When is a Motion Finite? Generating Finite Motions - and Their Transfer

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Outline

Failing the tests for rigidity

Tests for rigidity:

1. First-order rigidity (with matrices and rank);
2. necessary counts of edges and vertices (from first-order matrix, at regular points);
3. necessary counts under symmetry (under symmetry regular points);
4. shifts of dimension via coning;
5. global rigidity and stress matrices;

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Bottema's Mechanism

InfiniteGrid

Frameworks

Euclidean Metric $\mathbb{E}^d$ $\|(x_1, \ldots, x_d)\| = x_1^2 + \ldots + x_d^2$

- A framework (in $\mathbb{E}^d$) is a pair (G, p), where G is a graph and $p : V(G) \to \mathbb{E}^d$ is a map with $p(u) \neq p(v)$ for all $\{u, v\} \in E(G)$.

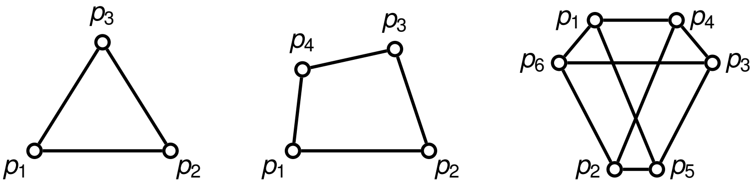

Figure: Some frameworks in $\mathbb{E}^2$

- finite motion of a framework (G, p), is an assignment continuous function $p(t)$, $0 \leq t < 1$ to the vertices such that: $|p_i(t) - p_j(t)| = |p_i(0) - p_j(0)|$, for all $(i, j) \in E$;
- non-trivial if $|p_h(t) - p_k(t)| \neq |p_h(0) - p_k(0)|$ for some $(h, k) \notin E$ and all $0 < t < 1$.

Infinitesimal motion

An *infinitesimal motion* of a framework (G, p) in $\mathbb{E}^d$ with $V(G) = \{1, \ldots, n\}$ is a function $u : V(G) \to \mathbb{R}^d$ such that

$$(p_i - p_j) \cdot (u_i - u_j) = 0 \quad \text{for all } \{i, j\} \in E(G), \tag{1}$$

where u_i denotes the vector $u(i)$ for each i.

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The rigidity matrix of (G, p) is the $|E(G)| \times dn$ matrix $\mathbf{R}(G, p)$

$$\begin{array}{cc} & \begin{array}{ccccccccccc} & & & i & & & & j & & & \end{array} \\ \{i,j\} & \begin{pmatrix} & & & & & \vdots & & & & & \\ 0 & \ldots & 0 & (p_i - p_j) & 0 & \ldots & 0 & (p_j - p_i) & 0 & \ldots & 0 \\ & & & & & \vdots & & & & & \end{pmatrix} \end{array}$$

kernel is space of infinitesimal motions.

Infinitesimal flexibility

An infinitesimal motion u of (G, p) is an infinitesimal rigid motion (or trivial infinitesimal motion) if it is an infinitesimal motion on a complete graph on vertices spanning the space - or is a restriction of such a motion to a subset of vertices.

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(G, p) is called infinitesimally rigid in $\mathbb{E}^d$ if every infinitesimal motion of (G, p) is an infinitesimal rigid motion.
Otherwise, (G, p) is called infinitesimally flexible.

An self-stress of a framework $G(p)$ is a row dependence ω of the rigidity matrix.

It is also viewed as a set of tensions ($\omega_{i,j} > 0$) and compressions ($\omega_{i,j} < 0$) in the bars which reach equilibrium at all vertices.

Examples of infinitesimal motions

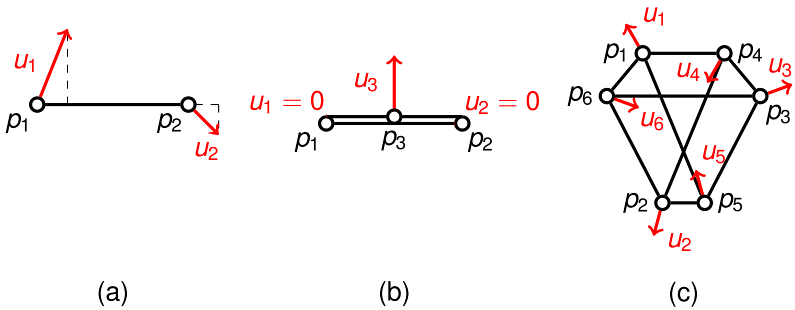

Figure: An infinitesimal rigid motion (a) and infinitesimal flexes (b, c) of frameworks in $\mathbb{R}^2$.

Configuration p is regular for G if the rigidity matrix of $G(p)$ has maximal rank for G. Otherwise p is singular for G.

Flexibility from Counts

Theorem (Asimov & Roth)

If a framework $G(p)$ on at least d vertices is at a regular point (maximum rank rigidity matrix) and the matrix has $rank < d|V| - \binom{d+1}{2}$ then the framework has a finite motion

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Corollary (Connelly)

If a framework $G(p)$ on at least d vertices is at a regular point (maximum rank rigidity matrix) has a finite motion, and $G(q)$ does not have a finite motion, then $G(q)$ has a self-stress.

Frameworks with symmetry

Guest, Kangwai, Fowler, Connelly, Schulze, W.

Let (G, p) be a (finite) framework with symmetry group $\mathcal{S}$.

An infinitesimal motion u of (G, p) is $\mathcal{S}$-symmetric if $s(u_i) = u_{s(i)}$ for all $i \in V(G)$ and all $s \in \mathcal{S}$.

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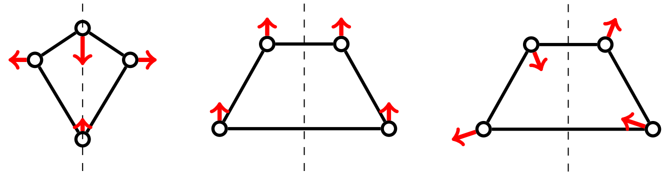
2D examples with mirror symmetry $\mathcal{C}_s$:



(a) (b) (c)

(a) $\mathcal{C}_s$-symmetric non-trivial infinitesimal motion;
(b) $\mathcal{C}_s$-symmetric trivial infinitesimal motion;
(c) non-trivial infinitesimal motion which is not $\mathcal{C}_s$-symmetric.

The orbit rigidity matrix for symmetric frameworks

Let (G, p) be a framework in $\mathbb{R}^d$ with symmetry group $\mathcal{S}$ (assume: no vertex or edge fixed by $s \in \mathcal{S}$, $s \neq id$)

$v_0 = \frac{v}{|\mathcal{S}|}$ (number of vertex orbits under the action of $\mathcal{S}$).
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The orbit rigidity matrix $\mathbf{O}(G, p, \mathcal{S})$ is the $e_0 \times dv_0$ matrix defined as

$$\begin{array}{ccccc} & a & & b & \\ \left(\begin{array}{ccccc} 0\dots0 & (p_a - s(p_b)) & 0\dots0 & (p_b - s^{-1}(p_a)) & 0\dots0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0\dots0 & (2p_a - s(p_a) - s^{-1}(p_a)) & 0\dots0 & 0 & 0\dots0 \end{array}\right) \end{array}$$

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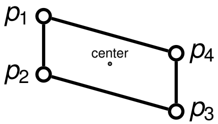

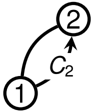

(G, p) with half-turn $(\mathcal{C}_2)$ symmetry and its $\mathcal{C}_2$ orbit graph

Sufficient condition for flexibility of symmetric frameworks

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Theorem (Schulze (2009); Schulze, W. (2010))

If (G, p) is $\mathcal{S}$-generic and

$$e_0 < 3v_0 - triv_{\mathcal{S}}$$

then (G, p) has a symmetry-preserving non-trivial finite motion.

Symmetry counts in 3D for finite motions

Counts for finite motions at symmetry regular configurations
Is $f_{\mathcal{S}} = (3v_0 - triv_{\mathcal{S}}) - e_0 > 0$?

$\mathcal{S}$	$triv_{\mathcal{S}}$	e	e_0	$3v_0 - triv_{\mathcal{S}}$	$f_{\mathcal{S}}$
$\mathcal{C}_1$	6	$3v - 6$	$3v_0 - 6$	$3v_0 - 6$	0
$\mathcal{C}_2$	2	$3v - 6$	$3v_0 - 3$	$3v_0 - 2$	1
$\mathcal{C}_s$	3	$3v - 6$	$3v_0 - 3$	$3v_0 - 3$	0
$\mathcal{D}_2$	0	$3v - 4$	$3v_0 - 1$	$3v_0$	1
$\mathcal{D}_3$	6	$3v - 6$	$3v_0 - 1$	$3v_0$	1

$\mathcal{C}_1$: no non-trivial symmetry
$\mathcal{C}_i$: inversion symmetry
$\mathcal{C}_2$: half-turn symmetry
$\mathcal{C}_s$: mirror symmetry
$\mathcal{D}_2$: 3 mutually perpendicular half-turn axes
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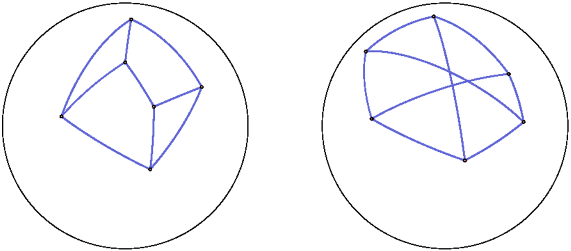
C_2, D_2 and D_3 occur in a number of proteins - more next week.

Coning and Frameworks on a Sphere

Generalizing earlier work of Saliola & W.



Describe the framework $G(q)$ on unit d-sphere $\mathbf{S}^d$ as extended cone framework $(G, q) * O$ in $\mathbb{E}^{d+1}$ with $|V|$ new edges. The added bars from the center to points on the sphere hold them at radius 1.

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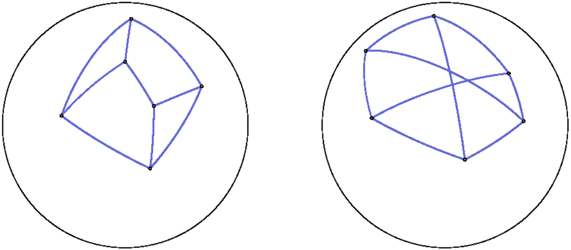

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Coning and Spheres

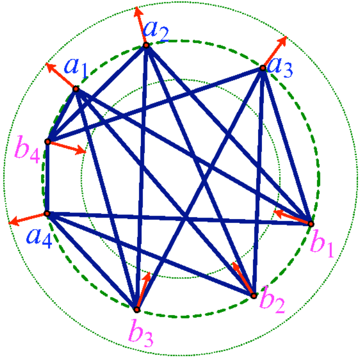
In general, coning does not transfer finite motions from $\mathbb{E}^d$ to $\mathbb{S}^d$. However,

- coning transfers undercounts, and therefore mechanisms due to undercounts;
- coning transfers regular points to regular points;
- coning vertically over the point of a point group, transfers symmetry to symmetry, and preserves rank of the orbit matrix;
- coning transfers finite symmetric motion at symmetry regular point to finite symmetric motion;
- in particular, coning transfers symmetry based finite motion between plane and the sphere.
- generalizes to transfer of symmetry based finite motion between Cayley-Klein metrics.
- transfer of finite motions of C_2-based flexible Bricard octahedra.

Infinitesimal Motions of Complete Bipartite Frameworks

Wunderlich (1977-1979) trilateration graphs;
Bolker and Roth (1980), W. (1980,1984)

Connections to failures of global rigidity (geodesy - map making)
Some Examples: over counted but infinitesimally flexible



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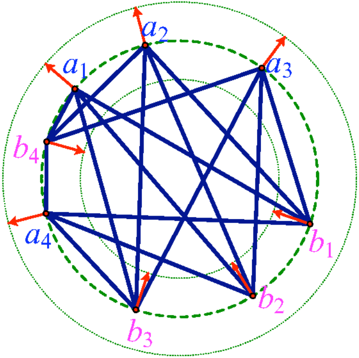

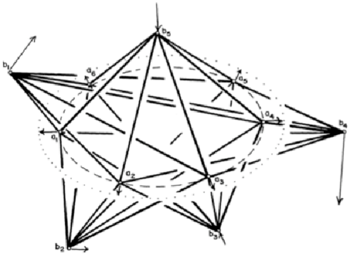

$\boxed{K_{4,4} \text{ motion, with conic}}$

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Theorem (Bolker&Roth, W.)

Given $K_{A,B}$ complete bipartite framework in dimension d, $|A|, |B| \geq d$, there is a non-trivial infinitesimal motion if, and only if either:

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When does this give finite motions?

$K_{7,7}$ in 4-space must lie on a quadric (14 points always do).

$$|E| = 49 > 46 = 4(14) - 10 = 4|V| - 10$$

overbraced - has a finite motion, with no two bars rigidity attached.

$K_{6,6}$ is a generic flexible circuit in 4-space.

Focus on Flatness

Key examples: - Wunderlich (1977-79),
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This property of flatness is projectively invariant, and as such the finite motion will transfer to the spherical metric, the hyperbolic metric, the Minkowskian metric ... .

Have extensions for added bars to a complete bipartite framework.

Second Step

For a finite motion within a class of frameworks defined by algebraic conditions:

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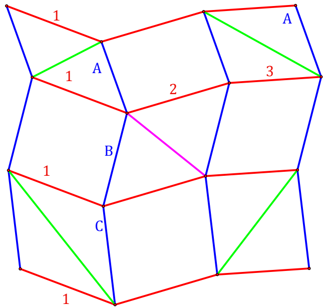
Theorem (Schulze & W. 2011)

f the configuration of a framework lies in an algebraic variety, and the velocities of a non-trivial motion are tangent to the algebraic variety at a regular point of the variety, then there is a finite motion within the variety.

Plane Parallelogram - Triangle Discs

Bolker & Crapo (1979) for grid of rectangles
Some extensions with York undergraduate students 2003-2011

Given any plane framework, use directions of edges to place points on a circle. Where edges are opposite sides of a parallelogram, place on same point. Where there is a triangle, place edges between the corresponding vertices on the circle. This is the Zone Graph.



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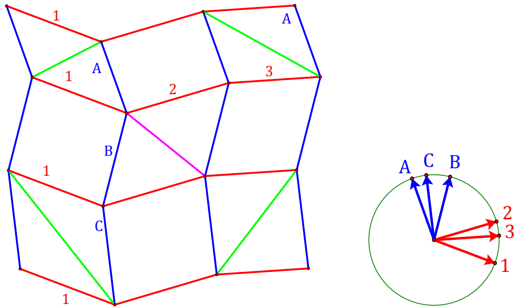

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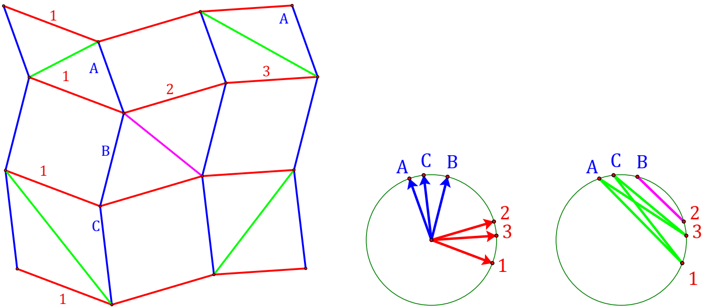

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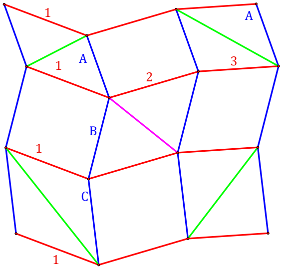

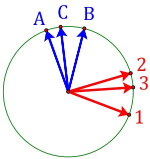

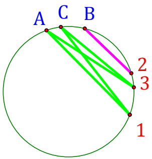

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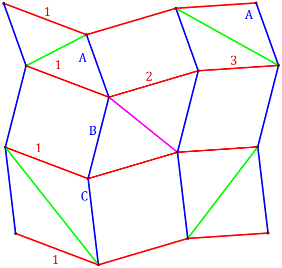
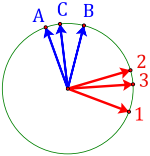
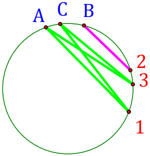






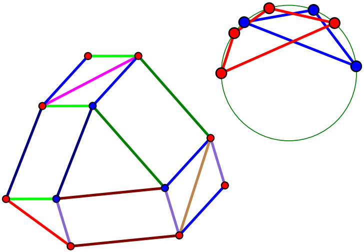
Theorem

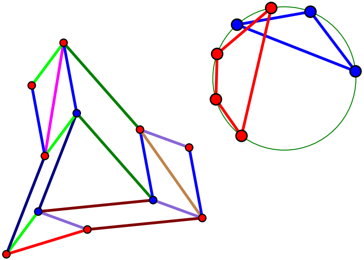
If the framework has an infinitesimal motion, then the framework on the circle is disconnected as a graph and has a finite motion. Conversely, If the original framework is a disc decomposed into parallelograms and triangles, then if the zone graph has a finite motion then the parallelogram disc has a finite motion.

3x3Grid

Plane Parallelogram - Triangle Discs

Need not be a grid - could be any disc of triangles and parallelograms





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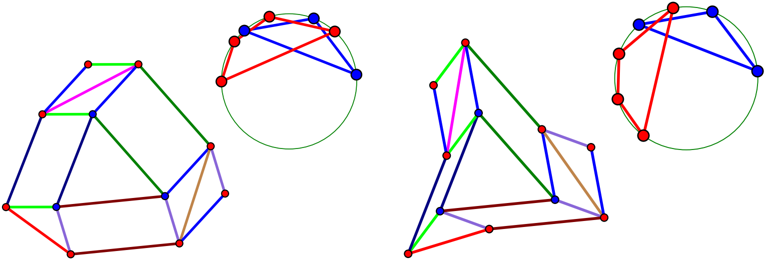


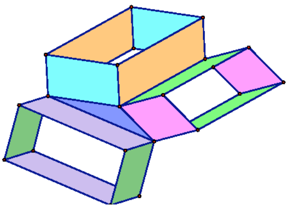
Can extend to symmetric zone graphs and symmetric frameworks,

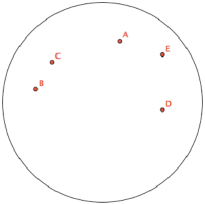
Can extend to periodic zone graphs and periodic frameworks,

Can extend to projected zonahedra (e.g. 5-cube)

3D Analog: Collared Spheres

Extensions to plane model
New - with some prior explorations of Camille Mittermeier





Consider a covering a disc or a sphere, composed of collared parallelograms, and rigid subpieces (eg. triangles) - a Collared Sphere. Use the directions of edges as points on the sphere, and place opposite edges of a collared parallelogram as a single point, and place edges on the sphere the angles in any triangles - the Spherical Zone Graph.

3D Analog: Collared Spheres

Theorem (W.)

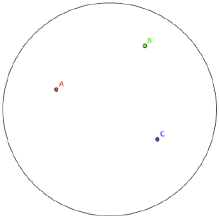
If the framework has an infinitesimal motion, then the spherical zone graph has an infinitesimal motion on the sphere. If the framework has a finite motion then the spherical zone graph has a finite motion. Conversely. If the spherical zone graph has a finite motion then the collared sphere has a finite motion.

3D Analog: Collared Spheres

Theorem (W.)

If the framework has an infinitesimal motion, then the spherical zone graph has an infinitesimal motion on the sphere. If the framework has a finite motion then the spherical zone graph has a finite motion. Conversely. If the spherical zone graph has a finite motion then the collared sphere has a finite motion.

Examples



3D Analog: Collared Spheres

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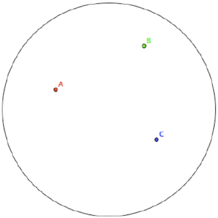
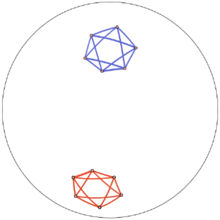




Transfer of motions on a sphere to motions of 3D frameworks

What Else? Extruded 2D Mechanisms

A “’2.5 dimensional” mechanism.

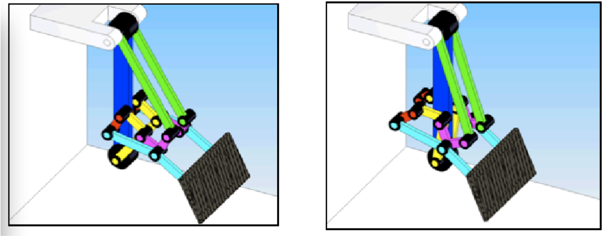
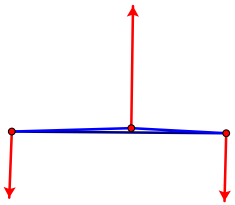


Figure: a 3-D mechanism which ‘cones’ a 2D mechanism

Many 3-D machines have an extruded version of a 2D mechanism - with two copies of some parts, sandwiching another part. It is related to a vertical extruded coned and avoids collisions etc. which would otherwise happen.

What Else? Slide joints and points at Infinity

Consider a collinear triangle with joints at infinity:



Collinear Triangle

What Else? Slide joints and points at Infinity

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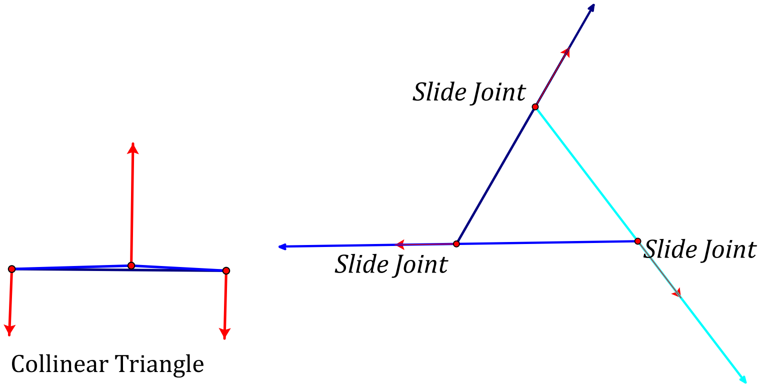
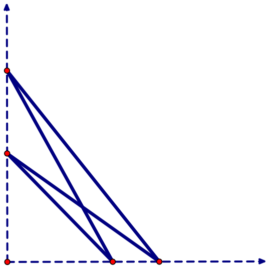


three slide joints - joints at infinity in the algebra, then a finite motion.

What is ‘enough’ infinity?

In general, an experience is that some points - ‘enough points’ at infinity, make a framework flexible.

Slider Mechanism



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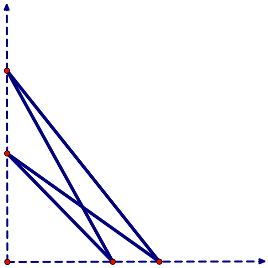

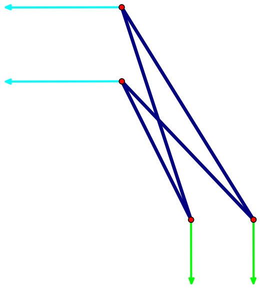

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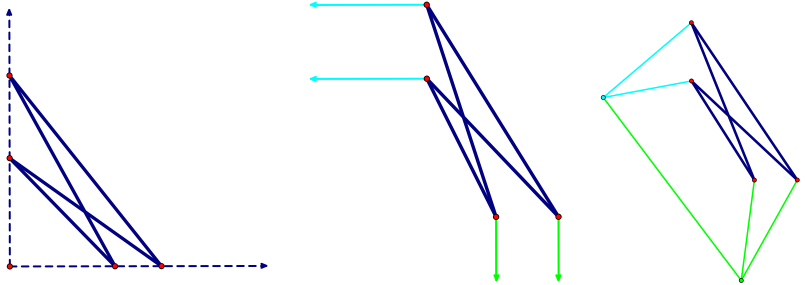

Slider framework - a flexible $K_{3,3}$ with two joints (and a bar) at infinity.

Other directions

1. body-bar frameworks;
2. molecular frameworks;
3. periodic frameworks;
4. infinite frameworks not restricted to periodic motions;
5. other symmetric with vertices or edges fixed by the symmetries;
6. more on transformations within and among metrics:
7. symmetry preserving polarity about sphere centered on point group center;
8. other algebraic varieties inducing finite motions at regular points;
9. continuing search for flexible polytopes in higher dimensions;
10. more on higher dimensions;
11. composition of finite motions;

Thanks

to many collaborators, students

and others who generated key examples and methods

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Questions?

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whiteley@mathstat.yorku.ca