On a strong version of the Kepler conjecture

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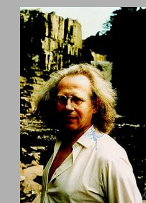
On a stronger form of Rogers's lemma and the minimum surface area of Voronoi cells in unit ball packings

By *Károly Bezdek*[1]) at Ithaca and Budapest

Abstract. Rogers's lemma is an essential tool for estimating the density of unit ball packings in Euclidean space. Also, it motivates several other techniques of the classical theory of packing. In this paper we prove a strengthening of Rogers's lemma and apply it to estimate the minimum surface area of Voronoi cells in unit ball packings. We prove a lower bound for the surface area of Voronoi cells of unit ball packings in d-dimensional Euclidean space in a rather short way. This bound is sharp for $d = 2$ and implies Rogers's upper bound for the density of unit ball packings in d-space for all $d \geqq 2$. Finally, we strengthen these results for $d = 3$.



Claude Ambrose Rogers FRS (1 November 1920 – 5 December 2005)

Theorem 1.4.7 *The surface volume of any Voronoi cell in a packing of unit balls in* $\mathbb{E}^d, d \geq 2$ *is at least* $\frac{d\omega_d}{\sigma_d}$.

Conjecture 1.4.3 *The surface area of any Voronoi cell in a packing with unit balls in* $\mathbb{E}^3$ *is at least as large as* $16.6508\ldots$, *the surface area of a regular dodecahedron of inradius* 1.

For d=3 the lower bound obtained is 16.1433… .

Discrete Comput Geom 28:75–106 (2002)
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Discrete & Computational
Geometry

Improving Rogers' Upper Bound for the Density of Unit Ball Packings via Estimating the Surface Area of Voronoi Cells from Below in Euclidean d-Space for All $d \geq 8$*

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Abstract. The sphere packing problem asks for the densest packing of unit balls in $\mathbf{E}^d$. This problem has its roots in geometry, number theory and information theory and it is part of Hilbert's 18th problem. One of the most attractive results on the sphere packing problem was proved by Rogers in 1958. It can be phrased as follows. Take a regular d-dimensional simplex of edge length 2 in $\mathbf{E}^d$ and then draw a d-dimensional unit ball around each vertex of the simplex. Let σ_d denote the ratio of the volume of the portion of the simplex covered by balls to the volume of the simplex. Then the volume of any Voronoi cell in a packing of unit balls in $\mathbf{E}^d$ is at least ω_d/σ_d, where ω_d denotes the volume of a d-dimensional unit ball. This has the immediate corollary that the density of any unit ball packing in $\mathbf{E}^d$ is at most σ_d. In 1978 Kabatjanskii and Levenštein improved this bound for large d. In fact, Rogers' bound is the presently known best bound for $4 \leq d \leq 42$, and above that the Kabatjanskii–Levenštein bound takes over. In this paper we improve Rogers' upper bound for the density of unit ball packings in Euclidean d-space for all $d \geq 8$ and improve the Kabatjanskii–Levenštein upper bound in small dimensions. Namely, we show that the volume of any Voronoi cell in a packing of unit balls in $\mathbf{E}^d$, $d \geq 8$, is at least $\omega_d/\hat{\sigma}_d$ and so the density of any unit ball packing in $\mathbf{E}^d$, $d \geq 8$, is at most $\hat{\sigma}_d$, where $\hat{\sigma}_d$ is a geometrically well-defined quantity satisfying the inequality $\hat{\sigma}_d < \sigma_d$ for all $d \geq 8$. We prove this by showing that the surface area of any Voronoi cell in a packing of unit balls in $\mathbf{E}^d$, $d \geq 8$, is at least $(d \cdot \omega_d)/\hat{\sigma}_d$.

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Finding the Best Face on a Voronoi Polyhedron – The Strong Dodecahedral Conjecture Revisited

By

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Abstract. In this paper we prove the following theorem. The surface area density of a unit ball in any face cone of a Voronoi cell in an arbitrary packing of unit balls of Euclidean 3-space is at most

$$\frac{-9\pi + 30\arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{5}\right)\right)}{5\tan\left(\frac{\pi}{5}\right)} = 0.77836\ldots,$$

and so the surface area of any Voronoi cell in a packing with unit balls in Euclidean 3-space is at least

$$\frac{20\pi \cdot \tan\left(\frac{\pi}{5}\right)}{-9\pi + 30\arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{5}\right)\right)} = 16.1445\ldots \,.$$

This result and the ideas of its proof support the Strong Dodecahedral Conjecture according to which the surface area of any Voronoi cell in a packing with unit balls in Euclidean 3-space is at least as large as $16.6508\ldots$, the surface area of a regular dodecahedron of inradius 1.

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A NEW LOWER BOUND ON THE SURFACE AREA OF A VORONOI POLYHEDRON

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This paper is dedicated to Károly Bezdek on occasion of his fiftieth birthday.

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Abstract

We prove that the mininum surface area of a Voronoi cell in a unit ball packing in $\mathbb{E}^3$ is at least 16.1977. This result provides further support for the Strong Dodecahedral Conjecture according to which the minimum surface area of a Voronoi cell in a 3-dimensional unit ball packing is at least as large as the surface area of a regular dodecahedron of inradius 1, which is about 16.6508 . . . .

Fejes Tóth Lecture Series

September 19, 21, 23, 2011

Thomas C. Hales (Univ. of Pittsburgh)

3:30 - 4:30 pm at the Fields Institute, Room 230

Sept. 19, Lecture 1.(abstract)

Mathematics in the Age of the Turing machine

Sept. 21, Lecture 2.(abstract)

The weak and strong Dodecahedral Conjectures.

Sept. 23, Lecture 3.(abstract)

Fejes Tóth's Contact Conjecture.

[13] T. C. Hales, The strong dodecahedral conjecture and Fejes Tóth's contact conjecture, *arXiv*:1110.0402v1 [math.MG] (2011), 1–11.

T. C. Hales, *Dense Sphere Packing - a blueprint for formal proofs*, Cambridge University Press (to appear), 1–256.

On a strong version of the Kepler conjecture

Karoly Bezdek

(Submitted on 14 Nov 2011)

In this short paper we raise and investigate the following problem: If the Euclidean 3-space is partitioned into convex cells each containing a unit ball, how should the shapes of the cells be designed to minimize the average surface area of the cells? In particular, we prove that the average surface area in question is always at least 13.8564....

Comments: 9 pages
Subjects: Metric Geometry (math.MG)
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1 Introduction

The central problem that we raise in this short paper can be phrased informally as follows: *if the Euclidean* 3*-space is partitioned into convex cells each containing a unit ball, how should the shapes of the cells be designed to minimize the average surface area of the cells*? In order to state our problem in more precise terms we proceed as follows. Let $\mathcal{T}$ be a tiling of the 3-dimensional Euclidean space $\mathbb{E}^3$ into convex polyhedra $\mathbf{P}_i, i = 1, 2, \ldots$ each containing a unit ball say, $\mathbf{P}_i$ containing the closed 3-dimensional ball $\mathbf{B}_i$ centered at the point $\mathbf{o}_i$ having radius 1 for $i = 1, 2, \ldots$. Also, we assume that there is a finite upper bound for the diameters of the convex cells in $\mathcal{T}$, i.e., $\sup\{\mathrm{diam}(\mathbf{P}_i) | i = 1, 2, \ldots\} < \infty$, where $\mathrm{diam}(\cdot)$ denotes the diameter of the corresponding set. In short, we say that $\mathcal{T}$ is a *normal tiling* of $\mathbb{E}^3$ with the underlying packing $\mathcal{P}$ of the unit balls $\mathbf{B}_i, i = 1, 2, \ldots$. Then we define the (lower) *average surface area* $\underline{s}(\mathcal{T})$ of the cells in $\mathcal{T}$ as follows:

$$\underline{s}(\mathcal{T}) := \liminf_{L\to\infty} \frac{\sum_{\{i|\mathbf{B}_i\subset\mathbf{C}_L\}} \mathrm{sarea}(\mathbf{P}_i \cap \mathbf{C}_L)}{\mathrm{card}\{i|\mathbf{B}_i \subset \mathbf{C}_L\}},$$

where $\mathbf{C}_L$ denotes the cube centered at the origin $\mathbf{o}$ with edges parallel to the coordinate axes of $\mathbb{E}^3$ and having edge length L furthermore, $\mathrm{sarea}(\cdot)$ and $\mathrm{card}(\cdot)$ denote the surface area and cardinality of the corresponding sets. (We note that it is rather straightforward to show that $\underline{s}(\mathcal{T})$ is independent from the choice of the coordinate system of $\mathbb{E}^3$.)

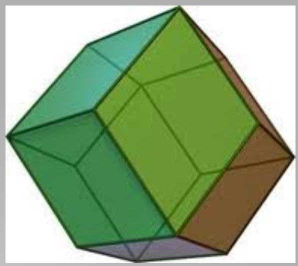
[10] G. Kertész, On totally separable packings of equal balls, *Acta Math. Hungar.* **51/3-4** (1988), 363–364.

In the second half of this paper, by adjusting Kertész's volume estimation technique ([10]) to our problem on estimating surface area, we give a proof of the following inequality. It is likely that our lower bound can be improved further however, any such improvement would require additional new ideas.

Theorem 1.1 *Let $\mathcal{T}$ be an arbitrary normal tiling of $\mathbb{E}^3$. Then the average surface area of the cells in $\mathcal{T}$ is always at least $\frac{24}{\sqrt{3}}$, i.e.,*

$$\underline{s}(\mathcal{T}) \geq \frac{24}{\sqrt{3}} = 13.8564\ldots .$$

Recall that in the face-centered cubic lattice packing of unit balls in $\mathbb{E}^3$, when each ball is touched by 12 others, the Voronoi cells of the unit balls are regular rhombic dodecahedra of inradius 1 and of surface area $12\sqrt{2}$ (for more details on the geometry involved see [4]). Thus, it is very natural to raise the following problem: *prove or disprove that if $\mathcal{T}$ is an arbitrary normal tiling of $\mathbb{E}^3$, then*



$$\underline{s}(\mathcal{T}) \geq 12\sqrt{2} = 16.9705\ldots . \quad (1)$$

Let us mention that an affirmative answer to (1) for the family of Voronoi tilings of unit ball packings would imply the Kepler conjecture. As is well known, the Kepler conjecture has been proved by Hales in a sequence of difficult papers ([5], [6], [7], [8], and [9]) concluding that the density of any unit ball packing in $\mathbb{E}^3$ is at most $\frac{\pi}{\sqrt{18}}$. Indeed, if $\underline{s}(\mathcal{T}) \geq 12\sqrt{2}$ were true for the Voronoi tilings $\mathcal{T}$ of unit ball packings $\mathcal{P}$ in $\mathbb{E}^3$, then based on the obvious inequalities

$$\sum_{\{i|\mathbf{B}_i\subset\mathbf{C}_L\}} \mathrm{vol}(\mathbf{P}_i \cap \mathbf{C}_L) \leq \mathrm{vol}(\mathbf{C}_L) \ \text{ and } \ \frac{1}{3}\mathrm{sarea}(\mathbf{P}_i \cap \mathbf{C}_L) \leq \mathrm{vol}(\mathbf{P}_i \cap \mathbf{C}_L),$$

(where $\mathrm{vol}(\cdot)$ denotes the volume of the corresponding set) we would get that the (upper) *density* $\overline{\delta}(\mathcal{P}) := \limsup_{L\to\infty} \frac{\frac{4\pi}{3}\mathrm{card}\{i|\mathbf{B}_i\subset\mathbf{C}_L\}}{\mathrm{vol}(\mathbf{C}_L)}$ of the packing $\mathcal{P}$ must satisfy the inequality

$$\overline{\delta}(\mathcal{P}) \leq \limsup_{L\to\infty} \frac{\frac{4\pi}{3}\mathrm{card}\{i|\mathbf{B}_i \subset \mathbf{C}_L\}}{\sum_{\{i|\mathbf{B}_i\subset\mathbf{C}_L\}} \mathrm{vol}(\mathbf{P}_i \cap \mathbf{C}_L)}$$

$$\leq \limsup_{L\to\infty} \frac{4\pi\mathrm{card}\{i|\mathbf{B}_i \subset \mathbf{C}_L\}}{\sum_{\{i|\mathbf{B}_i\subset\mathbf{C}_L\}} \mathrm{sarea}(\mathbf{P}_i \cap \mathbf{C}_L)} = \frac{4\pi}{\underline{s}(\mathcal{T})} \leq \frac{\pi}{\sqrt{18}}$$

Thus, one could regard the affirmative version of (1), stated for the Voronoi tilings of unit ball packings, as a *strong version of the Kepler conjecture.*

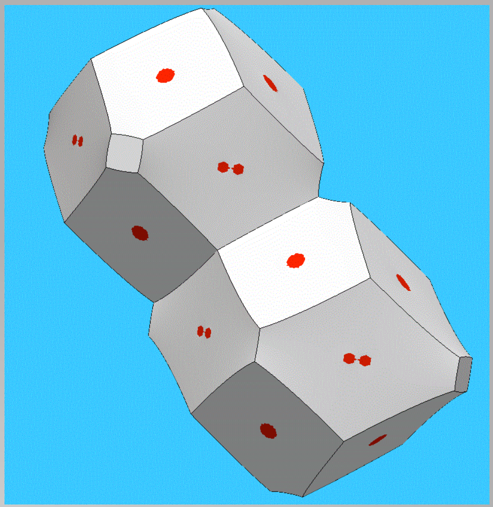
Last but not least, it is very tempting to further relax the conditions in our original problem by replacing convex cells with cells that are measurable and have measurable boundaries and ask the following more general question: *if the Euclidean* 3*-space is partitioned into cells each containing a unit ball, how should the shapes of the cells be designed to minimize the average surface area of the cells*? One can regard this question a *foam problem*, since foams are simply tilings of space that try to minimize surface area. Although foams are well studied (see the relevant sections of the highly elegant book [15] of Morgan), it is far not clear what would be a good candidate for the proper minimizer in the foam question just raised.

[11] F. Morgan, Geometric Measure Theory - A Beginner's Guide, Fourth edition, *Elsevier/Academic Press*, Amsterdam, 2009.

Brakke, Ken [brakke@susqu.edu]
Sent: October-31-11 6:41 AM
To:
Frank Morgan [Frank.Morgan@williams.edu]
Cc:
Karoly Bezdek; John M Sullivan [Sullivan@Math.TU-Berlin.DE]
Attachments:
williams-balls.gif (169 KB)
Frank,

I did a Williams cell foam with balls. The popping pattern for the rhombic tetrakaidecahedra turned out to be alternating planes of uniform direction, but with just two directions instead of all three possibilities. I've attached an image of two Williams cells with red where they drape over the balls. The hexagons get tilted symmetrically (as in the Kelvin version I sent you before), so there are two small contact circles near their centers. The pentagons get tilted just one way, so they get larger single contact patches. The area of a cell turns out to be 16.95753, compared to 16.958261 for Kelvin, so Williams seems to win by a hair.

Ken



2 Proof of Theorem 1.1

First, we prove the following "compact" version of Theorem 1.1. It is also a surface area analogue of the volume estimating theorem in [10].

[10] G. Kertész, On totally separable packings of equal balls, *Acta Math. Hungar.* **51/3-4** (1988), 363–364.

Theorem 2.1 *If the cube* $\mathbf{C}$ *is partitioned into the convex cells* $\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_n$ *each containing a unit ball in* $\mathbb{E}^3$*, then the sum of the surface areas of the* n *convex cells is at least* $\frac{24}{\sqrt{3}}n$*, i.e.,*

$$\sum_{i=1}^{n} \mathrm{sarea}(\mathbf{Q}_i) \geq \frac{24}{\sqrt{3}}n \ .$$

Proof: Let $E(\mathbf{Q}_i)$ denote the family of the edges of the convex polyhedron $\mathbf{Q}_i$ and let $\mathrm{ecurv}(\mathbf{Q}_i) := \sum_{e\in E(\mathbf{Q}_i)} L(e)\tan\frac{\alpha_e}{2}$ be the so-called *edge curvature* of $\mathbf{Q}_i$, where $L(e)$ denotes the length of the edge $e \in E(\mathbf{Q}_i)$ and α_e is the angle between the outer normal vectors of the two faces of $\mathbf{Q}_i$ meeting along the edge e, $1 \le i \le n$. It is well known that the Brunn-Minkowski inequality implies the following inequality (for more details we refer the interested reader to p. 287 in [4]):

$$\mathrm{sarea}^2(\mathbf{Q}_i) \ge 3\mathrm{vol}(\mathbf{Q}_i)\mathrm{ecurv}(\mathbf{Q}_i)\ . \tag{2}$$

Also, it will be more proper for us to use the inner dihedral angles $\beta_e := \pi - \alpha_e$ and the relevant formula

$$\mathrm{ecurv}(\mathbf{Q}_i) = \sum_{e\in E(\mathbf{Q}_i)} L(e)\cot\frac{\beta_e}{2}\ . \tag{3}$$

As, by assumption, $\mathbf{Q}_i$ contains a unit ball therefore

$$\mathrm{vol}(\mathbf{Q}_i) \ge \frac{1}{3}\mathrm{sarea}(\mathbf{Q}_i)\ . \tag{4}$$

Hence, (2), (3), and (4) imply in a straightforward way that

$$\mathrm{sarea}(\mathbf{Q}_i) \ge \sum_{e\in E(\mathbf{Q}_i)} L(e)\cot\frac{\beta_e}{2} \tag{5}$$

holds for all $1 \le i \le n$.

Now, let $s \subset \mathbf{C}$ be a closed line segment along which exactly k members of the family $\{\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_n\}$ meet having inner dihedral angles $\beta_1, \beta_2, \ldots, \beta_k$. There are the following three possibilities:

(a) s is on an edge of the cube $\mathbf{C}$;

(b) s is in the relative interior either of a face of $\mathbf{C}$ or of a face of a convex cell in the family $\{\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_n\}$;

(c) s is in the interior of $\mathbf{C}$ and not in the relative interior of any face of any convex cell in the family $\{\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_n\}$.

In each of the above cases we can make the following easy observations:

(a) $\beta_1 + \beta_2 + \cdots + \beta_k = \frac{\pi}{2}$ with $k \geq 1$;

(b) $\beta_1 + \beta_2 + \cdots + \beta_k = \pi$ with $k \geq 2$;

(c) $\beta_1 + \beta_2 + \cdots + \beta_k = 2\pi$ with $k \geq 3$.

As $y = \cot x$ is convex and decreasing over the interval $0 < x \leq \frac{\pi}{2}$ therefore the following inequalities must hold:

(a) $\cot \frac{\beta_1}{2} + \cot \frac{\beta_2}{2} + \cdots + \cot \frac{\beta_k}{2} \geq k \cot \frac{\pi}{4k} \geq k$;

(b) $\cot \frac{\beta_1}{2} + \cot \frac{\beta_2}{2} + \cdots + \cot \frac{\beta_k}{2} \geq k \cot \frac{\pi}{2k} \geq k$;

(c) $\cot \frac{\beta_1}{2} + \cot \frac{\beta_2}{2} + \cdots + \cot \frac{\beta_k}{2} \geq k \cot \frac{\pi}{k} \geq \frac{1}{\sqrt{3}}k$.

In short, the following inequality holds in all three cases:

$$\cot \frac{\beta_1}{2} + \cot \frac{\beta_2}{2} + \cdots + \cot \frac{\beta_k}{2} \geq \frac{1}{\sqrt{3}}k \ . \tag{6}$$

[1] A. S. Besicovitch and H. G. Eggleston, The total length of the edges of a polyhedron, *Quart. J. Math. Oxford Ser.* **2/ 8** (1957), 172–190.

Thus, by adding together the inequalities (5) for all $1 \leq i \leq n$ and using (6) we get that

$$\sum_{i=1}^{n} \mathrm{sarea}(\mathbf{Q}_i) \geq \frac{1}{\sqrt{3}} \sum_{i=1}^{n} \sum_{e \in E(\mathbf{Q}_i)} L(e) \ . \tag{7}$$

Finally, recall the elegant theorem of Besicovitch and Eggleston [1] claiming that the total edge length of any convex polyhedron containing a unit ball in $\mathbb{E}^3$ is always at least as large as the total edge length of a cube circumscribed a unit ball. This implies that

$$\sum_{e \in E(\mathbf{Q}_i)} L(e) \geq 24 \tag{8}$$

holds for all $1 \leq i \leq n$. Hence, (7) and (8) finish the proof of Theorem 2.1. $\square$

Second, we take a closer look of the given normal tiling $\mathcal{T}$ defined in details in the first Section of this paper and using Theorem 2.1 we give a proof of Theorem 1.1.

By assumption $D := \sup\{\mathrm{diam}(\mathbf{P}_i) | i = 1, 2, \dots\} < \infty$. Thus, clearly each closed ball of radius D in $\mathbb{E}^3$ contains at least one of the convex polyhedra $\mathbf{P}_i, i = 1, 2, \dots$ (forming the tiling $\mathcal{T}$ of $\mathbb{E}^3$). Now, let $\mathbf{C}_{L_N}, N = 1, 2, \dots$ be an arbitrary sequence of cubes centered at the origin $\mathbf{o}$ with edges parallel to the coordinate axes of $\mathbb{E}^3$ and having edge length $L_N, N = 1, 2, \dots$ with $\lim_{N\to\infty} L_n = \infty$. It follows that

$$0 < \liminf_{N\to\infty} \frac{\frac{4\pi}{3}\mathrm{card}\{i | \mathbf{B}_i \subset \mathbf{C}_{L_N}\}}{\mathrm{vol}(\mathbf{C}_{L_N})} \leq \limsup_{N\to\infty} \frac{\frac{4\pi}{3}\mathrm{card}\{i | \mathbf{B}_i \subset \mathbf{C}_{L_N}\}}{\mathrm{vol}(\mathbf{C}_{L_N})} < 1. \quad (9)$$

Note that clearly

$$\frac{\mathrm{card}\{i | \mathbf{P}_i \cap \mathrm{bd}\mathbf{C}_{L_N} \neq \emptyset\}}{\mathrm{card}\{i | \mathbf{B}_i \subset \mathbf{C}_{L_N}\}} \leq \frac{\big(\mathrm{vol}(\mathbf{C}_{L_N+2D}) - \mathrm{vol}(\mathbf{C}_{L_N-2D})\big)\mathrm{vol}(\mathbf{C}_{L_N})}{\mathrm{vol}(\mathbf{C}_{L_N})\frac{4\pi}{3}\mathrm{card}\{i | \mathbf{B}_i \subset \mathbf{C}_{L_N}\}} \quad (10)$$

moreover,

$$\lim_{N\to\infty} \frac{\mathrm{vol}(\mathbf{C}_{L_N+2D}) - \mathrm{vol}(\mathbf{C}_{L_N-2D})}{\mathrm{vol}(\mathbf{C}_{L_N})} = 0. \quad (11)$$

Thus, (9), (10), and (11) imply in a straightforward way that

$$\lim_{N\to\infty} \frac{\mathrm{card}\{i | \mathbf{P}_i \cap \mathrm{bd}\mathbf{C}_{L_N} \neq \emptyset\}}{\mathrm{card}\{i | \mathbf{B}_i \subset \mathbf{C}_{L_N}\}} = 0 \ . \quad (12)$$

Moreover, (5) yields that

$$\operatorname{sarea}(\mathbf{P}_i) \geq \operatorname{ecurv}(\mathbf{P}_i) = \sum_{e \in E(\mathbf{P}_i)} L(e) \cot \frac{\beta_e}{2} \tag{13}$$

holds for all $i = 1, 2, \ldots$. As a next step, using

$$\operatorname{sarea}(\mathbf{P}_i) = \operatorname{sarea}\left(\operatorname{bd}(\mathbf{P}_i \cap \mathbf{C}_L) \setminus \operatorname{bd}\mathbf{C}_L\right) + \delta_i \tag{14}$$

and

$$\operatorname{ecurv}(\mathbf{P}_i) \geq \sum_{e \in E(\operatorname{bd}(\mathbf{P}_i \cap \mathbf{C}_L) \setminus \operatorname{bd}\mathbf{C}_L)} L(e) \cot \frac{\beta_e}{2} \tag{15}$$

(with $\operatorname{bd}(\cdot)$ denoting the boundary of the corresponding set) we obtain the following from (13):

$$\operatorname{sarea}\left(\operatorname{bd}(\mathbf{P}_i \cap \mathbf{C}_L) \setminus \operatorname{bd}\mathbf{C}_L\right) + \delta_i \geq \sum_{e \in E(\operatorname{bd}(\mathbf{P}_i \cap \mathbf{C}_L) \setminus \operatorname{bd}\mathbf{C}_L)} L(e) \cot \frac{\beta_e}{2}\ , \tag{16}$$

where clearly $0 \leq \delta_i \leq \operatorname{sarea}(\mathbf{P}_i)$. Hence, (16) combined with (6) yields

Corollary 2.2

$$f(L) := \sum_{\{i | \operatorname{int}\mathbf{P}_i \cap \mathbf{C}_L \neq \emptyset\}} \operatorname{sarea}\Big(\operatorname{bd}(\mathbf{P}_i \cap \mathbf{C}_L) \setminus \operatorname{bd}\mathbf{C}_L\Big) + \sum_{\{i | \mathbf{P}_i \cap \operatorname{bd}\mathbf{C}_L \neq \emptyset\}} \delta_i$$

$$\geq g(L) := \frac{1}{\sqrt{3}} \sum_{\{i | \operatorname{int}\mathbf{P}_i \cap \mathbf{C}_L \neq \emptyset\}} \left(\sum_{e \in E(\operatorname{bd}(\mathbf{P}_i \cap \mathbf{C}_L) \setminus \operatorname{bd}\mathbf{C}_L)} L(e) \right)$$

Now, it is easy to see that

$$f(L) = \Delta(L) + \sum_{\{i|\mathbf{B}_i \subset \mathbf{C}_L\}} \operatorname{sarea}(\mathbf{P}_i \cap \mathbf{C}_L) \ , \tag{17}$$

where $0 \leq \Delta(L) \leq 2\sum_{\{i|\mathbf{P}_i \cap \operatorname{bd}\mathbf{C}_L \neq \emptyset\}} \operatorname{sarea}(\mathbf{P}_i)$.

Moreover, (8) implies that

$$g(L) \geq -\overline{\Delta}(L) + \sum_{\{i|\mathbf{B}_i \subset \mathbf{C}_L\}} \frac{24}{\sqrt{3}} \ , \tag{18}$$

where $0 \leq \overline{\Delta}(L) \leq \sum_{\{i|\mathbf{P}_i \cap \operatorname{bd}\mathbf{C}_L \neq \emptyset\}} \sum_{e \in E(\mathbf{P}_i)} L(e)$.

Lemma 2.3

$$A := \sup\{\operatorname{sarea}(\mathbf{P}_i) | i = 1, 2, \dots\} < \infty$$

and

$$E := \sup\{\sum_{e \in E(\mathbf{P}_i)} L(e) | i = 1, 2, \dots\} < \infty \ .$$

Proof: As $D = \sup\{\operatorname{diam}(\mathbf{P}_i) | i = 1, 2, \dots\} < \infty$ therefore according to Jung's theorem ([3]) each $\mathbf{P}_i$ is contained in a closed ball of radius $\sqrt{\frac{3}{8}}D$ in $\mathbb{E}^3$. Thus, $A \leq \frac{3}{2}\pi D^2 < \infty$.

For a proof of the other claim recall that $\mathbf{P}_i$ contains the unit ball $\mathbf{B}_i$ centered at $\mathbf{o}_i$. If the number of faces of $\mathbf{P}_i$ is f_i, then $\mathbf{P}_i$ must have at least f_i neighbours (i.e., cells of $\mathcal{T}$ that have at least one point in common with $\mathbf{P}_i$) and as each neighbour is contained in the closed 3-dimensional ball of radius $2D$ centered at $\mathbf{o}_i$ therefore the number of neighbours of $\mathbf{P}_i$ is at most $(2D)^3 - 1$ and so, $f_i \leq 8D^3 - 1$. (Here, we have used the fact that each neighbour contains a unit ball and therefore its volume is larger than $\frac{4\pi}{3}$.) Finally, Euler's formula implies that the number of edges of $\mathbf{P}_i$ is at most $3f_i - 6 \leq 24D^3 - 9$. Thus, $E \leq 24D^4 - 9D < \infty$ (because the length of any edge of $\mathbf{P}_i$ is at most D). $\square$

Thus, Corollary 2.2, (17), (18), and Lemma 2.3 imply the following inequality in a straightforward way.

Corollary 2.4

$$\frac{2A\mathrm{card}\{i|\mathbf{P}_i \cap \mathrm{bd}\mathbf{C}_L \neq \emptyset\} + \sum_{\{i|\mathbf{B}_i \subset \mathbf{C}_L\}} \mathrm{sarea}(\mathbf{P}_i \cap \mathbf{C}_L)}{\mathrm{card}\{i|\mathbf{B}_i \subset \mathbf{C}_L\}}$$

$$\geq \frac{-E\mathrm{card}\{i|\mathbf{P}_i \cap \mathrm{bd}\mathbf{C}_L \neq \emptyset\} + \sum_{\{i|\mathbf{B}_i \subset \mathbf{C}_L\}} \frac{24}{\sqrt{3}}}{\mathrm{card}\{i|\mathbf{B}_i \subset \mathbf{C}_L\}} \ .$$

Finally, Corollary 2.4 and (12) yield that

$$\liminf_{N\to\infty} \frac{\sum_{\{i|\mathbf{B}_i \subset \mathbf{C}_{L_N}\}} \mathrm{sarea}(\mathbf{P}_i \cap \mathbf{C}_{L_N})}{\mathrm{card}\{i|\mathbf{B}_i \subset \mathbf{C}_{L_N}\}} \geq \frac{24}{\sqrt{3}} \ , \tag{19}$$

finishing the proof of Theorem 1.1.