On the Volume of the Union and Intersection of Random Balls

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Workshop on Sphere Arrengements
November 14-18, 2011
Fields Institute, Toronto

Kneser–Poulsen Conjecture (1954-55)

Suppose that for the points $\mathbf{p}_1, \ldots, \mathbf{p}_N$ *and* $\mathbf{q}_1, \ldots, \mathbf{q}_N$ *in* $\mathbb{E}^n$,

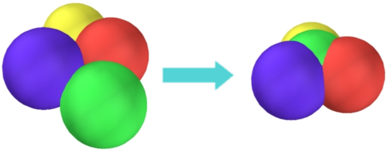
$$d(\mathbf{p}_i, \mathbf{p}_j) \geq d(\mathbf{q}_i, \mathbf{q}_j) \quad \forall\ 1 \leq i < j \leq N.$$

Then we say that the system $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ *is an expansion of the system* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$, *or that the system* $\mathbf{q}$ *is a contraction of the system* $\mathbf{p}$.

Do these inequalities imply the inequality

$$\mathrm{Vol}_n\left(\bigcup_{i=1}^{N} B(\mathbf{p}_i, r_i)\right) \geq \mathrm{Vol}_n\left(\bigcup_{i=1}^{N} B(\mathbf{q}_i, r_i)\right) \qquad (\cup)$$

for any $r_1 > 0, \ldots, r_N > 0$*?*



Remarks

- The conjecture was originally formulated for equal radii, but is seems to be true with different radii.
- The known special cases suggest that the conjecture is true in the spaces $\mathbb{H}^n$ and $\mathbb{S}^n$ as well.
- There is an analogous conjecture for intersections

$$\mathrm{Vol}_n\left(\bigcap_{i=1}^{N} B(\mathbf{p}_i, r_i)\right) \leq \mathrm{Vol}_n\left(\bigcap_{i=1}^{N} B(\mathbf{q}_i, r_i)\right), \qquad (\cap)$$

and also for bodies constructed from the balls with mixed $\cup$ and $\cap$ operations.

Most important results

Theorem (K. Bezdek, R. Connelly for $M^n = \mathbb{E}^n$, B. Cs. for $M^n = \mathbb{S}^n$ or $\mathbb{H}^n$)

If there exist continuous curves $\gamma_i \colon [0,1] \to M^{n+2}$ such that

- $\gamma_i(0) = \mathbf{p}_i \in M^n$,
- $\gamma_i(1) = \mathbf{q}_i \in M^n$, *and*
- $d(\gamma_i(t), \gamma_j(t))$ *is a decreasing function of* t *for all* $1 \leq i, j, \leq N$,

then the K-P inequalities $(\cup)$ *and* $(\cap)$ *hold in* M^n.

Leapfrog Lemma

- *If* $\mathbf{p} \in (\mathbb{E}^n)^N$ *is an expansion of* $\mathbf{q} \in (\mathbb{E}^n)^N$, *then there is a continuous expansion of* $\mathbf{q}$ *to* $\mathbf{p}$ *in* $\mathbb{E}^{2n}$.
- *If* $\mathbf{p} \in (\mathbb{S}^n)^N$ *is an expansion of* $\mathbf{q} \in (\mathbb{S}^n)^N$, *then there is a continuous expansion of* $\mathbf{q}$ *to* $\mathbf{p}$ *in* $\mathbb{S}^{2n+1}$.
- *No leapfrog lemma is known in the hyperbolic space.*

Corollary

The K-P conjecture is true in the Euclidean plane for both $\cup$ *and* $\cap$.

Fuzzy version of the K-P conjecture

Definition

A fuzzy subset of a set A is a map $m\colon A \to [0,1]$. For $x \in A$, $m(x)$ is the grade of membership of x in the subset,m is the membership function of the subset.

Definition

The volume of a fuzzy subset of $\mathbb{E}^n$ with measurable membership function m is $\int_{\mathbb{E}^n} m(\mathbf{x}) d\mathbf{x}$.

Definition

A fuzzy ball in $\mathbb{E}^n$ centered at $\mathbf{p}$ is a fuzzy subset of $\mathbb{E}^n$ whose membership function is of the form $m(\mathbf{x}) = f(||\mathbf{x} - \mathbf{p}||)$, where $f\colon \mathbb{R}_+ \to [0,1]$ is a decreasing function.

What is the union and intersection of fuzzy sets?

Axioms for fuzzy intersection

An intersection operation on fuzzy subsets m_1, m_2 of a set is specified by binary operation

$$I : [0,1] \times [0,1] \to [0,1]$$

on the unit interval. Given I the corresponding operation is

$$\big(m_1 \underset{I}{\cap} m_2\big)(x) = I(m_1(x), m_2(x)).$$

I defines a fuzzy intersection if the following axioms are fulfilled:

Axiom i1. Boundary condition: $I(a,1) = a$;
Axiom i2. Monotonicity: $b \leq d$ implies $I(a,b) \leq I(a,d)$;
Axiom i3. Commutativity: $I(a,b) = I(b,a)$;
Axiom i4. Associativity: $I(a, I(b,d)) = I(I(a,b),d)$;
Axiom i5. Continuity: I is a continuous function;
Axiom i6. Subidempotency: $I(a,a) \leq a$.

Axioms for fuzzy union

Similarly, a union operation on fuzzy subsets of a set is specified by binary operation

$$U : [0,1] \times [0,1] \to [0,1]$$

on the unit interval. satisfying the axioms for fuzzy union:

Axiom u1. Boundary condition: $U(a,0) = a$;
Axiom u2. Monotonicity: $a \leq b$ implies $U(a,c) \leq U(b,c)$;
Axiom u3. Commutativity: $U(a,b) = U(b,a)$;
Axiom u4. Associativity: $U(a, U(b,c)) = U(U(a,b),c)$;
Axiom u5. Continuity: U is a continuous function;
Axiom u6. Superidempotency: $U(a,a) \geq a$;
Axiom u7. Strict monotonicity: $a_1 < a_2$ and $b_1 < b_2$ implies $U(a_1,b_1) < U(a_2,b_2)$.

Observation

- I defines a fuzzy intersection $\iff$ $U(a,b) = 1 - I(1-a, 1-b)$ satisfies axioms u1–u6.
- Axiom u7 is fulfilled for U $\iff$ I is strictly monotonous.

Examples of fuzzy intersections

- Zadeh's intersection: $I(a,b) = \min\{a,b\}$, $U(a,b) = \max\{a,b\}$.
- Probabilistic intersection: $I(a,b) = ab$, $U(a,b) = a + b - ab$.
- Bold intersection: $I(a,b) = \max\{0, a+b-1\}$ (not strictly monotonous!)
- Drastic intersection: $I(a,b) = \begin{cases} \min\{a,b\} & \text{if } \max\{a,b\} = 1 \\ 0 & \text{otherwise.} \end{cases}$ (not continuous, not strictly monotonous!)
- Hamacher's intersection: $I(a,b) = \dfrac{ab}{\gamma + (1-\gamma)(a+b-ab)}$, where $\gamma \in [0,\infty)$.
- Yager's intersection: $I(a,b) = 1 - \min\{1, ((1-a)^w + (1-b)^w)^{1/w}\}$, where $w \in (0,\infty)$.
- Dombi's intersection: $I(a,b) = \{1 + [(1/a - 1)^w + (1/b - 1)^w]^{1/w}\}^{-1}$, where $w \in (0,\infty)$. $U(a,b) = \{1 + [(1/a - 1)^{-w} + (1/b - 1)^{-w}]^{-1/w}\}^{-1}$

- For a fuzzy intersection I set

$$I_N(a_1, \dots, a_N) = I(a_1, I(a_2, \dots, I(a_{N-1}, a_N) \dots)).$$

- Define U_N for a fuzzy union U similarly.
- One can also define other aggregated fuzzy operations $H : [0,1]^N \to [0,1]$ built up from a combination of fuzzy intersections and unions of N fuzzy sets.

Question

Suppose the system $\mathbf{p} \in (E^n)^N$ is an expansion of the system $\mathbf{q} \in (E^n)^N$. Is it true that for any measurable decreasing "fuzzy radii" $f_i : \mathbb{R}_+ \to [0,1]$ and any fuzzy intersection I or fuzzy union U

$$\int_{\mathbb{R}^n} I_N(f_1(||\mathbf{x} - \mathbf{p}_1||), \dots, f_N(||\mathbf{x} - \mathbf{p}_N||))d\mathbf{x} \leq \int_{\mathbb{R}^n} I_N(f_1(||\mathbf{x} - \mathbf{q}_1||), \dots, f_N(||\mathbf{x} - \mathbf{q}_N||))d\mathbf{x}$$

and

$$\int_{\mathbb{R}^n} U_N(f_1(||\mathbf{x} - \mathbf{p}_1||), \dots, f_N(||\mathbf{x} - \mathbf{p}_N||))d\mathbf{x} \geq \int_{\mathbb{R}^n} U_N(f_1(||\mathbf{x} - \mathbf{q}_1||), \dots, f_N(||\mathbf{x} - \mathbf{q}_N||))d\mathbf{x}?$$

Theorem

Suppose that

- I *is a smooth fuzzy intersection satisfying* $I(0,a) = a$*,*
- $f_i\colon \mathbb{R}_+ \to [0,1]$*,* $1 \le i \le N$ *are smooth decreasing functions with* $\lim_{t\to\infty} f_i(t) = 0$*.*

Then

$$\int_{\mathbb{R}^n} I_N(f_1(||\mathbf{x}-\mathbf{p}_1||),\ldots,f_N(||\mathbf{x}-\mathbf{p}_N||))d\mathbf{x}$$

$$= \int_{\mathbb{R}^n_+} \partial_1 \ldots \partial_N I_N(f_1(r_1),\ldots,f_N(r_N)) \prod_{i=1}^{N}(-f_i'(r_i)) \cdot$$

$$\cdot\ Vol\left(\bigcap_{i=1}^{N}(B(\mathbf{p}_i, r_i)\right) d\mathbf{r}$$

Observation

If $\partial_1 \ldots \partial_N I_N \ge 0$, then the K-P inequality for the intersections of "crisp" balls implies the inequality for the fuzzy intersection of fuzzy balls.

In this case, $\partial_1 \dots \partial_N I_N \geq 0$ is a probability density on the unit cube and if $X = (X_1, \dots, X_N)$ is a random vector chosen with this density, then I_N is the CDF of X

$$I_N(x_1, \dots, x_N) = P(X_1 \leq x_1, \dots, X_N \leq x_N). \qquad (*)$$

Theorem

Suppose that

- *I is a smooth fuzzy intersection satisfying $I(0, a) = a$,*
- *there is a probability density on $[0,1]^N$ for which $(*)$ holds.*
- *$f_i \colon \mathbb{R}_+ \to [0,1]$, $1 \leq i \leq N$ are decreasing functions such that the fuzzy balls $\mathcal{B}_i(\mathbf{x}) = f_i(||\mathbf{x} - \mathbf{p}_i||)$ have finite volume.*

Then

$$\int_{\mathbb{R}^n} I_N(\mathcal{B}_1(\mathbf{x}), \dots, \mathcal{B}_N(\mathbf{x})) d\mathbf{x} = E\Big(\mathrm{Vol}\big(\bigcap_{i=1}^{N} B(\mathbf{p}_i, g_i^{max}(X_i))\big)\Big),$$

$$\int_{\mathbb{R}^n} U_N(\mathcal{B}_1(\mathbf{x}), \dots, \mathcal{B}_N(\mathbf{x})) d\mathbf{x} = E\Big(\mathrm{Vol}\big(\bigcup_{i=1}^{N} B(\mathbf{p}_i, g_i^{min}(1 - X_i))\big)\Big),$$

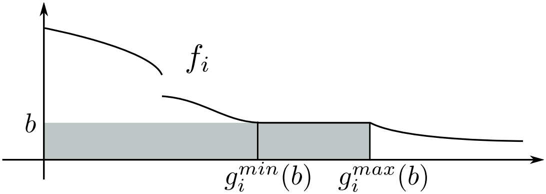
where $g_i^{max}(b) := \sup\{a : f(a) \geq b\}$ and $g_i^{min}(b) := \inf\{a : f(a) \leq b\}$.

... *Then*

$$\int_{\mathbb{R}^n} I_N(\mathcal{B}_1(\mathbf{x}), \ldots, \mathcal{B}_N(\mathbf{x}))d\mathbf{x} = E\Big(\mathrm{Vol}(\bigcap_{i=1}^{N} B(\mathbf{p}_i, g_i^{max}(X_i))\Big),$$

$$\int_{\mathbb{R}^n} U_N(\mathcal{B}1(\mathbf{x}), \ldots, \mathcal{B}_N(\mathbf{x}))d\mathbf{x} = E\Big(\mathrm{Vol}(\bigcup_{i=1}^{N} B(\mathbf{p}_i, g_i^{min}(1 - X_i))\Big),$$

where $g_i^{max}(b) := \sup\{a : f(a) \geq b\}$ *and* $g_i^{min}(b) := \inf\{a : f(a) \leq b\}$.



Corollary

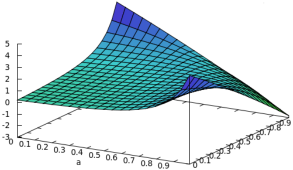
If I_N *is the CDF of a random vector distribution* $(X_1, \ldots, X_N) \in [0, 1]^N$, *then the K-P inequalities* $(\cap)$ *and* $(\cup)$ *for intersections and unions of "crisp" balls imply analogous inequalities for fuzzy balls with the same centers.*

Examples

- For the uniform distribution on $[0,1]^N$,
  $P(X_1 \leq x_1, \dots, X_N \leq x_n) = x_1 \cdots x_N$ is the probabilistic intersection.
  If X is a uniformly distributed random variable in $[0,1]$, then $(X_1, \dots, X_n) = (X, \dots, X)$ is a random vector uniformly distributed on the main diagonal of $[0,1]^N$.
  $P(X_1 \leq x_1, \dots, X_N \leq x_n) = \min\{x_1, \dots, x_N\}$ is Zadeh's min intersection.
- If $(X_1, \dots, X_N)$ is uniformly distributed on the simplex spanned by the vertices of $[0,1]^N$ neighbouring $(1, \dots, 1)$, then

  $$P(X_1 \leq x_1, \dots, X_N \leq x_n) = \min\{0, x_1 + x_2 + \cdots + x_N - (N-1)\}$$

  is the bold intersection.
- For Hamacher's intersection: $I(a,b) = \dfrac{ab}{\gamma + (1-\gamma)(a+b-ab)}$, where $\gamma \in [0, \infty)$, the candidate $\partial_1 \partial_2 I$ has negative values for $\gamma > 2$.



Why should we consider fuzzy/probabilistic versions?

- They are corollaries of the "crisp" conjecture, so they might be easier to prove.
- If we can prove the probabilistic version for a sufficiently large family of distributions, we can get a proof for the Kneser-Poulsen conjecture.
- If the conjecture is not true, it might be easier to show existence of a counterexample by the probabilistic method then by presenting the coordinates of the centers.

Case study

Theorem

If $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ *is a decreasing* L_1*-function, and* $\mathbf{p}_i \in \mathbb{R}^n$, $i = 1 \ldots N$, *then*

$$\prod_{i=1}^{N}(-f_i'(r_i))\mathrm{Vol}\left(\bigcap_{i=1}^{N}(B(\mathbf{p}_i, r_i)\right) = \int_{\mathbb{R}^n}\prod_{i=1}^{N} f_i(||\mathbf{x}-\mathbf{p}_i||)d\mathbf{x}$$

Probabilistic interpretation

- *The (generalized) function* $g_i = -\frac{1}{f(0)} f_i' \geq 0$ *is a probability distribution supported on* $\mathbb{R}_+$ *i.e.* $\int_0^\infty -\frac{1}{f(0)} f_i'(t)dt = 1$.
- *Any probability distribution* g_i *supported on* $\mathbb{R}_+$ *can be obtained this way from* $f_i(t) = \int_t^\infty g_i(t)dt$.
- *If the radius* r_i *is a random variable with this distribution, then*

$$E\Big(Vol(\bigcap_{i=1}^{N}(B(\mathbf{p}_i, r_i))\Big) = \int_{\mathbb{R}^n}\prod_{i=1}^{N} f_i(0) f_i(||\mathbf{x}-\mathbf{p}_i||)d\mathbf{x}$$

Case study

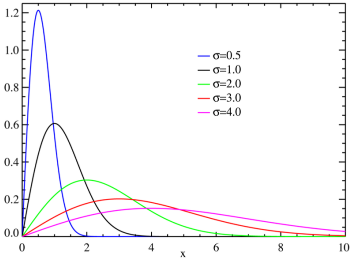
Corollary

By the inclusion exclusion formula we get

$$E\Big(Vol(\bigcup_{i=1}^{N}(B(\mathbf{p}_i, r_i))\Big) = \sum_{\emptyset \neq I \subset \underline{N}} (-1)^{|I|+1} \int_{\mathbb{R}^n} \prod_{i \in I} f_i(0) f_i(||\mathbf{x} - \mathbf{p}_i||) d\mathbf{x}$$

Definition

A random variable $R \in [0, \infty)$ has Rayleigh distribution with parameter σ if its density function is $f(x; \sigma) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}$.



Case study

Theorem

Assume that the radius r_i of the ith ball is a random variable with Rayleigh distribution with parameter $\sigma_i = \frac{1}{\sqrt{2m_i}}$. Then

- $$E(r_i) = \sqrt{\frac{\pi}{2}}\sigma_i = \frac{1}{2}\sqrt{\frac{\pi}{m_i}},$$
- $$E(\mathrm{Vol}(B(\mathbf{p}_i, r_i)) = (\pi/m_i)^{n/2},$$
- $$E(\mathrm{Vol}(\bigcap B(\mathbf{p}_i, r_i))) = \left(\frac{\pi}{M}\right)^{n/2} e^{-\frac{\sum_{i<j} m_i m_j ||\mathbf{p}_i - \mathbf{p}_j||^2}{M}},$$

  where $M = \sum_{i=1}^{N} m_i$.

Corollary

When the centers are contracted, the expectation of the volume of the intersection does not decrease.

Case study

Theorem (R. Alexander)

For $M = \sum_{i=1}^{N} m_i$*, then If the smallest ball containing the point set* $\mathbf{p}_i$ *is centered at* $\mathbf{o}$ *and has radius* R*, then*

$$\frac{\sum_{i<j} m_i m_j ||\mathbf{p}_i - \mathbf{p}_j||^2}{M} \leq MR^2,$$

and equality holds if and only if the mass center of the weighted point set $(\mathbf{p}_i, m_i)$ *is* $\mathbf{o}$*.*

Corollary (Kirszbraun's theorem)

When the point set is contracted, R *does not increase.*

Corollary

$$E(\mathrm{Vol}(\bigcap B(\mathbf{p}_i, r_i))) \geq \left(\frac{\pi}{M}\right)^{n/2} e^{-MR^2},$$

and equality holds if and only if the mass center of the weighted point set $(\mathbf{p}_i, m_i)$ *is* $\mathbf{o}$*.*

Case study

Monotonicity of the expectation

$$E(\mathrm{Vol}(\bigcup B(\mathbf{p}_i, r_i))) = \sum_{\emptyset \neq I \subseteq N} (-1)^{|I|+1} \left(\frac{\pi}{M_I}\right)^{n/2} e^{-\frac{\sum_{i<j \in I} m_i m_j ||\mathbf{p}_i - \mathbf{p}_j||^2}{M_I}}$$

of the volume of the union is not obvious.

Question

Is the RHS a monotone function of the distances $d_{ij} = ||\mathbf{p}_i - \mathbf{p}_j||$?

Assume all the $d_{i,j}$'s depend smoothly on a parameter t. Then

$$\frac{d}{dt}\mathrm{RHS} = \sum_{i<j} \Big(\sum_{\{i,j\} \subseteq I \subset N} (-1)^{|I|} \frac{\pi^{n/2} m_i m_j}{M_I^{n/2+1}} e^{-\frac{\sum_{k<l \in I} m_k m_l d_{kl}^2}{M_I}} \Big) (d_{ij}^2)'$$

The RHS is monotone function of the distances if the coefficient of $(d_{ij}^2)'$ is non-negative.

So the question is

$$\sum_{\{i,j\}\subseteq I\subset N} (-1)^{|I|}\frac{\pi^{n/2}m_i m_j}{M_I^{n/2+1}}e^{-\frac{\sum_{k<l\in I} m_k m_l d_{kl}^2}{M_I}} \geq 0?$$

Observation

If the distances $d_{kl}(t_0)$ can be realized as distances between points $\mathbf{u}_1,\ldots,\mathbf{u}_N \in \mathbb{E}^{n+2}$, then for the random balls $B_i^{n+2} = B_i^{n+2}(\mathbf{u}_i, r_i)$ we have

$$E(\mathrm{Vol}_{n+2}(B_i^{n+2} \cap B_j^{n+2} \setminus \bigcup_{k\notin\{i,j\}} B_k^{n+2})$$

$$= \sum_{\{i,j\}\subseteq I\subset N} (-1)^{|I|}\left(\frac{\pi}{M_I}\right)^{n/2+1} e^{-\frac{\sum_{k<l\in I} m_k m_l |r_k-r_l|^2}{M_I}} \geq 0.$$

Corollary

$E(\mathrm{Vol}(\bigcup_i B_i))$ *does not decrease if there is an expansive homotopy of the centers in* $\mathbb{E}^{n+2}$.

For balls moving in $\mathbb{E}^n$, compare the formulae

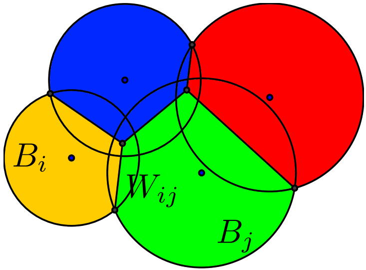
$$\frac{d}{dt} E(\mathrm{Vol}(\bigcup_{i=1}^{N} B(\mathbf{p}_i, r_i)))$$

$$= \sum_{i<j} \Big(\sum_{\{i,j\} \subseteq I \subset N} (-1)^{|I|} \frac{\pi^{n/2} m_i m_j}{M_I^{n/2+1}} e^{-\frac{\sum_{k<l \in I} m_k m_l d_{kl}^2}{M_I}} \Big) (d_{ij}^2)'$$

and

$$\frac{d}{dt} \mathrm{Vol}(\bigcup_{i=1}^{N} B(\mathbf{p}_i, r_i))) = \sum_{i<j} \mathrm{Vol}_{n-1}(W_{ij}) d_{ij}',$$

where W_{ij} is the wall between the truncated Voronoi cells of the ith and jth balls.



Question

Does the equality

$$E(\mathrm{Vol}_{n-1}(W_{ij})) = \Big(\sum_{\{i,j\} \subseteq I \subset N} (-1)^{|I|} \frac{\pi^{n/2} m_i m_j}{M_I^{n/2+1}} e^{-\frac{\sum_{k<l \in I} m_k m_l d_{kl}^2}{M_I}} \Big) 2 d_{ij}$$

hold?