The Rigidity Transition in Random Graphs

Louis Theran
(joint work with Shiva Kasiviswanathan and Cris Moore)

Frameworks

- Planar structure
- ... made of *fixed-length bars*
- ... connected by *universal joints* with full rotational degrees of freedom
- Allowed continuous motions preserve *length* and *connectivity* of the bars
  - No "stretching" no "breaking"
- *Rigid* if only Euclidean motions allowed

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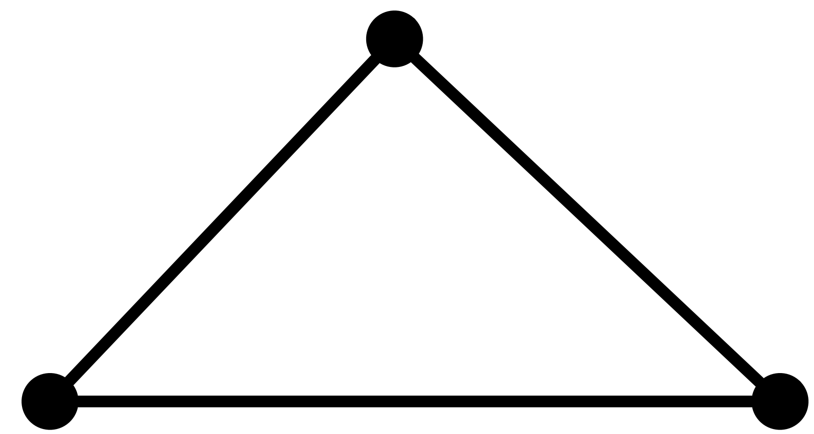

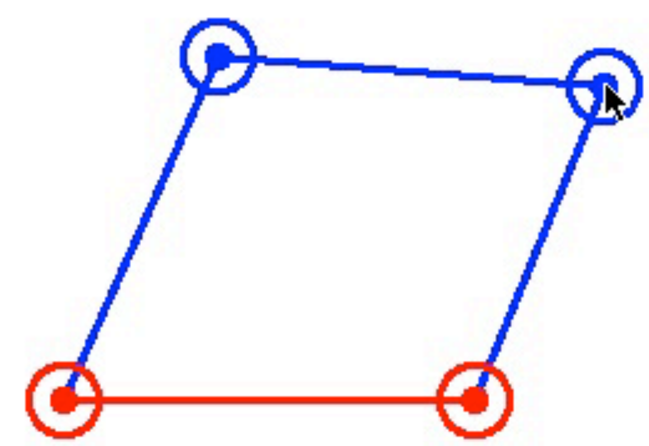

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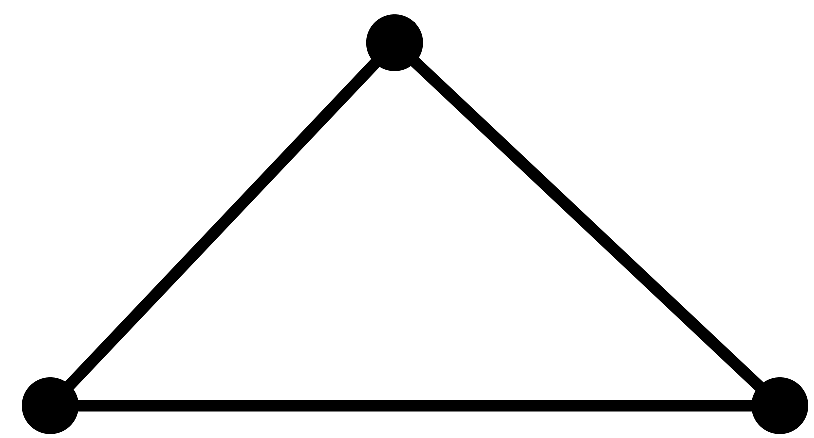
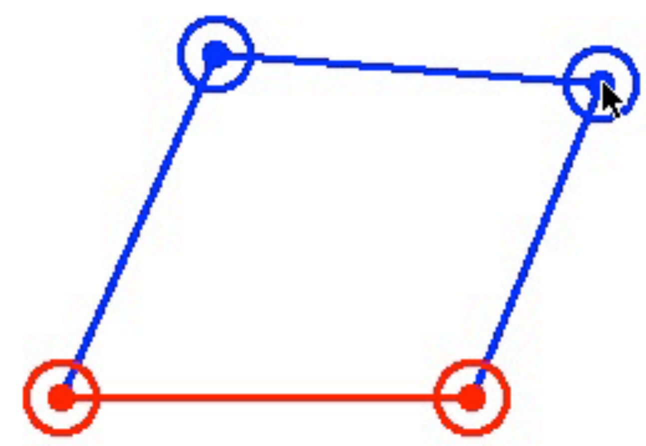


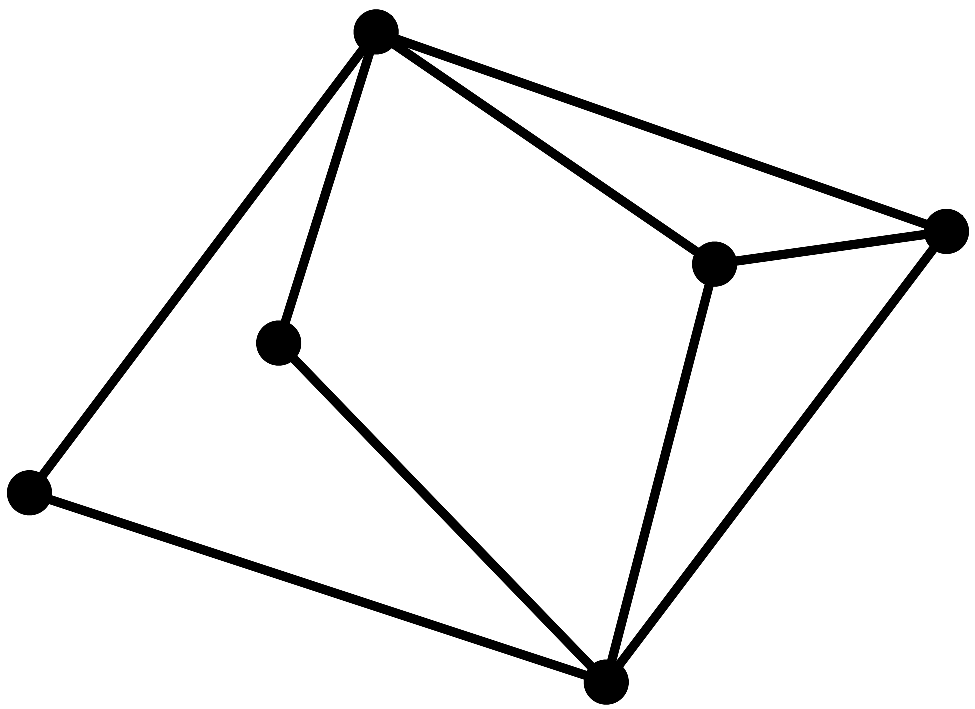
Rigid



Flexible

Combinatorial rigidity

- A graph G=(V,E) with n vertices and m edges is a *Laman graph* if m=2n - 3 and all subgraphs satisfy $m' \leq 2n'-3$
- G is *Laman-spanning* if it has a Laman graph as a spanning subgraph
- Maxwell-Laman Theorem ('64,'70): Generically, *rigid blocks* of frameworks correspond to Laman-spanning subgraphs, *rigid components* correspond to *maximal* Laman-spanning subgraphs



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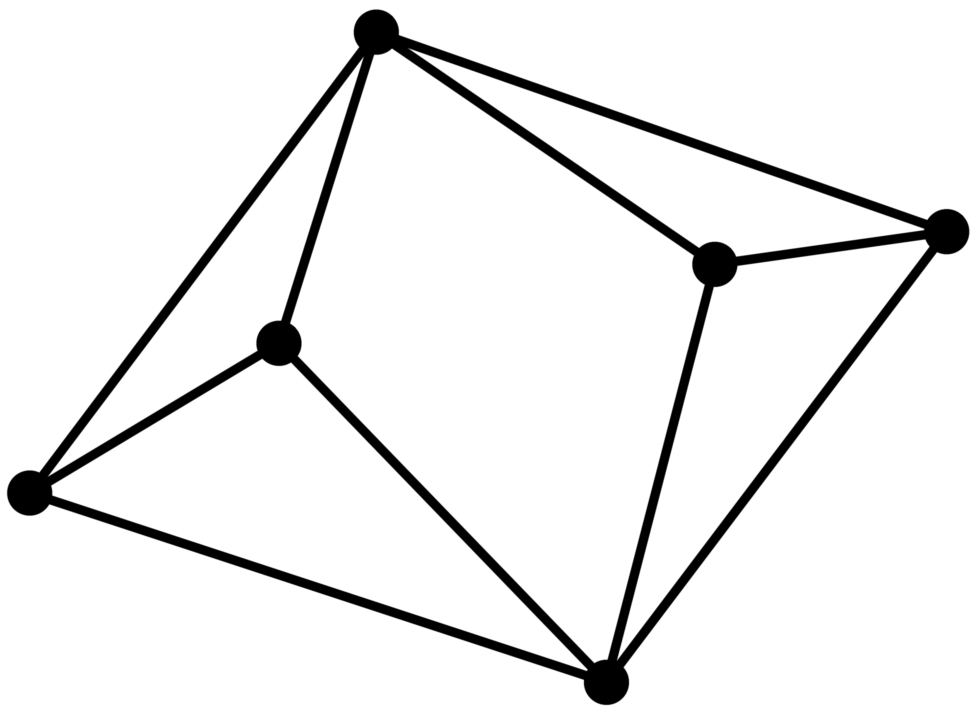

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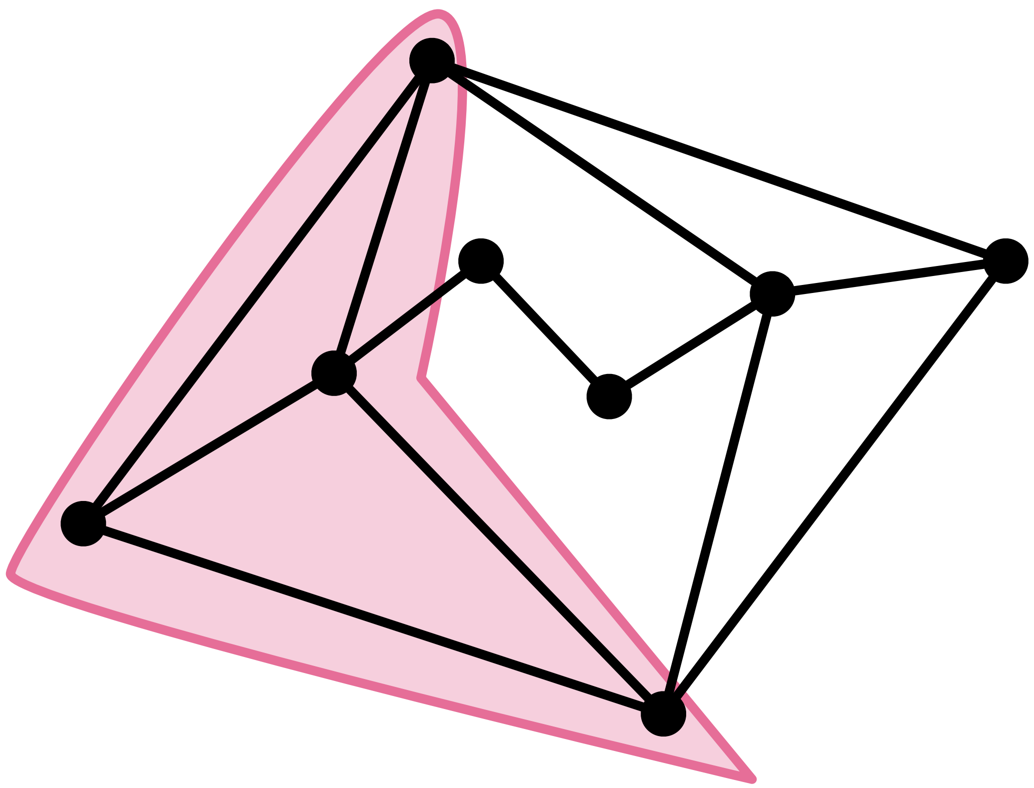

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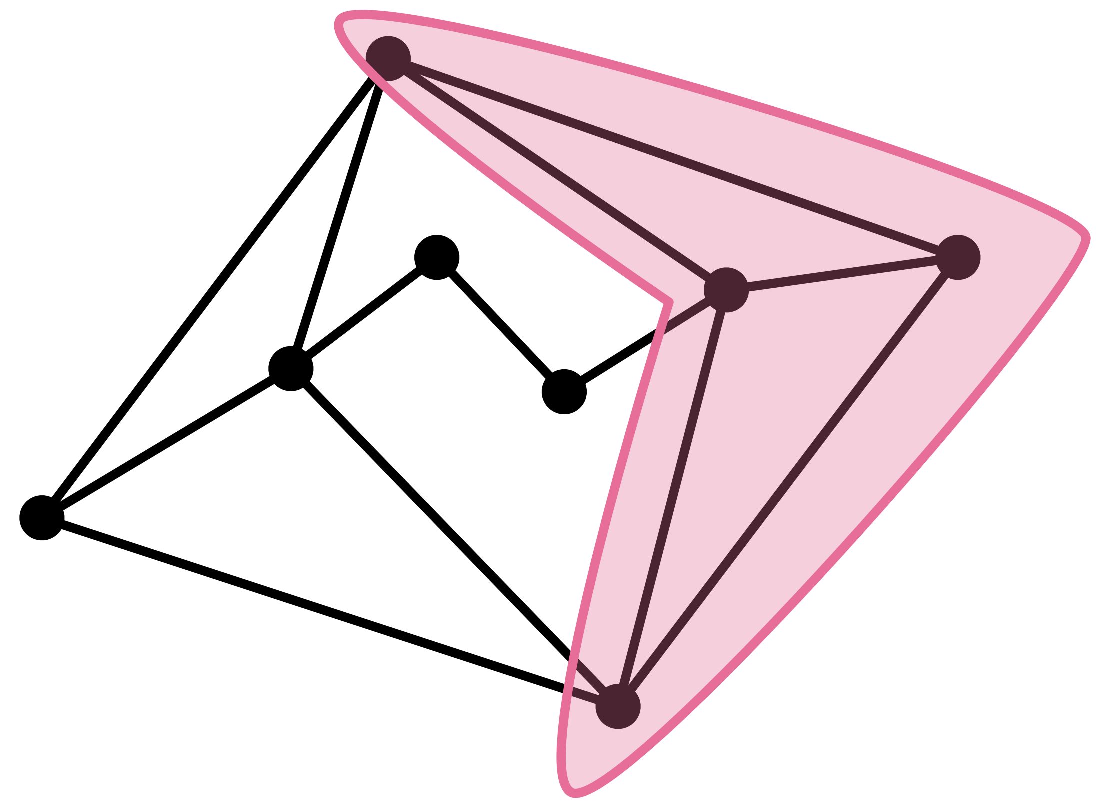

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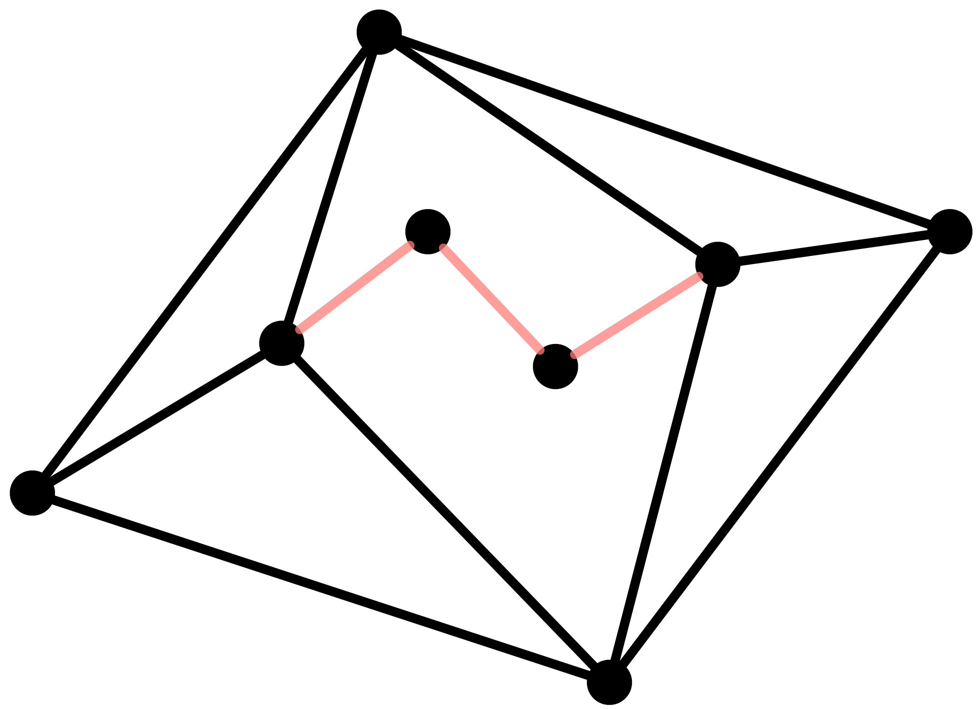
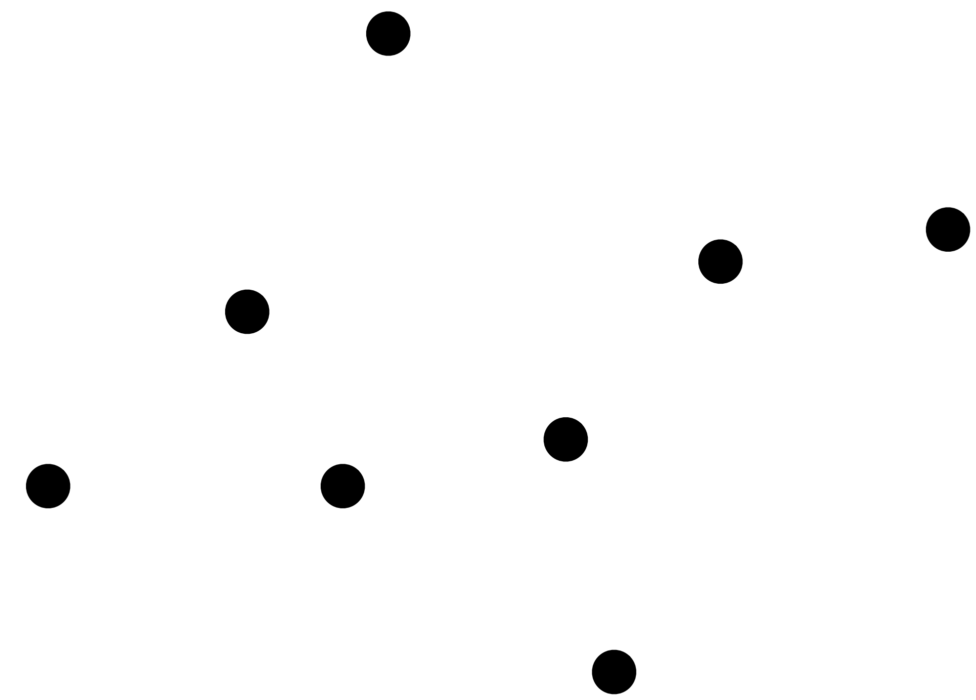


A random process

- Start with n "generic" points in the plane and then...
- Uniformly at random:
  - select a pair of points and fix the distance between them (removing a potential motion)
- *Rigid components* form
  - Maximal rigid blocks



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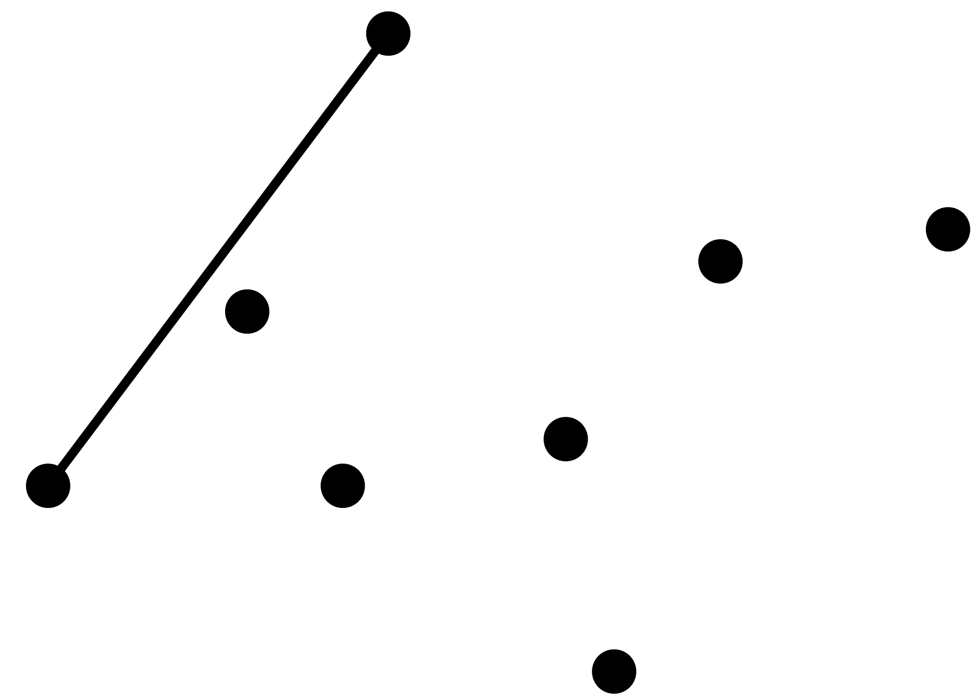

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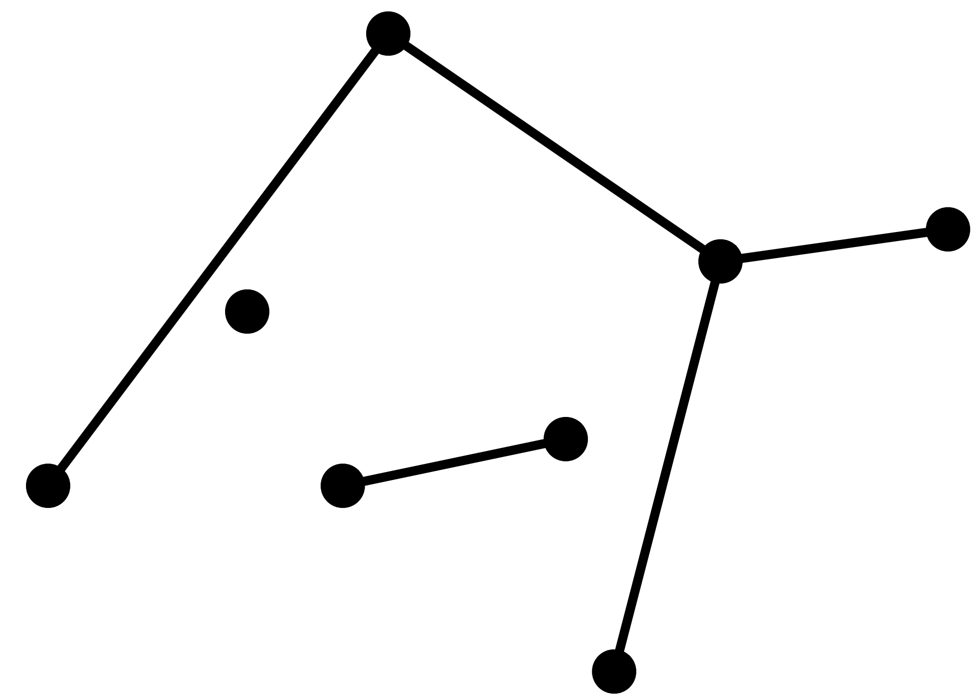

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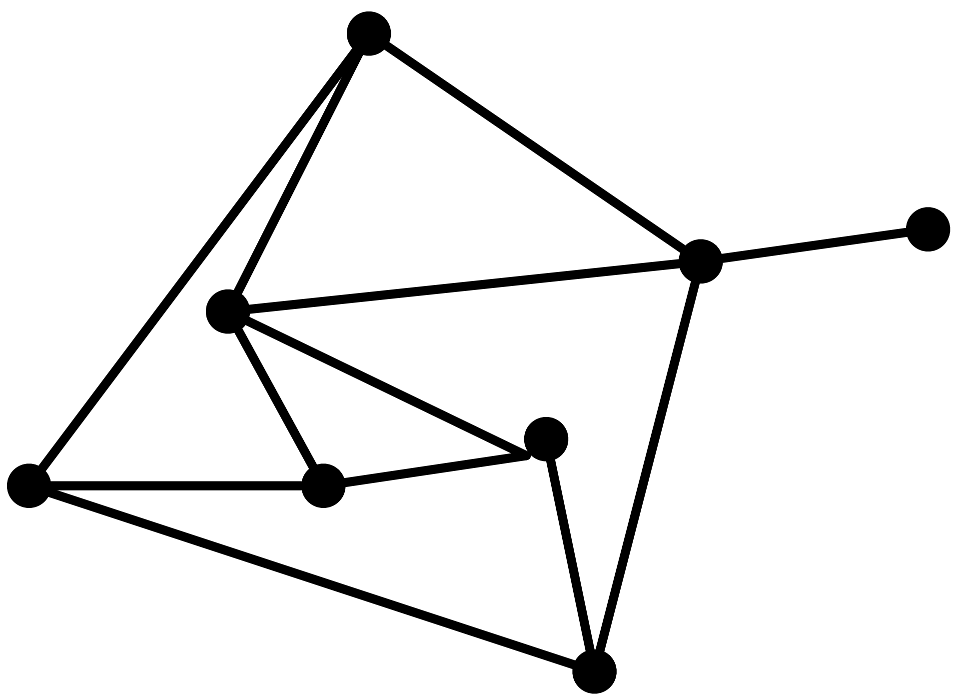

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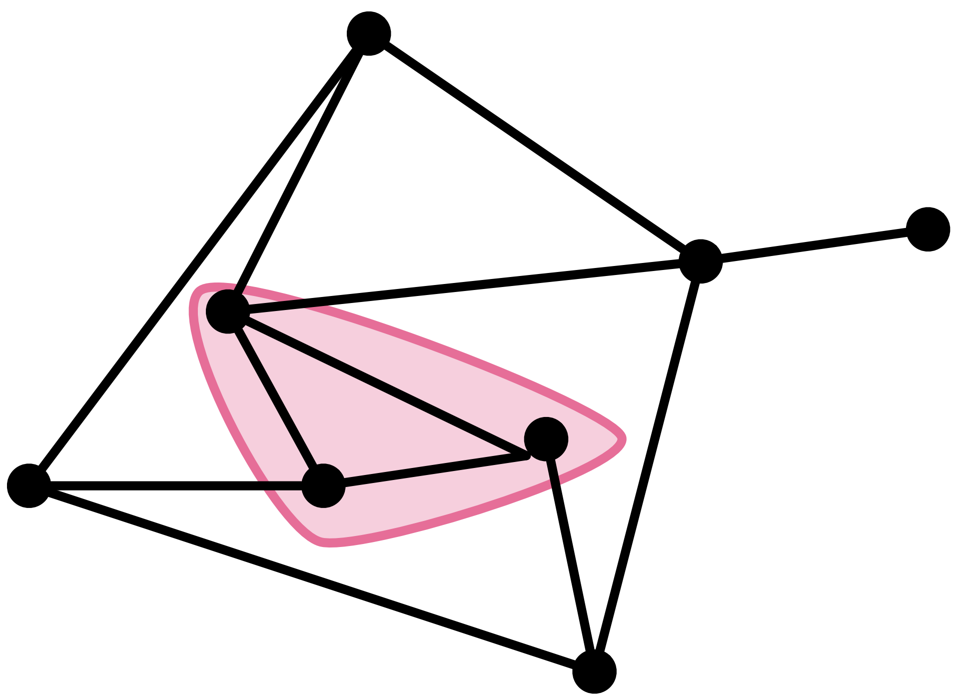

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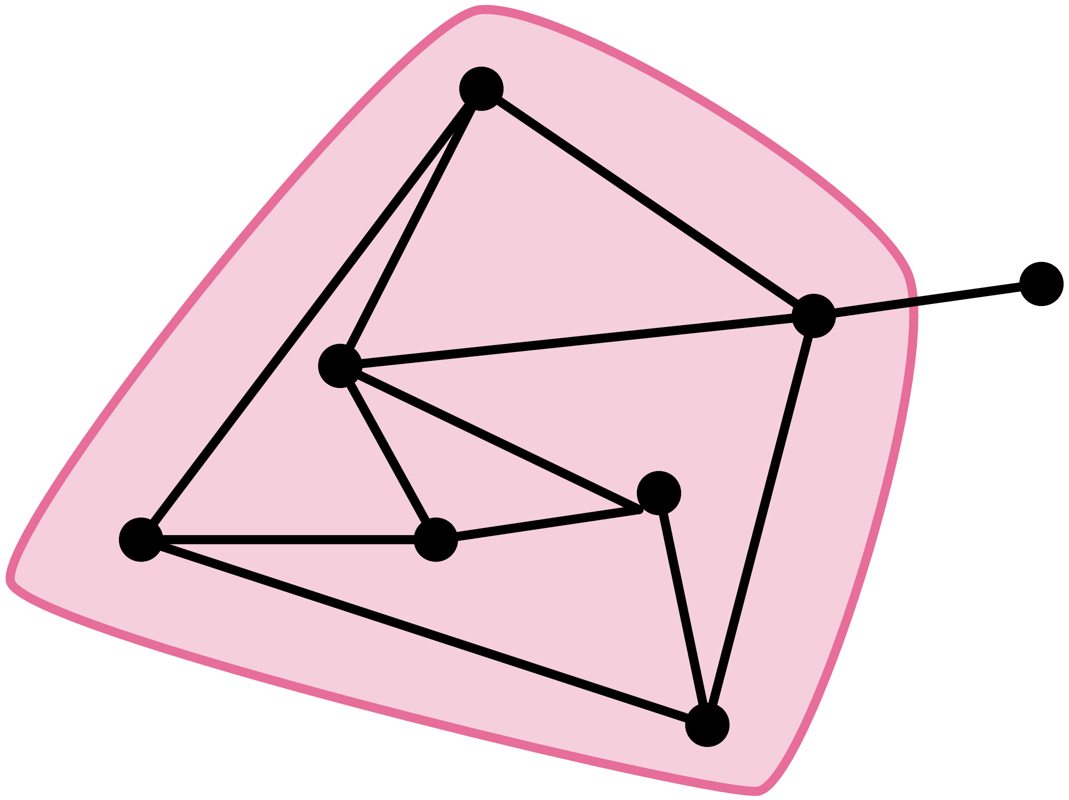
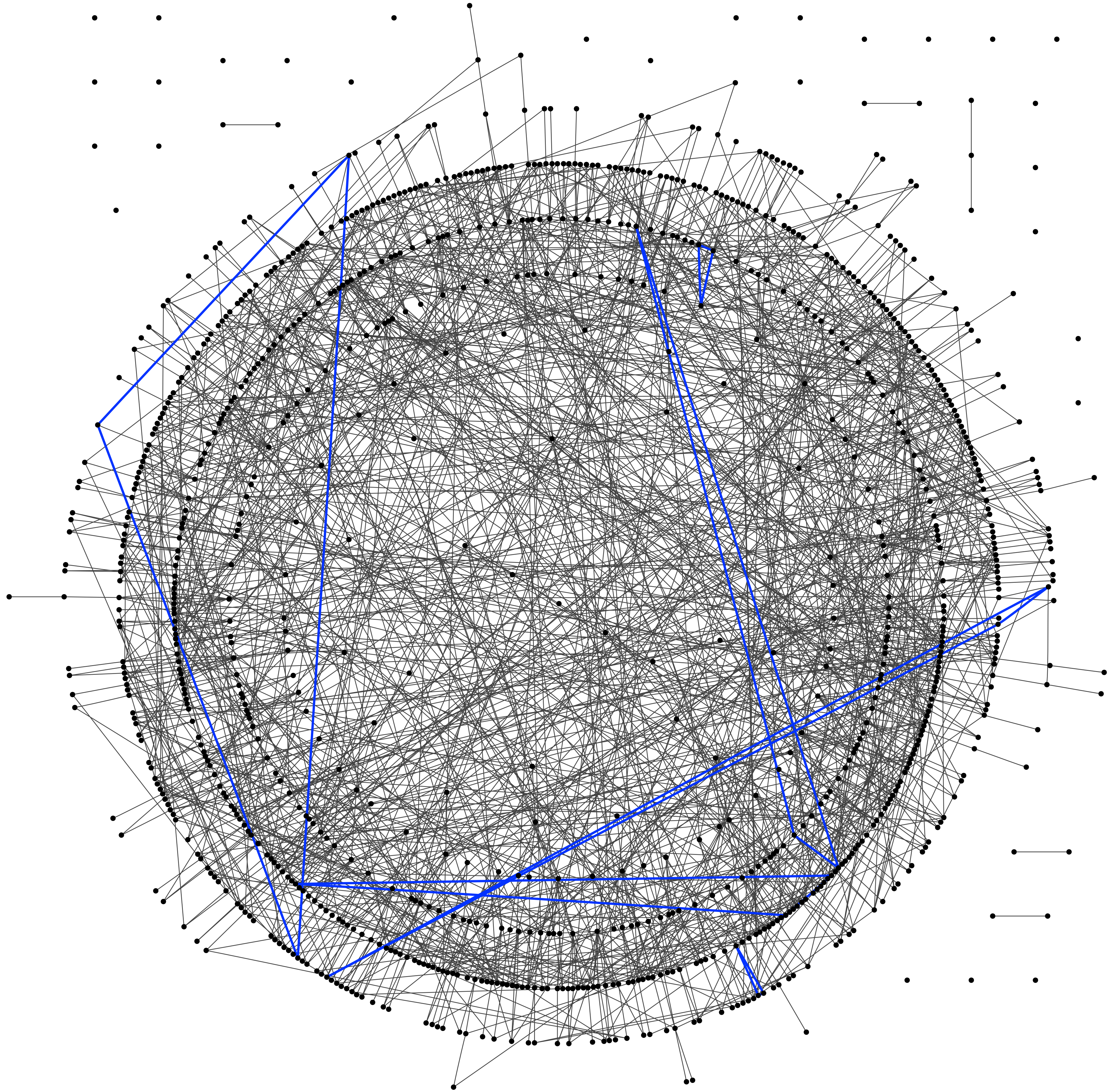
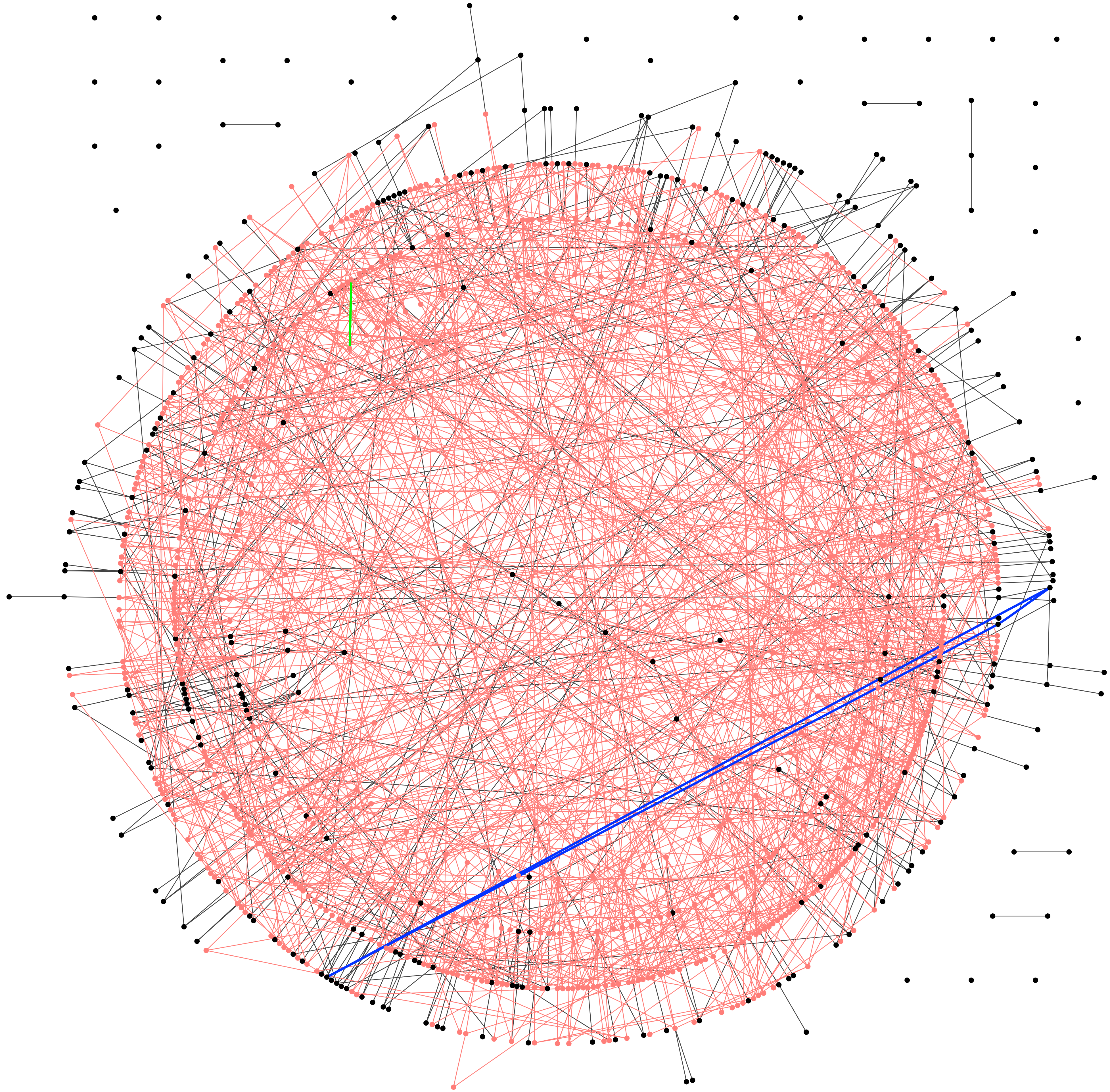
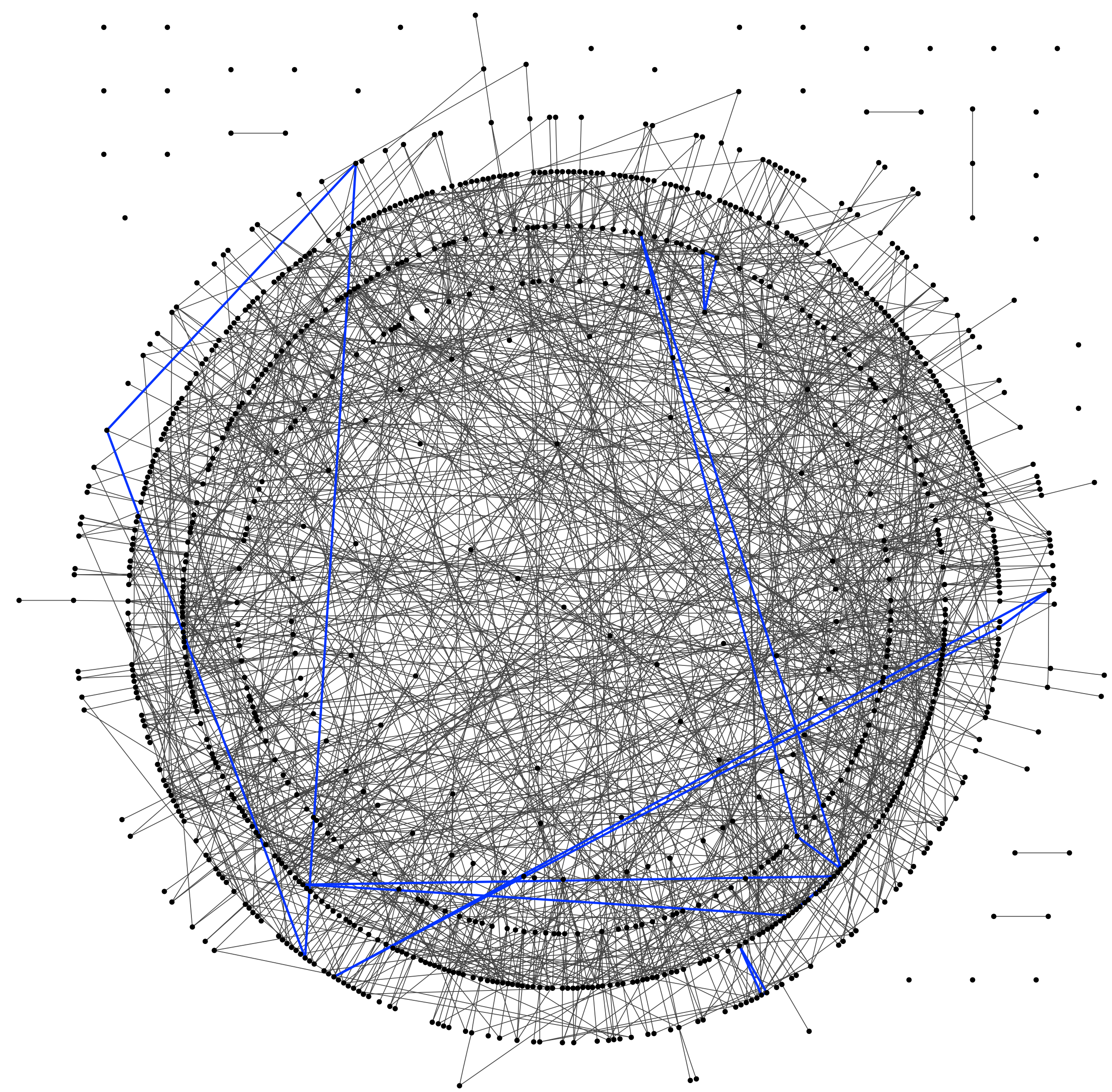






Random graphs

- A *random graph* is really a distribution over graphs with *n* vertices
- Simplest model is "Erdös-Rényi" G(*n*,*p*)
  - Each edge in with prob. *p* independently
- Here we want "w.h.p." statements like
  - Prob[big comp.] = 1 - o(1)

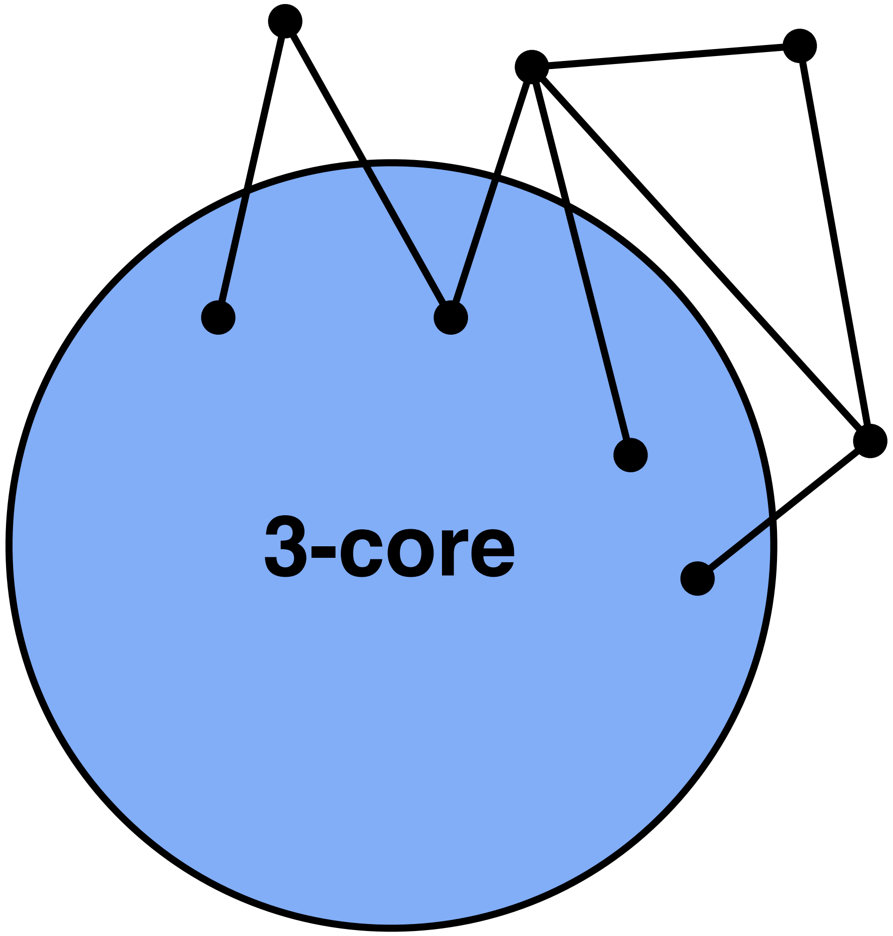


Questions

- **Q:** After *how many random edges* will we see a "big" component (w.h.p.)?
  - #edges $\approx$ avg. degree c in G(n,c/n)
- **Q:** *How many big rigid components* will there be?
- **Q:** *How many vertices* will the component span?

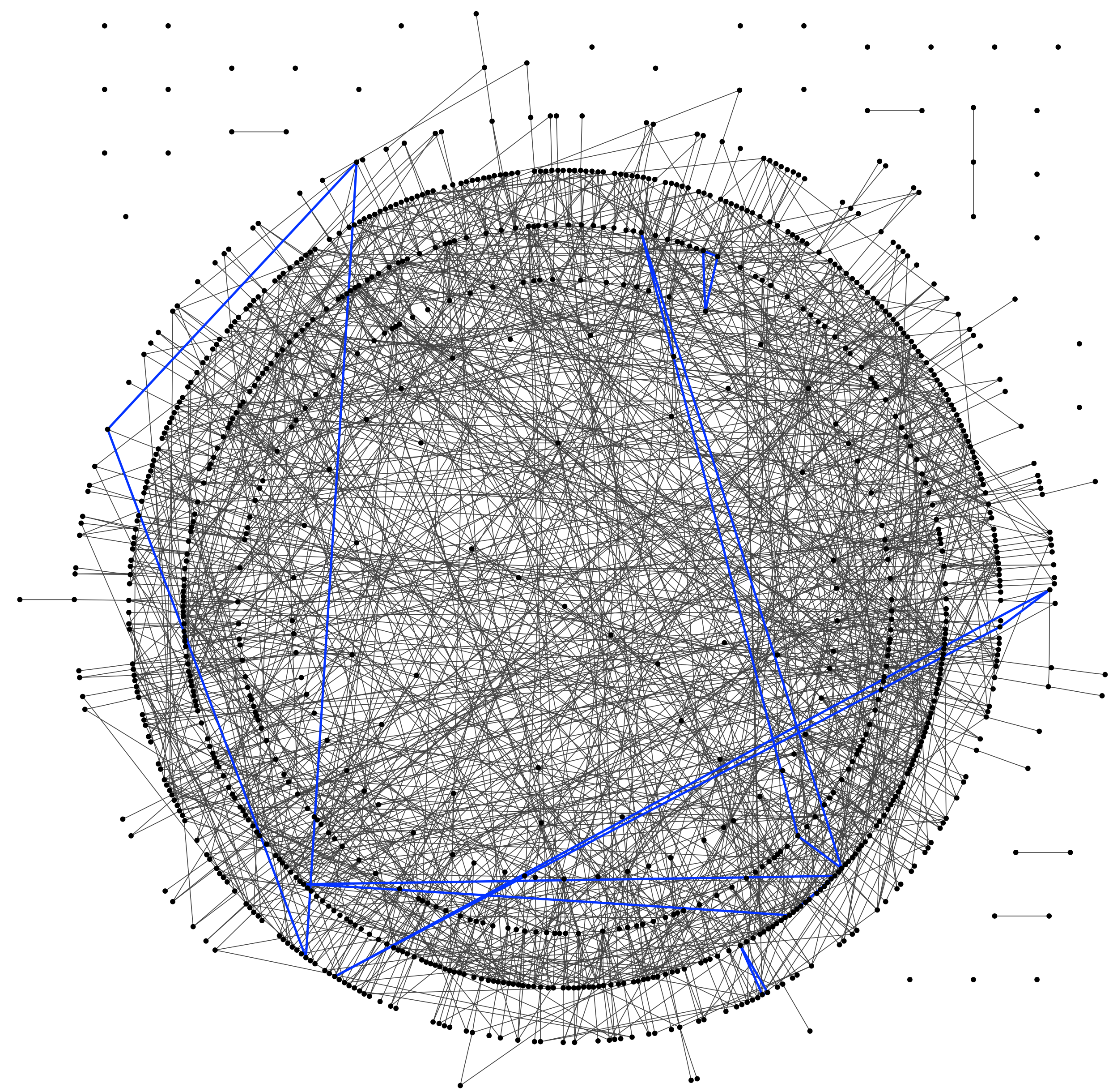
The Rigidity Transition

- At *what average degree* will we see a "big" component?
- A *linear-sized rigid component* appears w.h.p. in G(*n*,*c*/*n*) when c > 3.588...
- c < 3.588 all *tiny* rigid comp.
- *How many big components* will there be?
- The giant component is *unique*.
- *How many vertices* will the components span?
- The giant component spans a (1-o(1))-fraction of the "(3+2)"-core



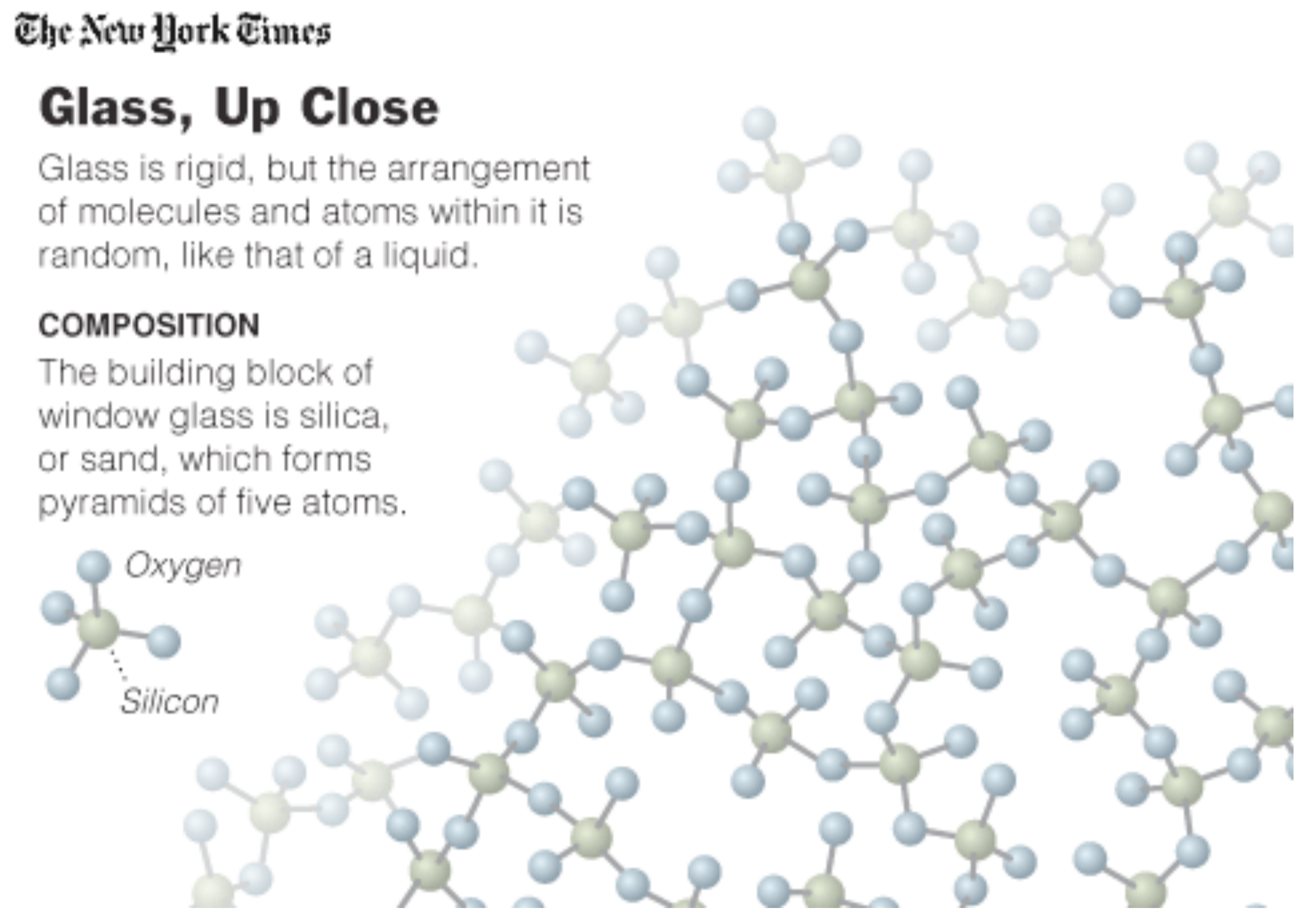
A sparse G(n,p)...

- w.h.p. has isolated vertices and leaves
  - *so no hope to be rigid*
- For c = 3.588 w.h.p.
- *no* induced subgraph of minimum degree 4
  - i.e., below the threshold for the "4-core"
- *linear size* (~30%) *3-core*
- larger *(3+2)-core*
  - *conjecturally* 75%



Physics

- These problems have been studied, via simulation, in the *statistical physics* community
- Motivation is phase transition in *network glasses*
- Explosive growth of a rigid component in the 3-core at 1.749n edges [Rivoire & Barré]
- Other work by [Moukarzel; Thorpe, et al; Jacobs & Thorpe]

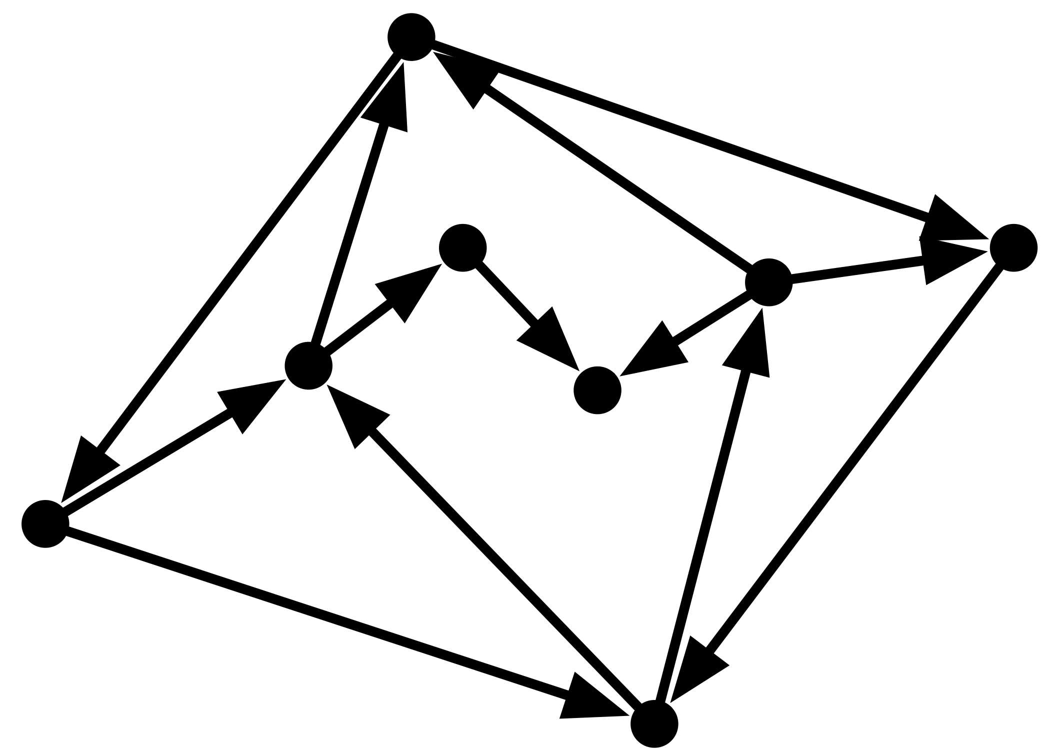


Rigidity

- [Jackson, Servatius, Servatius'07]
  - Random 4-reg. graph is w.h.p. *globally rigid*
  - G(n,p) w.h.p. *rigid* ~ avg. deg. log n
- [T. '09] G(n,c/n) all rigid components are *tiny* (≤3 vertices) or *giant*

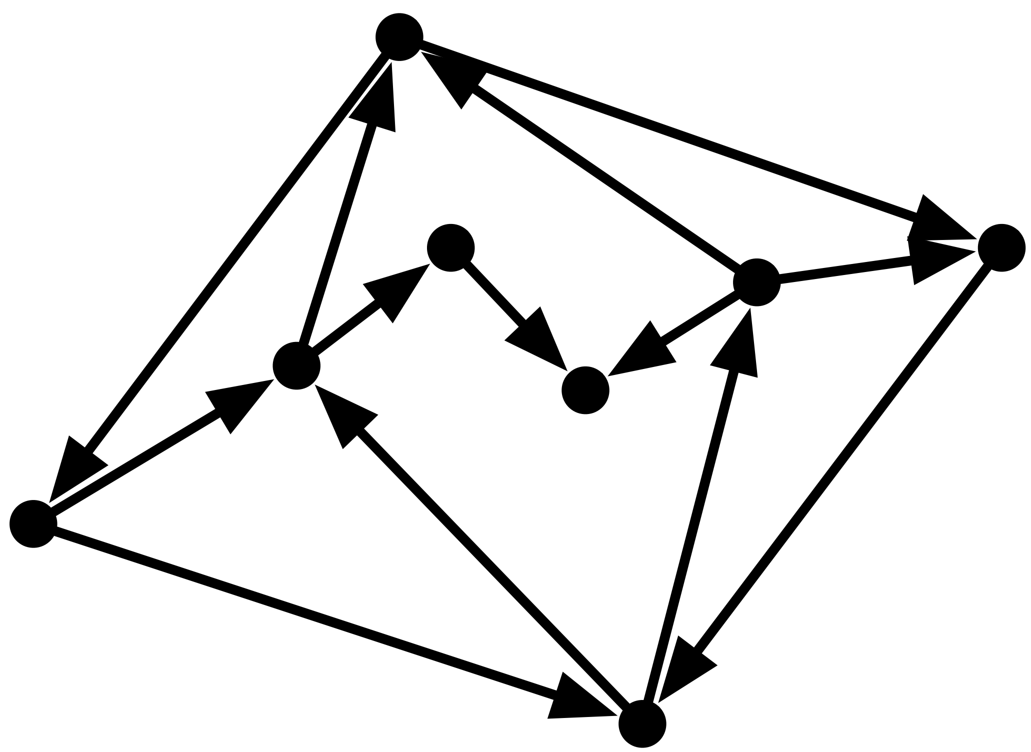
Random graphs

- The constant 3.588... is the threshold for *2-orientability* in G(n,c/n) [Fernholz & Ramanchandran; Cain, et al. SODA'07]
- 2-orientability means G can be oriented s.t. out-degree ≤ 2 for all vertices
- 3.588 is also the threshold for the 3-core to reach avg. degree 4 [FR;CSW]
- We're going to use this later



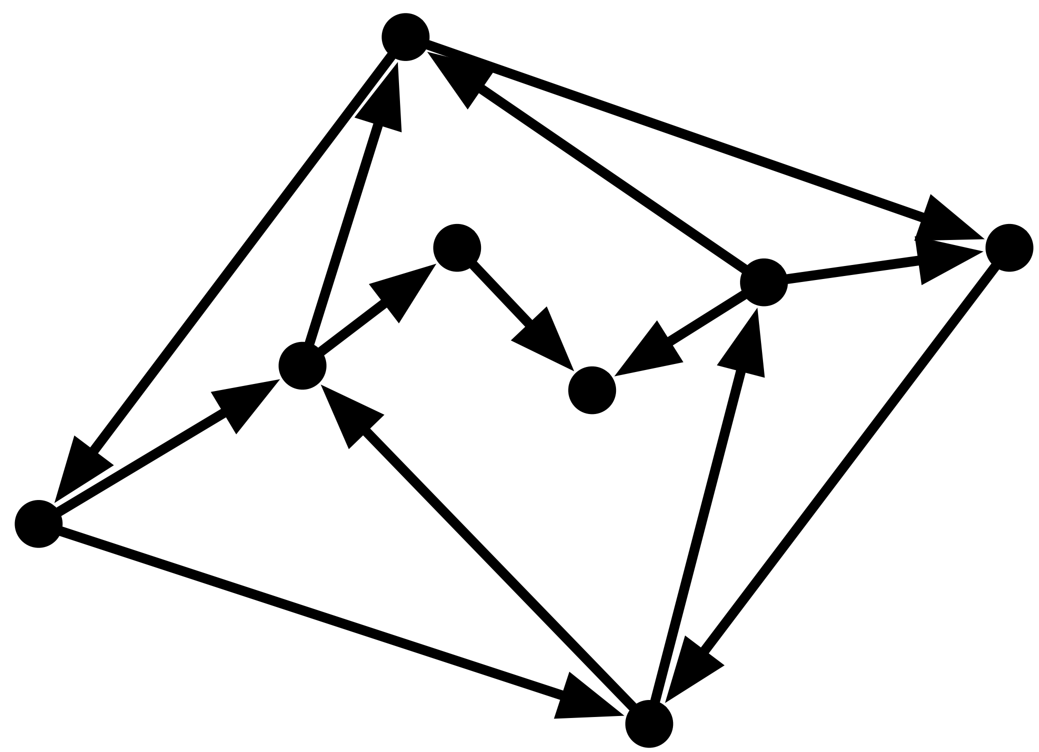
2-orientability

- To see the relevance of 2-orientability, recall the Laman counts:
  - $m' \leq 2n' - 3$
- Alternative way to say "G is 2-orientable" is "(2,0)-sparse"
  - $m' \leq 2n'$
- Any large rigid component implies not 2-orientable *with const. probability*
- Basic intuition: with enough randomness and edges, these conditions behave similarly



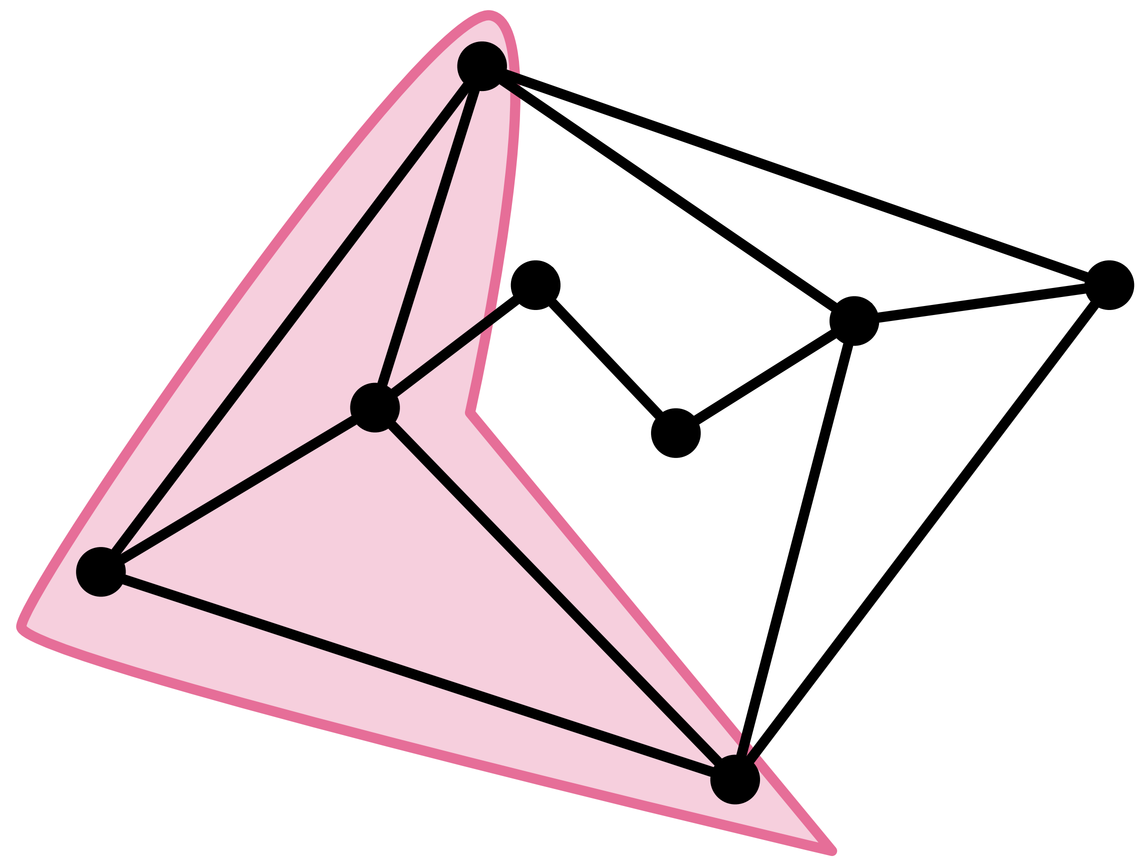
Above the threshold

- [FR; CSW] don't say what happens *above* the threshold for 2-orientability
- We give a bound on how many vertices can get out-degree *at least* 2
- **Theorem:** If $c > 3.588$ then, w.h.p., the 3-core of $G(n,c/n)$ has an orientation such that all but $o(n)$ vertices have out-degree ≥ 2.
- Implies the same for the (3+2)-core



Standard facts

- Rigid blocks and components are *vertex induced subgraphs*
- If G has $m \geq 2n' - 3$ edges and is simple, then G has a rigid block on ≥ 4 vertices
  - "non-trivial block"
- This block has minimum degree 3, if not all of G
- Adding a vertex of degree two with both neighbors in a rigid block makes a larger rigid block
- Adding ≥ 3 edges between rigid blocks makes a larger rigid block



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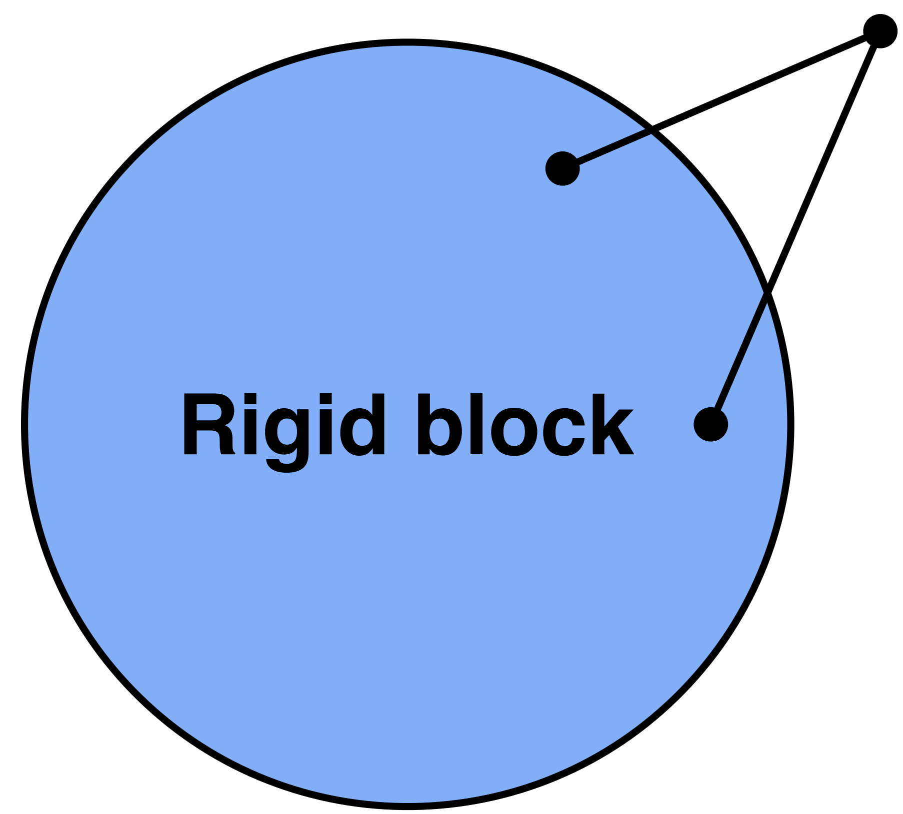

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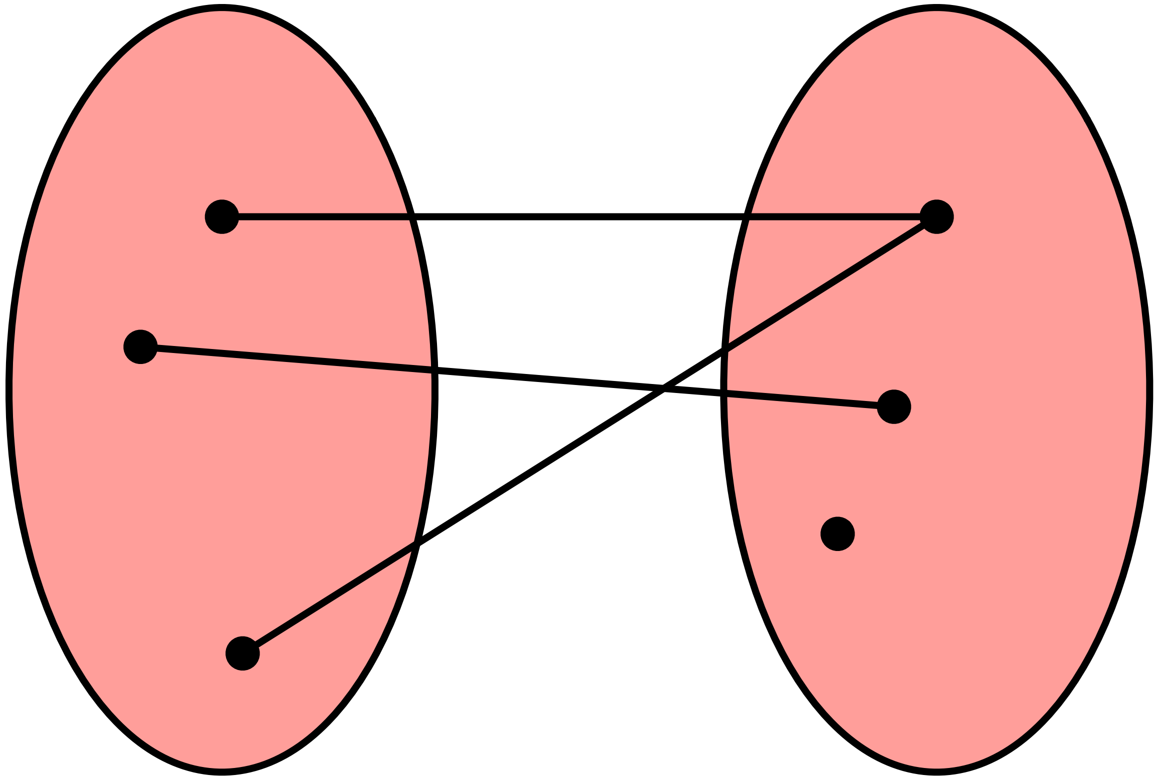
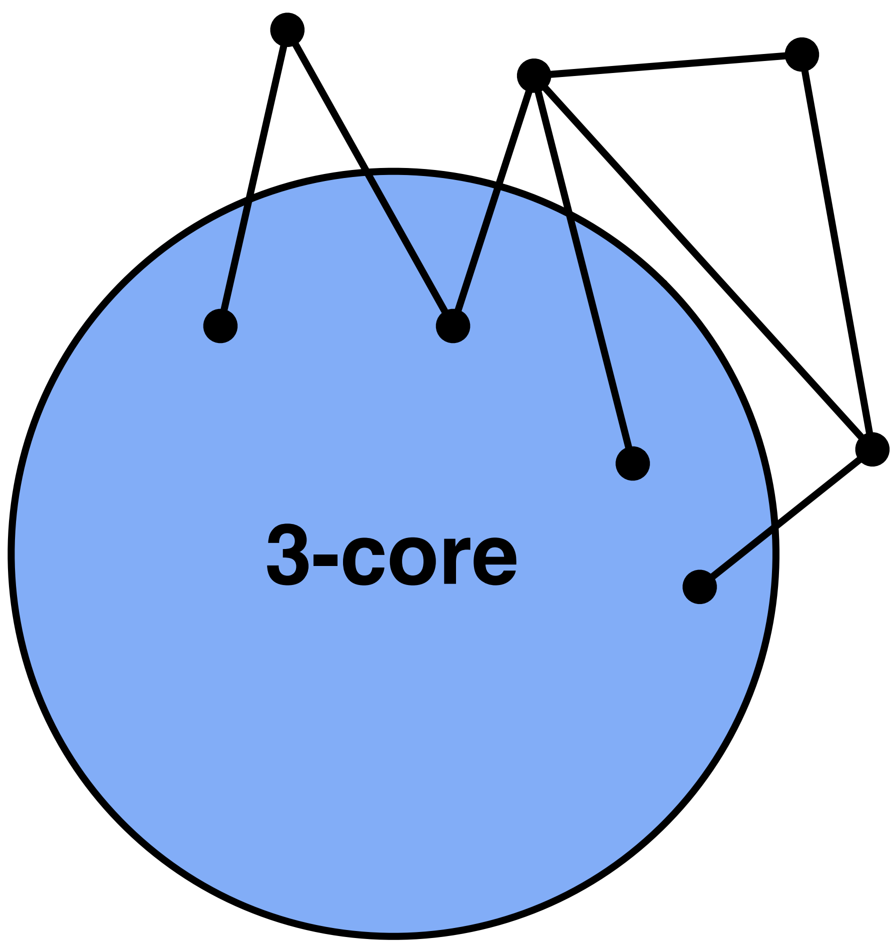


Graph theoretic lemma

- The *(3+2)-core* is inductively defined starting from the 3-core and adding degree 2 vertices
- If G is simple and coincides with its (3+2)-core, has a rigid component G' on n' vertices and G\G' is incident on at least 2(n-n') edges, then either:
  - G is Laman spanning
  - G has a rigid component that is not G'
- "Can't avoid rigidity or multiple components"



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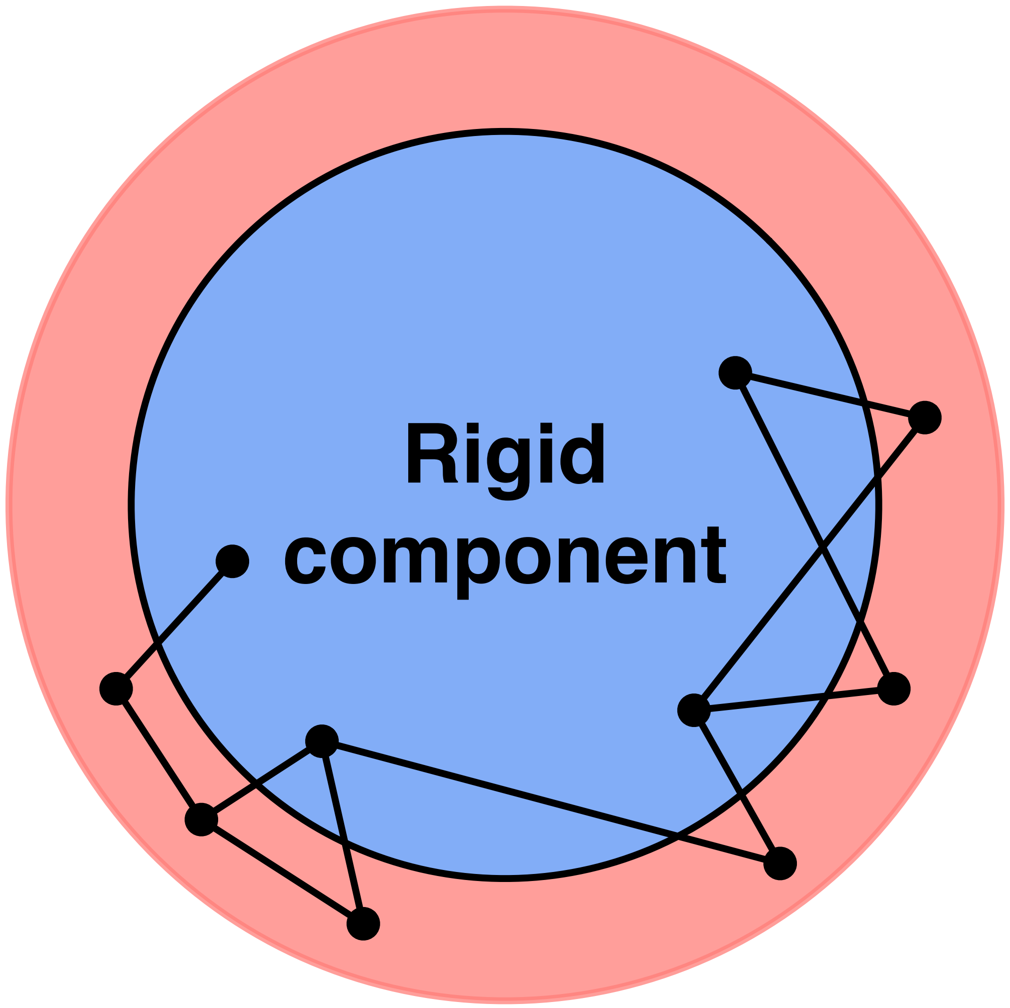
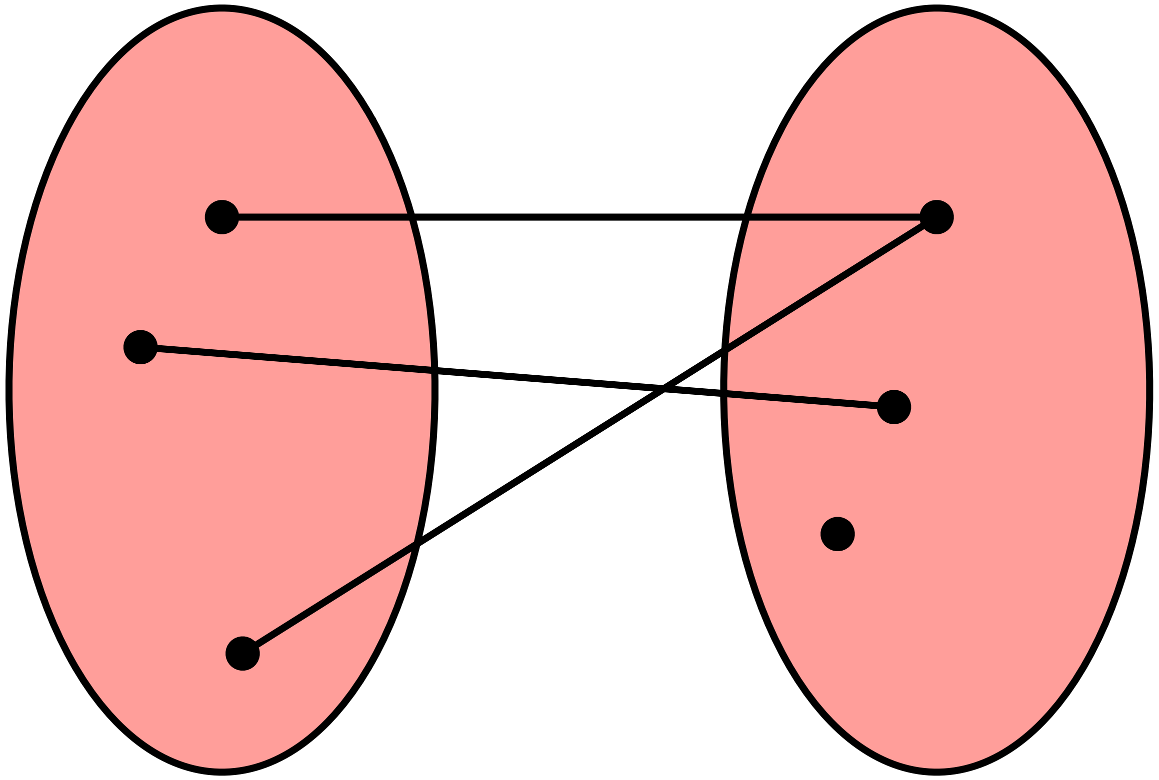


4-reg. case revisited

- If a graph is 4-regular, the Lemma *always* applies
- Let G be a random 4-regular graph, we'll recover [JSS]
- Trick 1: *exploit density*
  - rigid components have m/n ≥ 1.5, counting arguments imply all tiny or linear sized [T]
- Trick 2: *use expansion*
  - rigid blocks with 3 edges between them make a larger block
  - two big components survive with prob. o(1)



Size proof: warm up

- Fix c > 3.588, and assume the 3-core of G(n,c/n) has an *out-degree ≥ 2 orientation*
- This time look at the 3-core:
  - avg. deg. ≥ 4 implies giant rigid component and it's unique
- Counting edges by their tails Lemma applies
- Uniqueness of the giant component says that w.h.p., the (3+2)-core *is* Laman-spanning

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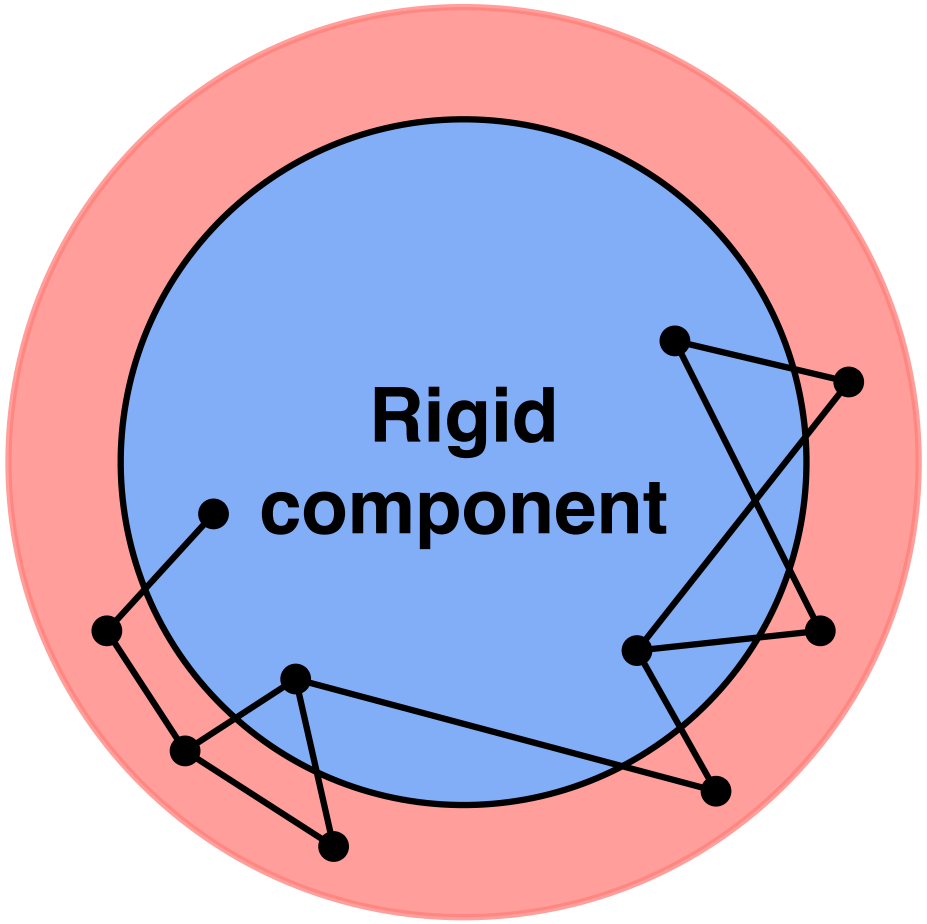

Putting it together

- Let c > 3.588 and G=G(n,c/n)
- We *don't know* that the 3-core has an out deg. exactly 2 orientation
- Lemma doesn't apply
- By 2-orientability theorem, w.h.p. need only o(n) more edges
- Add o(n) more uniform edges
- Lemma applies to the original (3+2)-core; w.h.p. it's Laman-spanning
- Show that the new (3+2)-core grows by o(n) vertices w.h.p.

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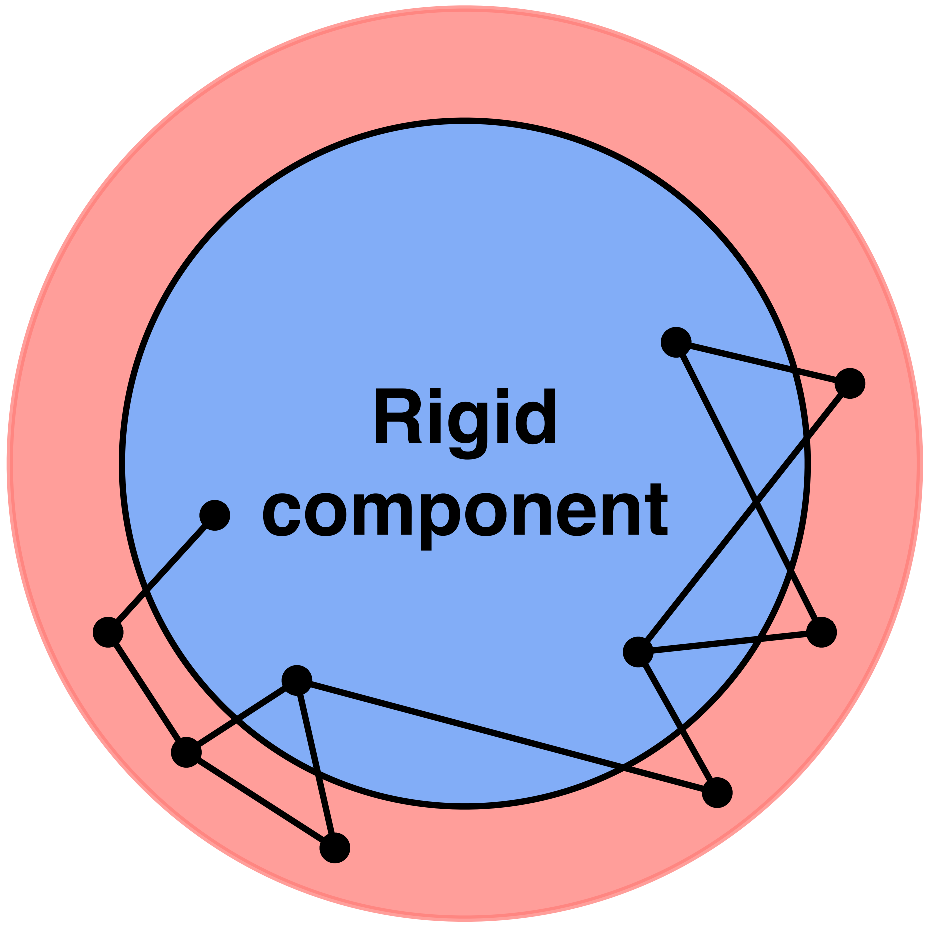
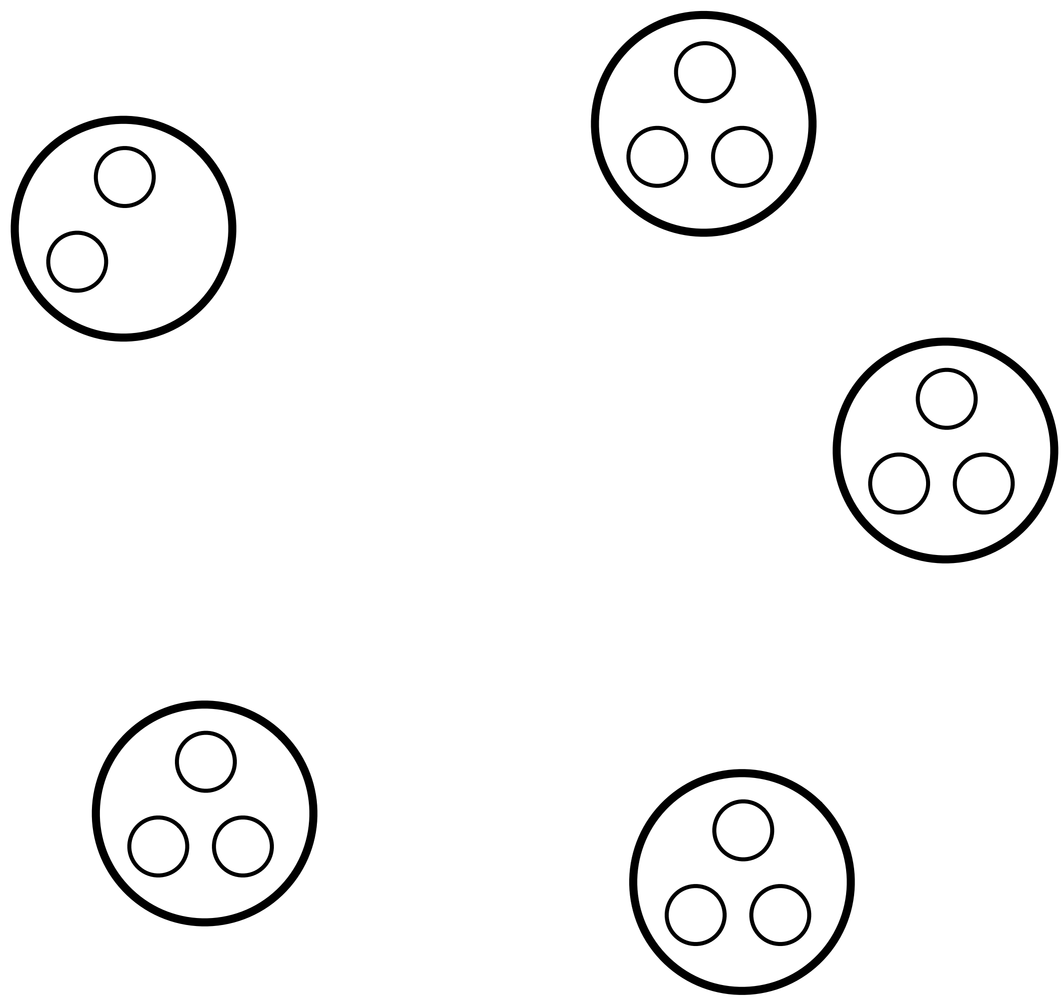


Configurations

- What's left to do is prove the 2-orientation theorem
- This is easier to analyze in an equivalent model called *configurations*
  - Really a random multigraph
  - Simple with prob. > 0
- Recall: Erdös-Renyi was "flip a coin for every edge"
- Configuration model:
  - generate degrees
  - match up the "copies" of vertices
- Poisson degrees eqv. to Erdös-Renyi



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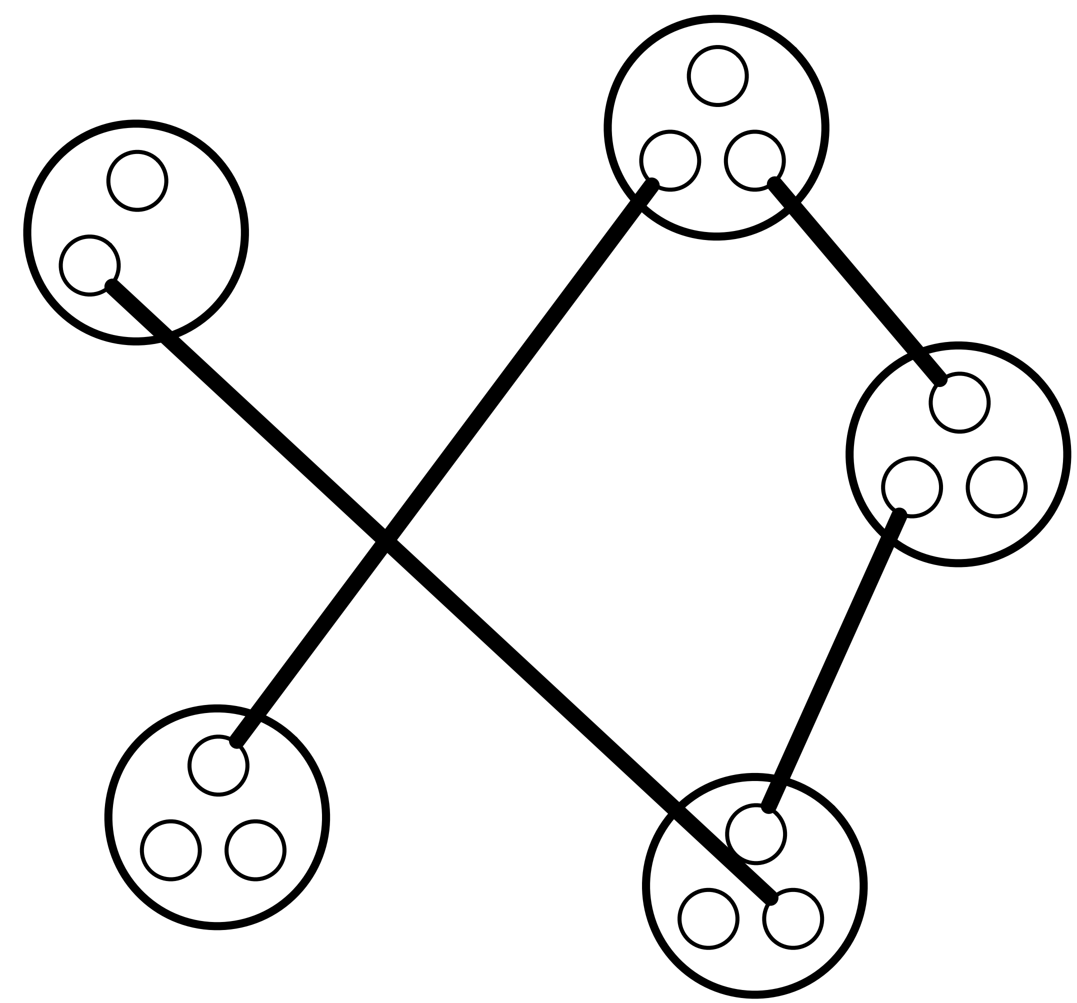

Generating configurations

- Can match up the vertex copies with *any* algorithm that gives a uniform matching
- Fernholz and Ramachandran define two "moves"
- **FR1:** Remove a degree d $\leq$ 2 vertex and two uniformly selected copies, recurse
- **FR2:** Remove a degree 3 vertex:
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  - "split a uniformly selected edge"
- **FR2** is just a Henneberg 2 move!

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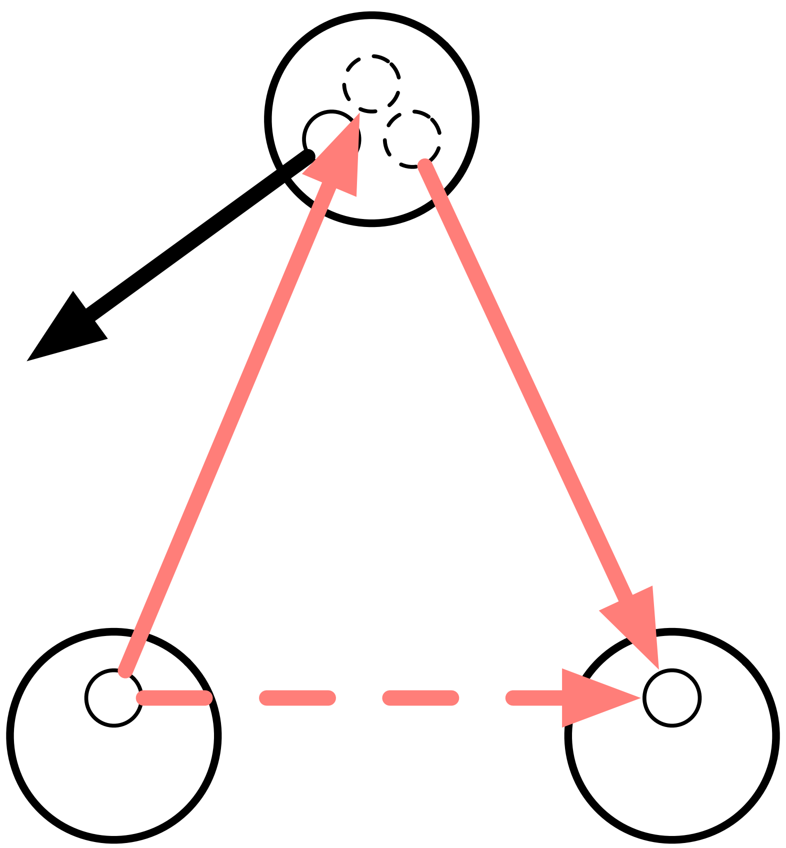
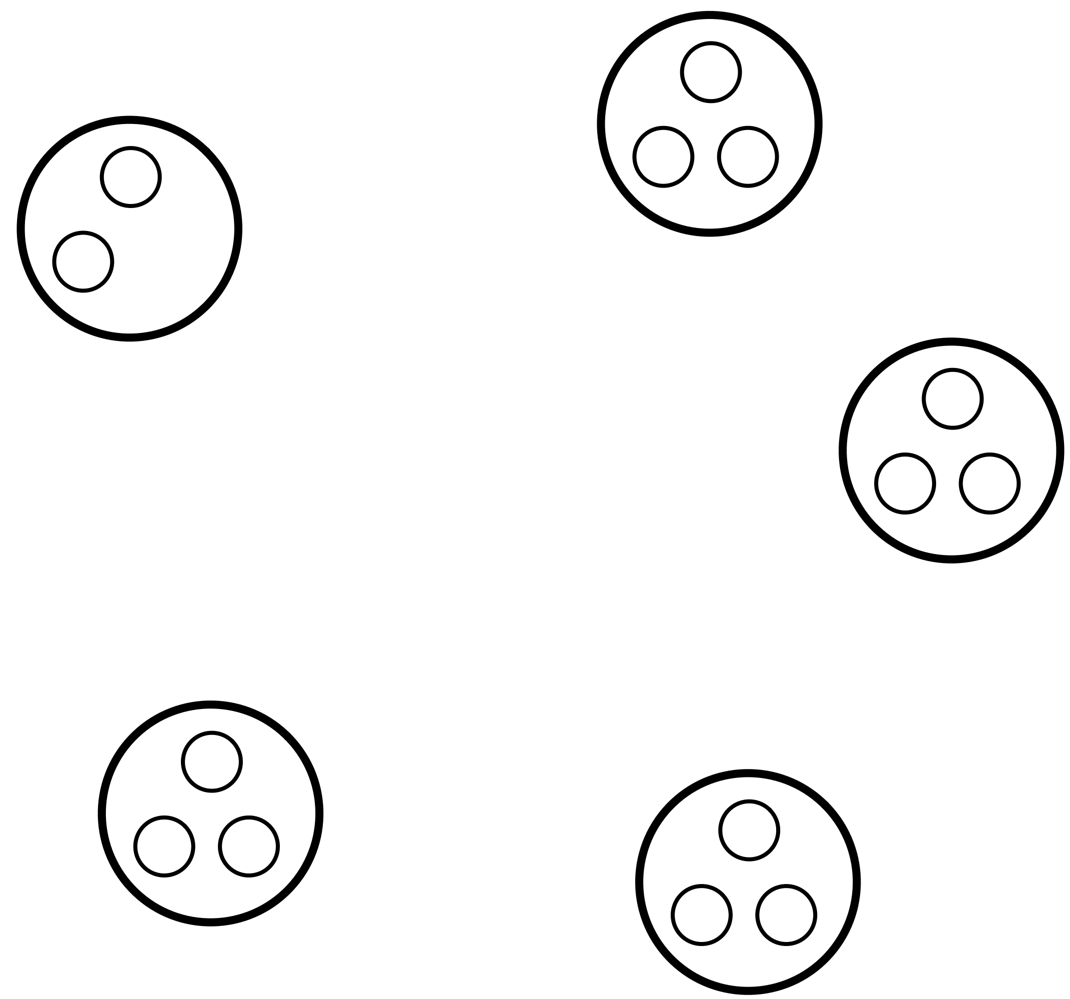


2-orientation process

- Configuration model, Po(c) degrees, which implies results on G(n,c/n)
- Modified [F&R] algorithm
- When min. "degree" is ≤ 2, just discard that vertex and ≤ 2 random copies
- When min. "degree" is 3, discard that vertex and one random copy (this preserves 2-orientability)
- When min. "degree" is 4, done.
- Run until o(n) vertices left



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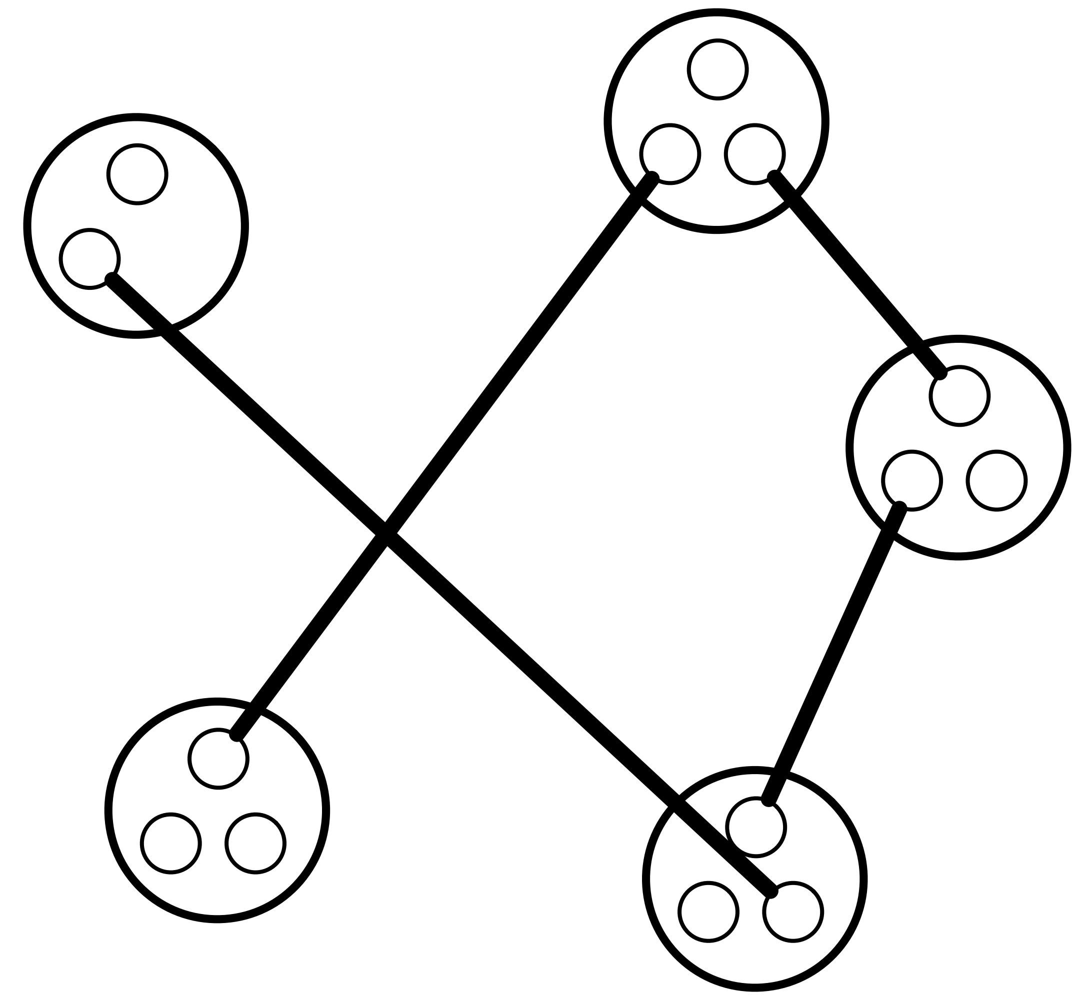

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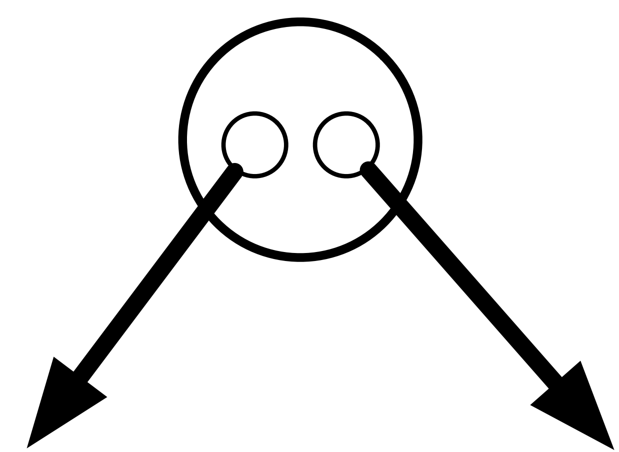
- When min. "degree" is 3, discard that vertex and one random copy (this preserves 2-orientability)
- When min. "degree" is 4, done.
- Run until o(n) vertices left



2-orientation process

- Configuration model, Po(c) degrees, which implies results on G(n,c/n)
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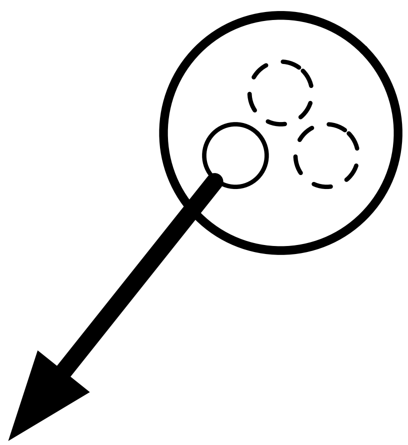

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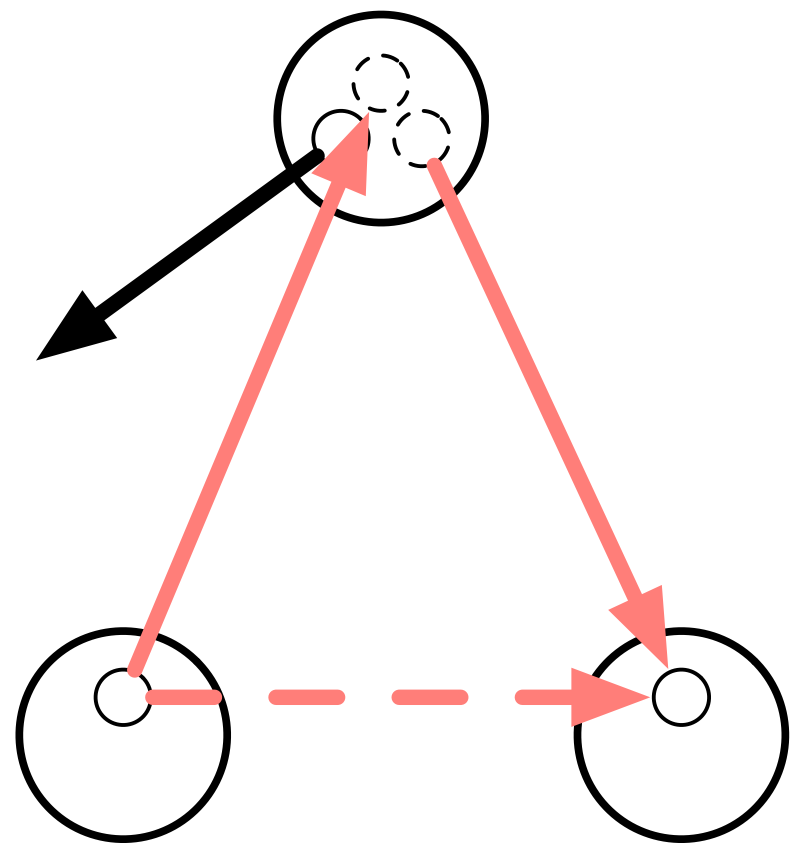
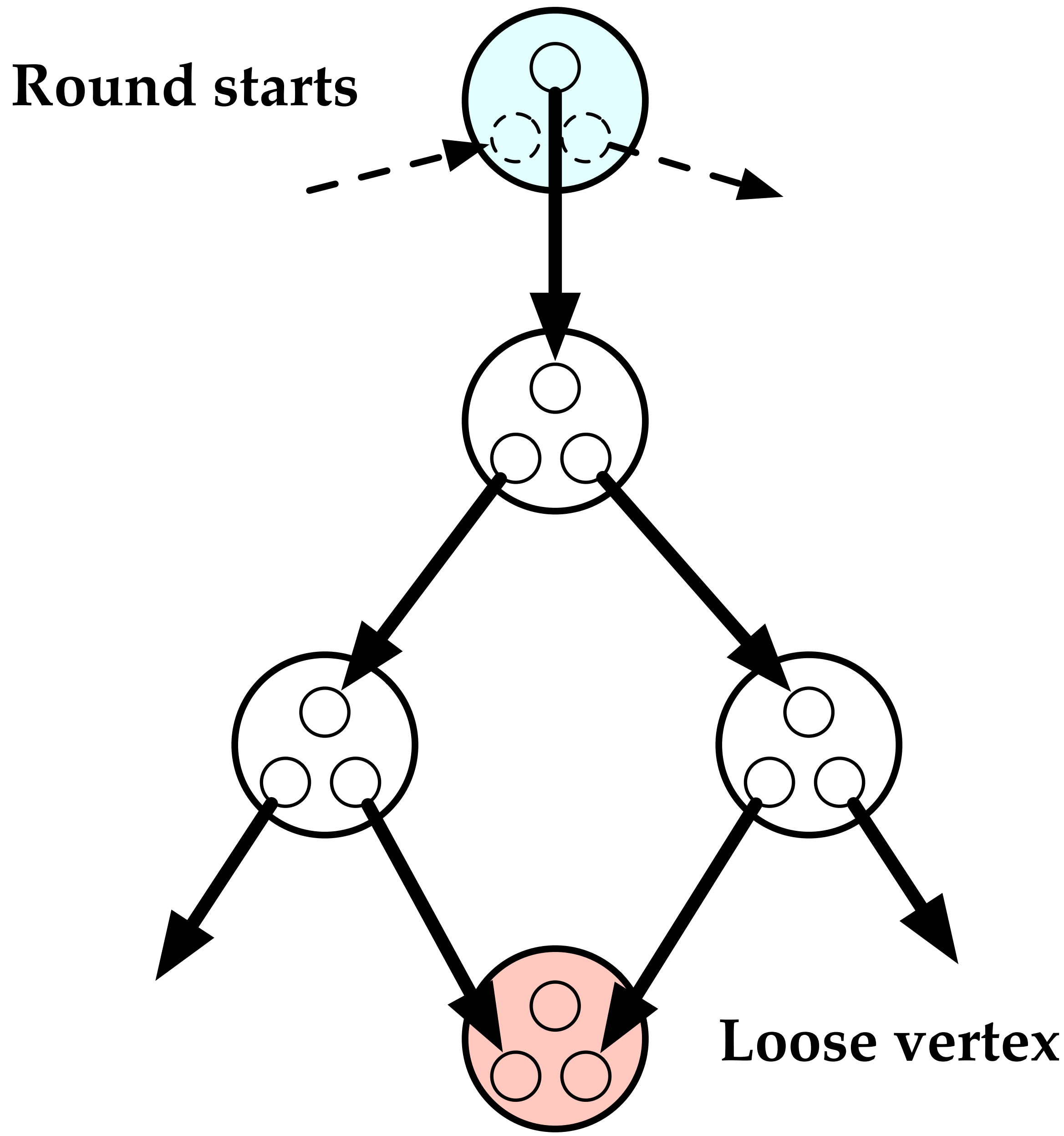


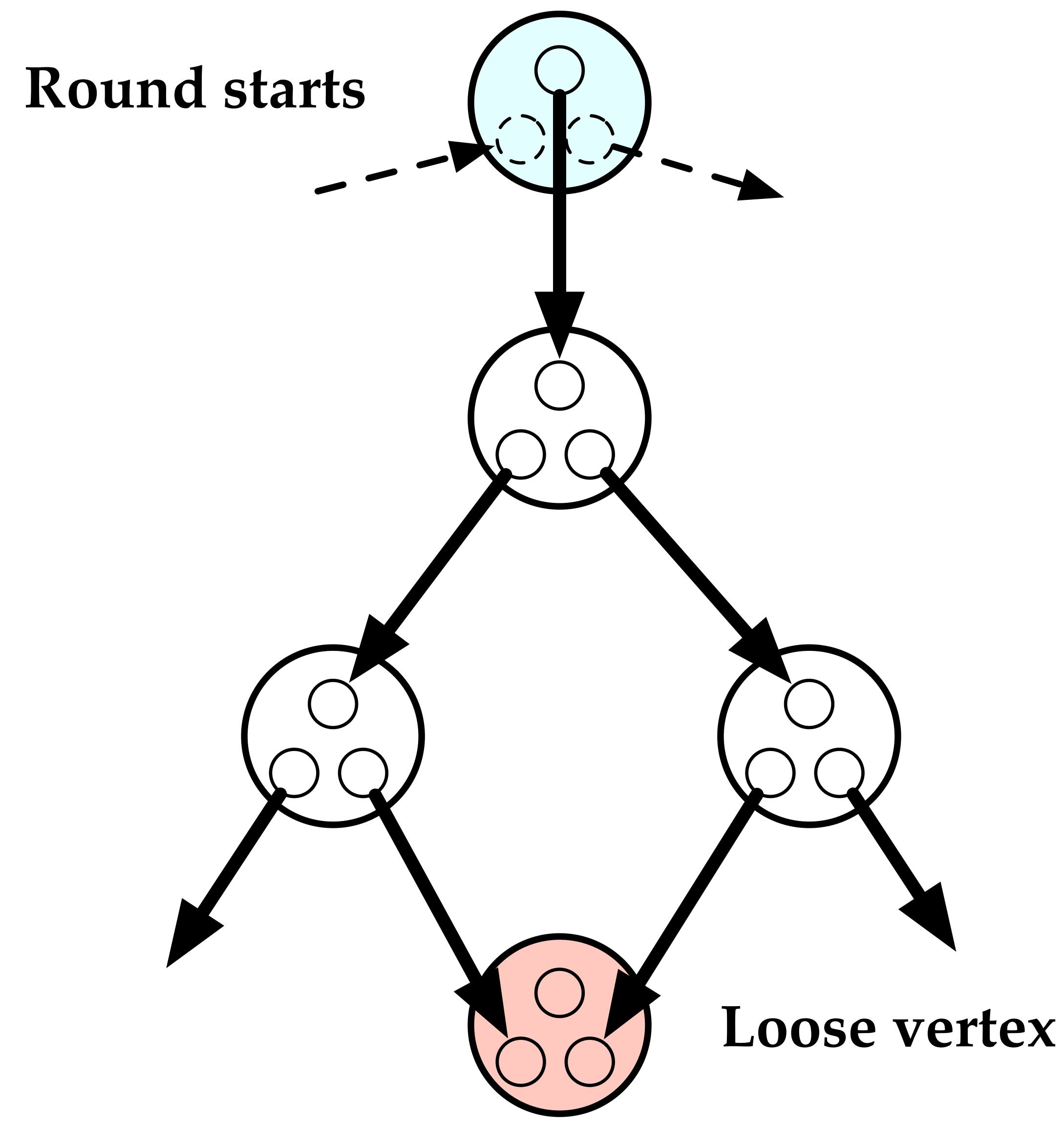
Min. deg. 3 rounds

- When min. "degree" is ≤ 2, just discard that vertex and $\leq$ 2 random copies
- When min. "degree" is 3, discard that vertex and one random copy (this preserves 2-orientability)
- Steps in between min. deg. 3 steps are a *round*
- A vertex get out deg. < 2 iff
  - It is hit twice in a round
  - It had degree 2 and is hit at random
- Both events happen with prob $\leq n^{-.5}$



Min. deg. 3 rounds

- Steps in between min. deg. 3 steps are a *round*
- All rounds last O(log n) steps w.h.p.
- This implies o(n) vertices of out degree < 2
- Rounds are analyzed as a *branching process*
- We use the *method of differential equations* to control expected number of children



Questions

- **Conjecture:** The size of the (3+2)-core is w.h.p. approx. 0.75n
  - Comes from a branching process heuristic, solution to $q = 1 - \exp(-q \cdot c)(1 + q \cdot c)$
- Improve the analysis so only O(1) loose vertices
- Is a similar statement true for more general degree sequences than Poisson?
- Is the 3-core "globally rigid" w.h.p?
  - Need to show 3-connected and *redundantly rigid*
  - 3-connectivity is standard