# Kneser-Poulsen conjecture for large radii

Igors Gorbovickis

Cornell University

November 17, 2011

# Kneser-Poulsen conjecture

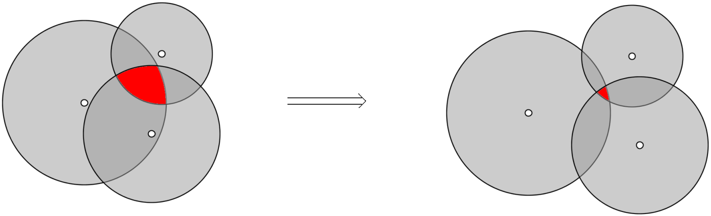
## Conjecture

*If a finite set of balls in* $\mathbb{E}^d$ *is rearranged so that the distance between each pair of centers does not decrease, then the volume of the union does not decrease, and the volume of the intersection does not increase.*

# Kneser-Poulsen conjecture

## Conjecture

*If a finite set of balls in $\mathbb{E}^d$ is rearranged so that the distance between each pair of centers does not decrease, then the volume of the union does not decrease, and the volume of the intersection does not increase.*



**Theorem (K.Bezdek, R.Connelly 2002)**

*The Kneser-Poulsen conjecture holds in the plane (when $d = 2$).*

# Kneser-Poulsen conjecture

**Theorem (K.Bezdek, R.Connelly 2002)**

*The Kneser-Poulsen conjecture holds in the plane (when $d = 2$).*

**Problem**

*Prove or disprove the conjecture when $d \geq 3$.*

# Notation

## Definition

Let $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$ and $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ be two configurations of $N$ points, where each $\mathbf{p}_i \in \mathbb{E}^d$ and each $\mathbf{q}_i \in \mathbb{E}^d$. If for all $1 \leq i < j \leq N$, $|\mathbf{p}_i - \mathbf{p}_j| \leq |\mathbf{q}_i - \mathbf{q}_j|$, we say that $\mathbf{q}$ is an **expansion** of $\mathbf{p}$.

# Notation

## Definition

Let $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ and $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ be two configurations of $N$ points, where each $\mathbf{p}_i \in \mathbb{E}^d$ and each $\mathbf{q}_i \in \mathbb{E}^d$. If for all $1 \leq i < j \leq N$, $|\mathbf{p}_i - \mathbf{p}_j| \leq |\mathbf{q}_i - \mathbf{q}_j|$, we say that $\mathbf{q}$ is an **expansion** of $\mathbf{p}$.

## Definition

We say that there exists a **continuous expansion** from a configuration of centers $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ to another configuration of centers $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$, if there exists a smooth motion $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_N(t))$, with $\mathbf{p}_i(t) \in \mathbb{E}^d$ for all $0 \leq t \leq 1$ such that $\mathbf{p}(0) = \mathbf{p}$ and $\mathbf{p}(1) = \mathbf{q}$, and for each pair of indices $(i, j)$ the distance $|\mathbf{p}_i(t) - \mathbf{p}_j(t)|$ is a non-decreasing function of $t$.

# Known results

The following theorems were proved with the assumption that there exists a continuous expansion between the initial and the final configurations of centers:

Theorem (Bollobás 1968)

*The KP Conjecture is true for the union of equal balls in* $\mathbb{E}^2$.

Theorem (Capoyleas 1996)

*The KP Conjecture is true for the intersection of equal balls in* $\mathbb{E}^2$.

Theorem (Bern, Sahai 1998)

*The KP Conjecture for nonequal balls is true in* $\mathbb{E}^2$.

Theorem (Csikós 1998)

*The KP Conjecture for nonequal balls is true in* $\mathbb{E}^d$.

# Known results

**Theorem (Gromov 1987)**

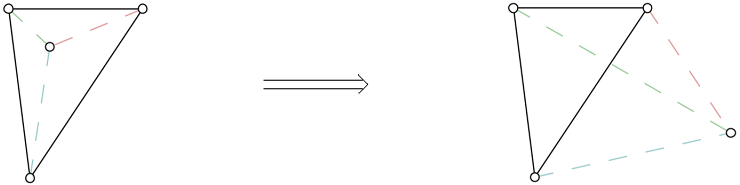
*The KP Conjecture for the intersection of $d+1$ balls is true in $\mathbb{E}^d$.*

**Theorem (Capoyleas, Pach 1991)**

*The KP Conjecture for the union of $d+1$ balls is true in $\mathbb{E}^d$.*

# Expansion that is not continuous

There are examples of expansions that are not continuous, when $N \geq d+2$.



# Continuous expansion in $\mathbb{E}^{2d}$

Suppose that $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ is an expansion of a point configuration $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$ in $\mathbb{E}^d$. We regard $\mathbb{E}^d$ as the subset $\mathbb{E}^d = \mathbb{E}^d \times \{0\} \subset \mathbb{E}^d \times \mathbb{E}^d = \mathbb{E}^{2d}$. Then the motion $\mathbf{p}(t)$ defined in $\mathbb{E}^{2d}$ by the formula

$$\mathbf{p}_i(t) = \left( \frac{\mathbf{p}_i + \mathbf{q}_i}{2} + (\cos \pi t) \frac{\mathbf{p}_i - \mathbf{q}_i}{2}, (\sin \pi t) \frac{\mathbf{p}_i - \mathbf{q}_i}{2} \right), \quad 1 \le i \le N$$

is a continuous expansion from $\mathbf{p} = \mathbf{p}(0)$ to $\mathbf{q} = \mathbf{p}(1)$.

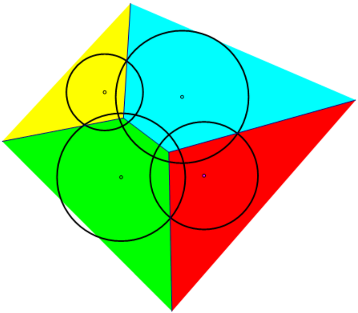
# Steps of the proof of KP conjecture for $d = 2$

1. Consider a continuous expansion of the centers in $\mathbb{E}^{d+2}$. (Its existence is guaranteed when $d = 2$)
2. Show that the "weighted surface volume" of the union (intersection) of $(d + 2)$-dimensional balls is weakly increasing (decreasing).
3. Show that when the centers lie in a $d$-dimensional subspace (beginning and end configurations), the "weighted surface volume" is equal to the volume of the $d$-dimensional ball configuration.

# Voronoi regions

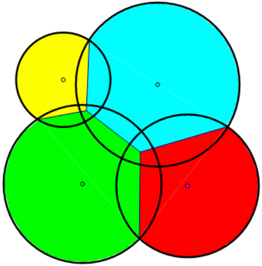
Given a configuration of centers $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N) \subset \mathbb{E}^d$ and corresponding radii $\mathbf{r} = (r_1, \ldots, r_N)$, we can define the **(generalized) Voronoi regions**:

$$C_i(\mathbf{p}, \mathbf{r}) = \{\mathbf{p}_0 \in \mathbb{E}^d \mid \text{for all } j, |\mathbf{p}_0 - \mathbf{p}_i|^2 - r_i^2 \leq |\mathbf{p}_0 - \mathbf{p}_j|^2 - r_j^2\}.$$



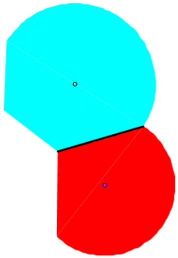
# Truncated Voronoi regions

For each point $\mathbf{p}_i \in \mathbb{E}^d$ we define the **truncated Voronoi region** $\hat{C}_i(\mathbf{p}, \mathbf{r}) = C_i(\mathbf{p}, \mathbf{r}) \cap B_d(\mathbf{p}_i, r_i)$.



# Walls between truncated Voronoi regions

For each pair of truncated Voronoi regions $\hat{C}_i(\mathbf{p}, \mathbf{r})$, $\hat{C}_j(\mathbf{p}, \mathbf{r})$ define the wall $W_{ij}(\mathbf{p}, \mathbf{r}) = \hat{C}_i(\mathbf{p}, \mathbf{r}) \cap \hat{C}_j(\mathbf{p}, \mathbf{r})$.



# The change in the volume

Suppose that $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_N(t))$ is a smooth motion of a configuration of centers in $\mathbb{E}^d$. Let $d_{ij} = |\mathbf{p}_i(t) - \mathbf{p}_j(t)|$, and let $d'_{ij}$ be the $t$-derivative of $d_{ij}$.

**Theorem (Csikós 1998)**

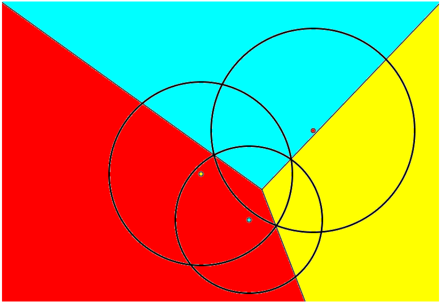
*Let $d \geq 2$ and let $\mathbf{p}(t)$ be a smooth motion of centers in $\mathbb{E}^d$ such that for each $t$, the points of the configuration are pairwise distinct. Then regarding the following as a function of $t$, $V_d(\mathbf{p}(t), \mathbf{r}) = \mathrm{Vol}_d[\cup_{i=1}^N B_d(\mathbf{p}_i(t), r_i)]$ is differentiable and,*

$$\frac{d}{dt} V_d(\mathbf{p}(t), \mathbf{r}) = \sum_{1 \leq i < j \leq N} d'_{ij} \mathrm{Vol}_{d-1}[W_{ij}(\mathbf{p}(t), \mathbf{r})]$$

# Voronoi regions

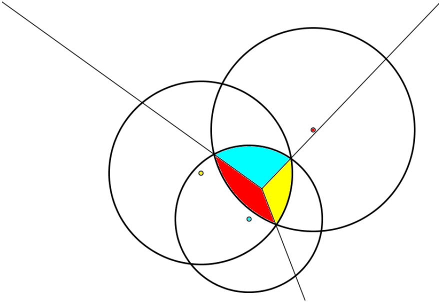
Given a configuration of centers $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N) \subset \mathbb{E}^d$ and corresponding radii $\mathbf{r} = (r_1, \dots, r_N)$, we can define the **(generalized) farthest point Voronoi regions**:

$$C^i(\mathbf{p}, \mathbf{r}) = \{\mathbf{p}_0 \in \mathbb{E}^d \mid \text{for all } j, |\mathbf{p}_0 - \mathbf{p}_i|^2 - r_i^2 \geq |\mathbf{p}_0 - \mathbf{p}_j|^2 - r_j^2\}.$$



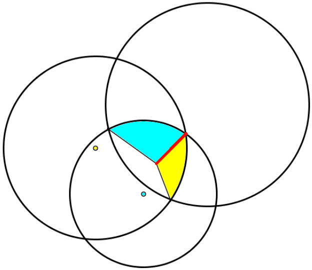
# Truncated Voronoi regions

For each point $\mathbf{p}_i \in \mathbb{E}^d$ we define the **truncated farthest point Voronoi region** $\hat{C}^i(\mathbf{p}, \mathbf{r}) = C^i(\mathbf{p}, \mathbf{r}) \cap B_d(\mathbf{p}_i, r_i)$.



# Walls between truncated Voronoi regions

For each pair of truncated Voronoi regions $\hat{C}^i(\mathbf{p}, \mathbf{r})$, $\hat{C}^j(\mathbf{p}, \mathbf{r})$ define the wall $W^{ij}(\mathbf{p}, \mathbf{r}) = \hat{C}^i(\mathbf{p}, \mathbf{r}) \cap \hat{C}^j(\mathbf{p}, \mathbf{r})$.



# The change in the volume

Then the function $V^d(\mathbf{p}(t), \mathbf{r}) = \mathrm{Vol}_d[\cap_{i=1}^N B_d(\mathbf{p}_i(t), r_i)]$ is differentiable with respect to $t$ and,

$$\frac{d}{dt} V^d(\mathbf{p}(t), \mathbf{r}) = \sum_{1 \le i < j \le N} -d'_{ij} \mathrm{Vol}_{d-1}[W^{ij}(\mathbf{p}(t), \mathbf{r})].$$

# Kneser-Poulsen for large radii

### Theorem (I.G.)

*If* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ *is an expansion of* $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ *in* $\mathbb{E}^d$*, then there exists* $r_0 > 0$ *such that for any* $r \geq r_0$

$$\mathrm{Vol}_d \left[ \bigcup_{i=1}^{N} B_d(\mathbf{p}_i, r) \right] \leq \mathrm{Vol}_d \left[ \bigcup_{i=1}^{N} B_d(\mathbf{q}_i, r) \right],$$

*and*

$$\mathrm{Vol}_d \left[ \bigcap_{i=1}^{N} B_d(\mathbf{p}_i, r) \right] \geq \mathrm{Vol}_d \left[ \bigcap_{i=1}^{N} B_d(\mathbf{q}_i, r) \right],$$

*and if the point configurations* $\mathbf{q}$ *and* $\mathbf{p}$ *are not congruent, then the inequalities are strict.*

# Monotonicity of the mean width

**Theorem (Sudakov, Alexander, Capoyleas, Pach)**

*Let* $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ *be an expansion of* $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$ *in* $\mathbb{E}^d$*,* $d \geq 2$*. Then*

$$M_d[\mathrm{Conv}\{\mathbf{q}_1, \dots, \mathbf{q}_N\}] \geq M_d[\mathrm{Conv}\{\mathbf{p}_1, \dots, \mathbf{p}_N\}], \qquad (1)$$

*where* $\mathrm{Conv}$ *stands for the convex hull, and* $M_d$ *is the mean width or the total mean curvature of a compact convex set in* $\mathbb{E}^d$*.*

# Monotonicity of the mean width

**Theorem (Sudakov, Alexander, Capoyleas, Pach)**

*Let* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ *be an expansion of* $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ *in* $\mathbb{E}^d$*,* $d \geq 2$*. Then*

$$M_d[\mathrm{Conv}\{\mathbf{q}_1, \ldots, \mathbf{q}_N\}] \geq M_d[\mathrm{Conv}\{\mathbf{p}_1, \ldots, \mathbf{p}_N\}], \tag{1}$$

*where* $\mathrm{Conv}$ *stands for the convex hull, and* $M_d$ *is the mean width or the total mean curvature of a compact convex set in* $\mathbb{E}^d$*.*

**Theorem (I.G.)**

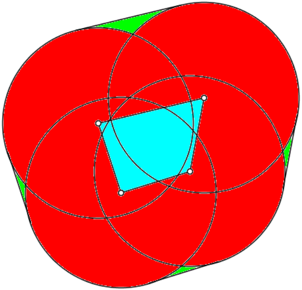
*If in the above theorem the point configurations* $\mathbf{p}$ *and* $\mathbf{q}$ *are not congruent, then the inequality (1) is strict.*

# Why the mean width helps

Lemma (Capoyleas, Pach)

$$\mathrm{Vol}_d\left[\bigcup_{i=1}^{N} B_d(\mathbf{p}_i, r)\right] = \delta_d r^d + M_d[\mathrm{Conv}\{\mathbf{p}_1, \ldots, \mathbf{p}_N\}] r^{d-1} + o(r^{d-1}),$$

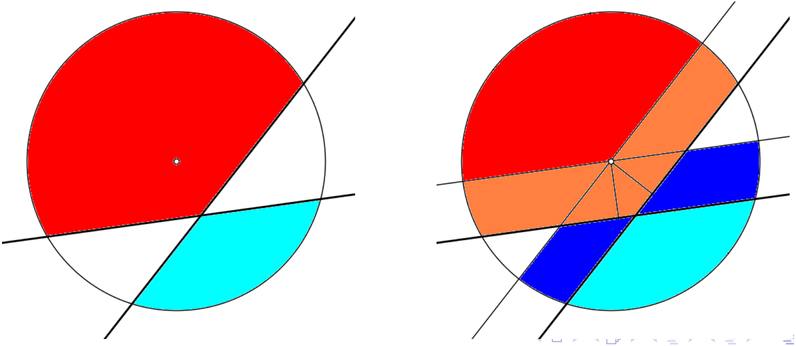
*where* $\delta_d = \mathrm{Vol}_d[B_d(0,1)]$. *(Compare with Steiner's formula!)*



# Mean width and the volume of the intersection

## Lemma (I.G.)

$$\mathrm{Vol}_d\left[\bigcap_{i=1}^{N} B_d(\mathbf{p}_i, r)\right] = \delta_d r^d - M_d[\mathrm{Conv}\{\mathbf{p}_1, \ldots, \mathbf{p}_N\}]r^{d-1} + o(r^{d-1}),$$



## Strong inequality for the mean width

1. Consider a continuous expansion $\mathbf{p}(t)$ in a higher dimensional space $\mathbb{E}^n$
2. Use Schläfli's formula

$$\frac{d}{dt} M_n[\mathrm{Conv}\{\mathbf{p}_1(t), \dots, \mathbf{p}_N(t)\}] = c_n \sum_{(i,j)\in E(t)} \beta_{ij}(t) d'_{ij}(t),$$

   where $E(t)$ is the set of edges of the convex hull, and $\beta_{ij}(t)$ is a curvature of the edge.
3. $\beta_{ij}(t) > 0$ for all edges of the convex hull, so it is enough to show that at least for some $t$ the length of at least one edge of the convex hull has a positive derivative. The last statement follows from the rigidity result of Whiteley (modulo technical details).

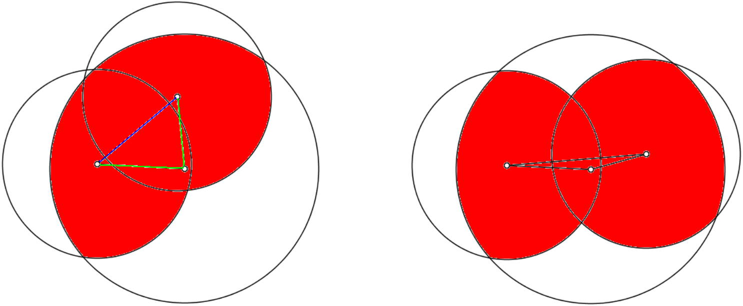
# Strong inequality for the mean width

Finally, if $K \subset \mathbb{E}^d \subset \mathbb{E}^n$, then

$$M_n[K] = cM_d[K].$$

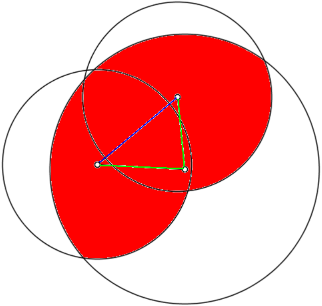
# Remark about flowers

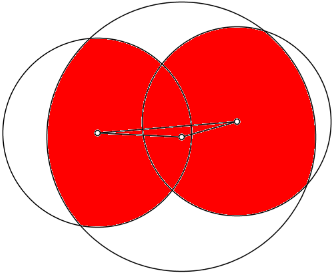
Flowers were introduced by Gordon and Meyer.



# Remark about flowers

Flowers were introduced by Gordon and Meyer.





### Question

*Is it possible to generalize the large radii result for flowers?*

# Monotonicity of the surface volume

## Theorem (I.G.)

*If* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ *is an expansion of* $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ *in* $\mathbb{E}^d$*, for* $d \geq 2$*, then there exists* $r_0 > 0$ *such that for any* $r \geq r_0$

$$\mathrm{Vol}_{d-1}\left[\mathrm{Bdy}\left[\bigcup_{i=1}^{N} B_d(\mathbf{p}_i, r)\right]\right] \leq \mathrm{Vol}_{d-1}\left[\mathrm{Bdy}\left[\bigcup_{i=1}^{N} B_d(\mathbf{q}_i, r)\right]\right],$$

*and*

$$\mathrm{Vol}_{d-1}\left[\mathrm{Bdy}\left[\bigcap_{i=1}^{N} B_d(\mathbf{p}_i, r)\right]\right] \geq \mathrm{Vol}_{d-1}\left[\mathrm{Bdy}\left[\bigcap_{i=1}^{N} B_d(\mathbf{q}_i, r)\right]\right],$$

*and if the point configurations* $\mathbf{q}$ *and* $\mathbf{p}$ *are not congruent, then the inequalities are strict.*