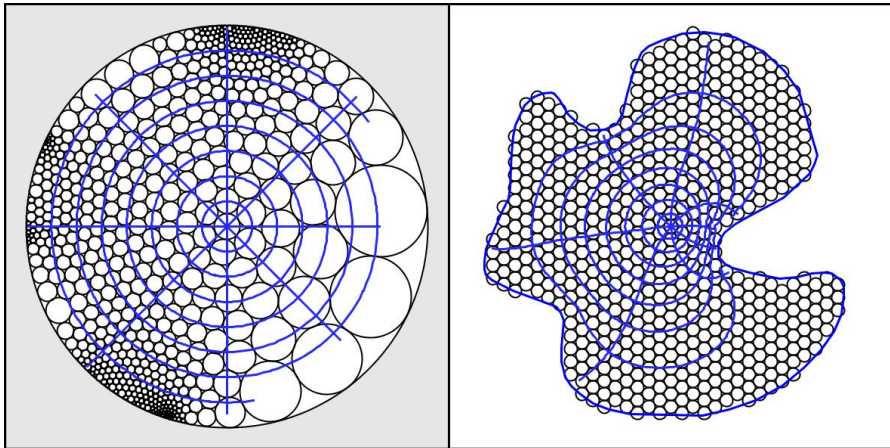




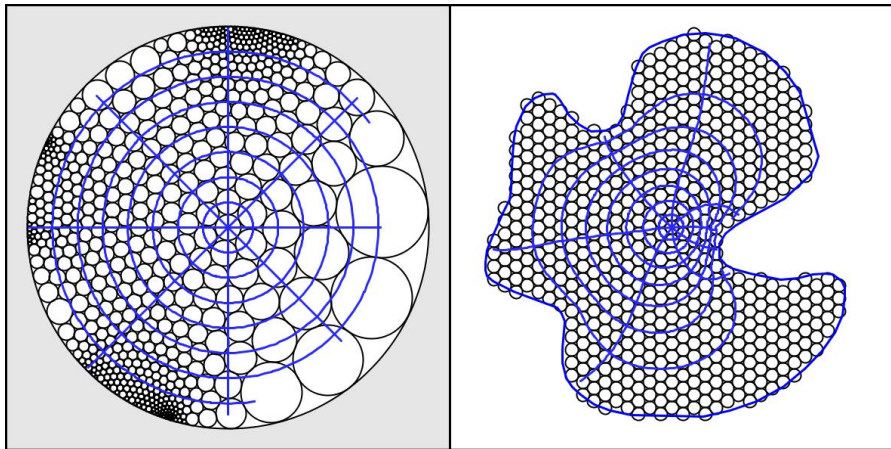


Knot embeddings, Morwen Thistlethwaite; tree embeddings

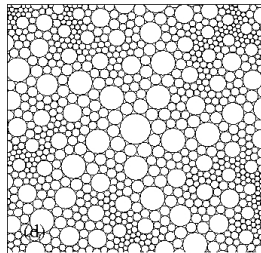
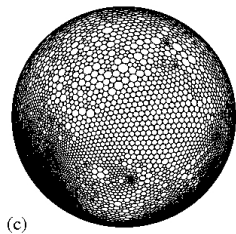
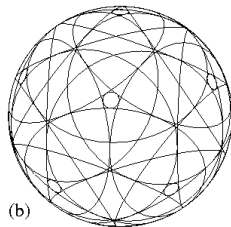
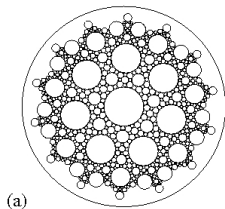


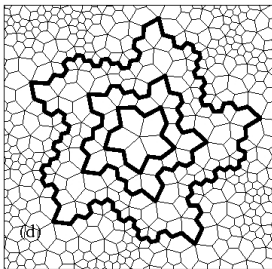
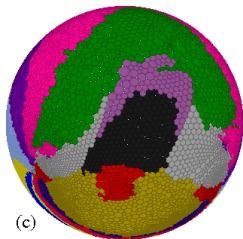
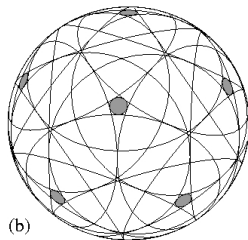
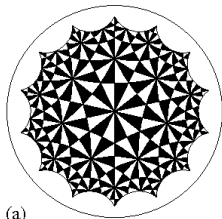


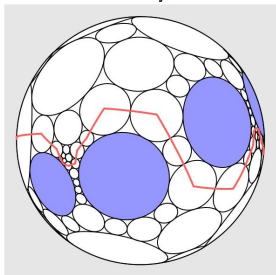
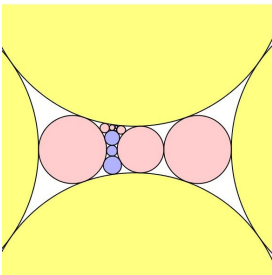
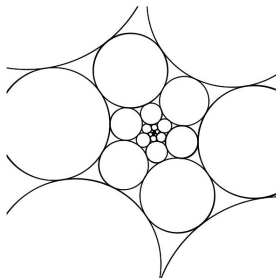
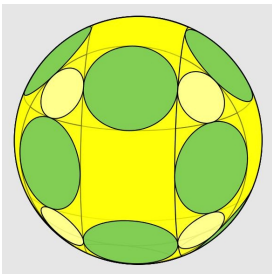
Conformal geometry



Conformal geometry—the main thrust of circle packing since Bill Thurston introduced it

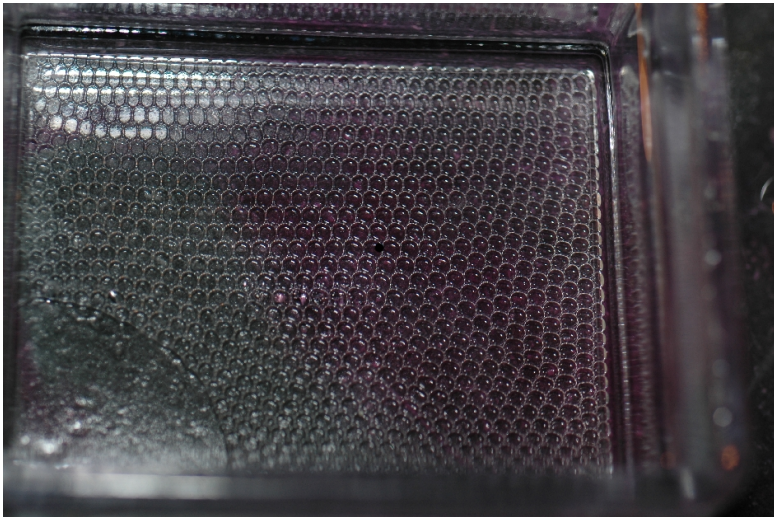


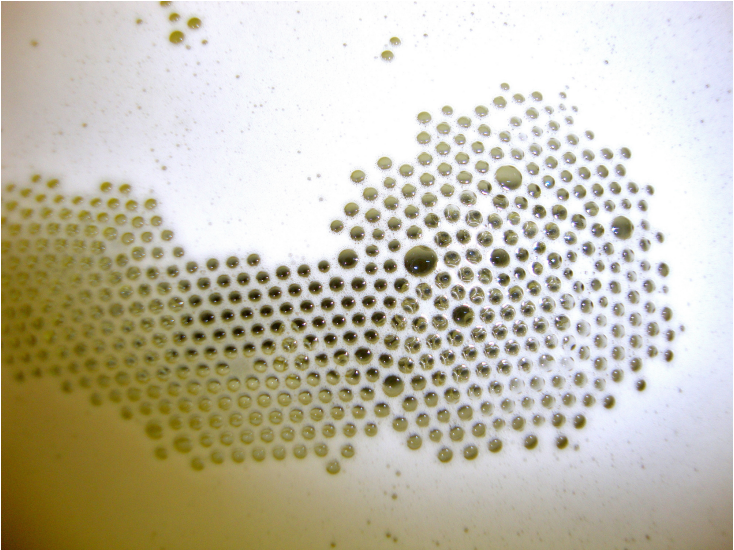


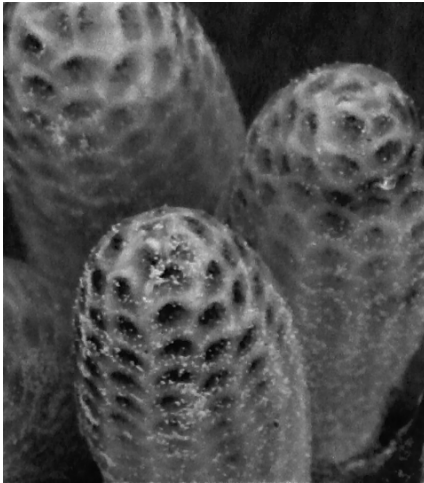






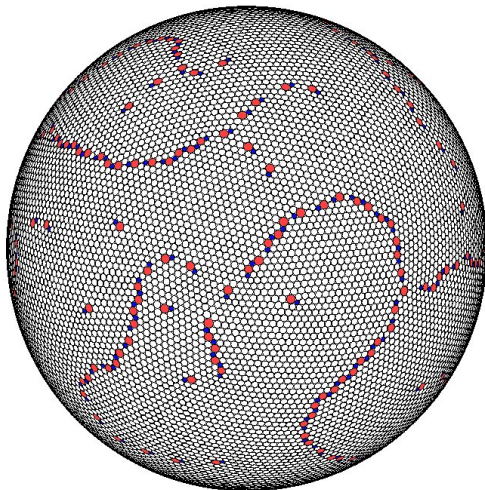


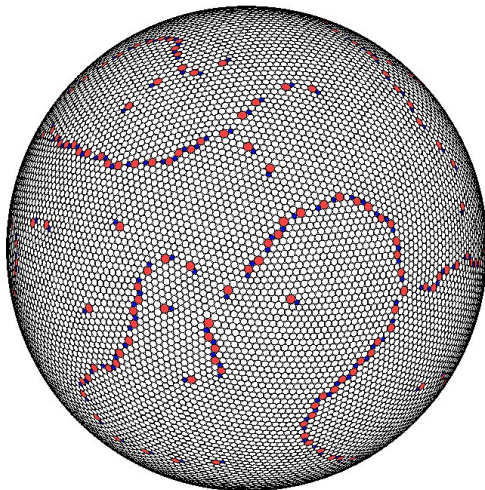






# Sphere Packing: Possibilities?





Point distributions





- Random combinatorics

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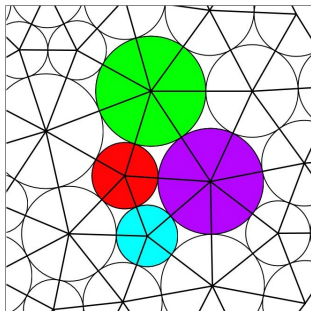
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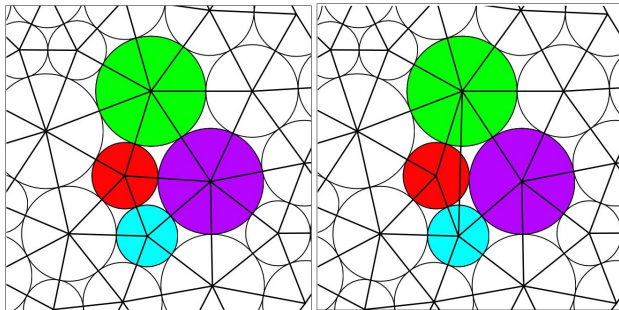
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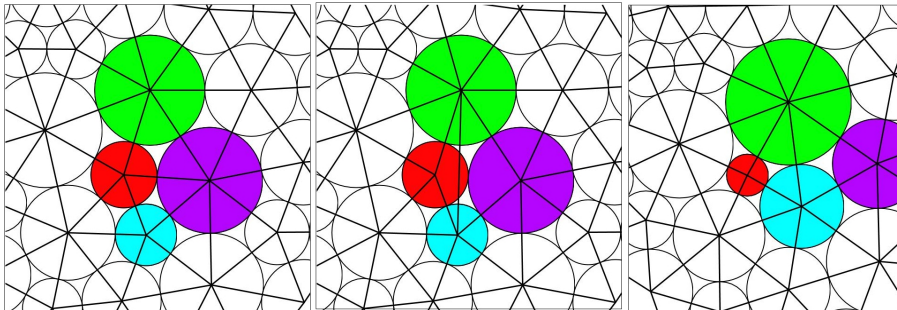
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Scripts

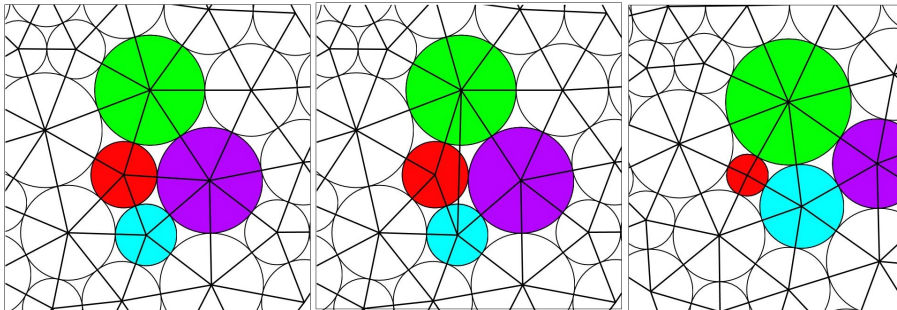








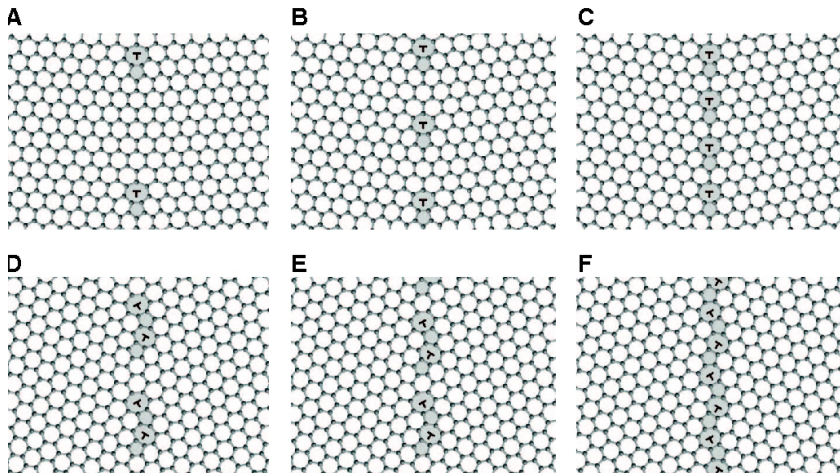




Scripts

GSR\_Examples.xmd

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**Fig. 1.** The structures of grain boundaries in (A to C) zigzag-oriented and (D to F) armchair-oriented graphene sheets with varying mismatch angles.



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- So, how does “circle” packing connect with “sphere” packing???



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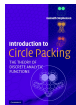


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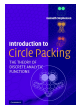
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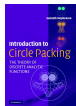
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