Rooted-tree Decomposition with Matroid Constraints and the Infinitesimal Rigidity of Frameworks with Boundaries

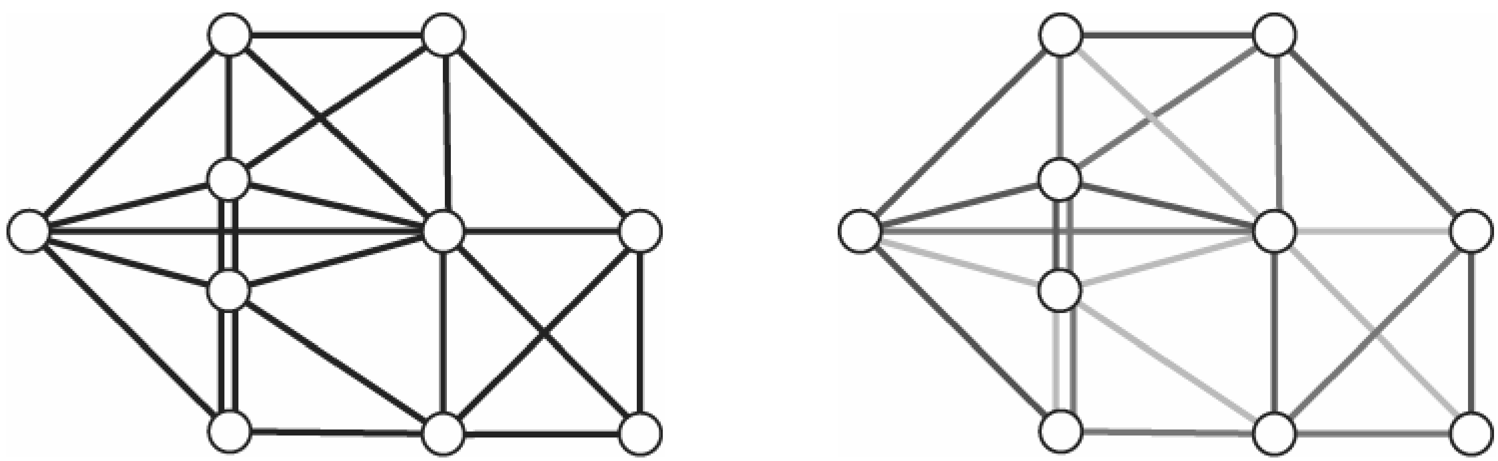
Shin-ichi Tanigawa
RIMS, Kyoto University
Joint work with N. Katoh

List of Talks

- Nash-Williams' tree-decomposition theorem
- Connection to rigidity theory
- An extension of Nash-Williams' theorem
  - Rooted-tree decomposition with matroid constraints
- Applications to rigidity
  - Extensions of Laman's theorem and Tay's theorem
- Algorithms

Nash-Williams’ tree-decomposition theorem

- Theorem (Nash-Williams 1964). An undirected graph $G = (V, E)$ can be decomposed into k edge-disjoint spanning trees $\Leftrightarrow$
  - $|E| = k|V| - k$
  - $\emptyset \neq \forall X \subseteq V, |E(X)| \leq k|X| - k$
    - where $E(X) := \{uv \in E \mid u, v \in X\}$

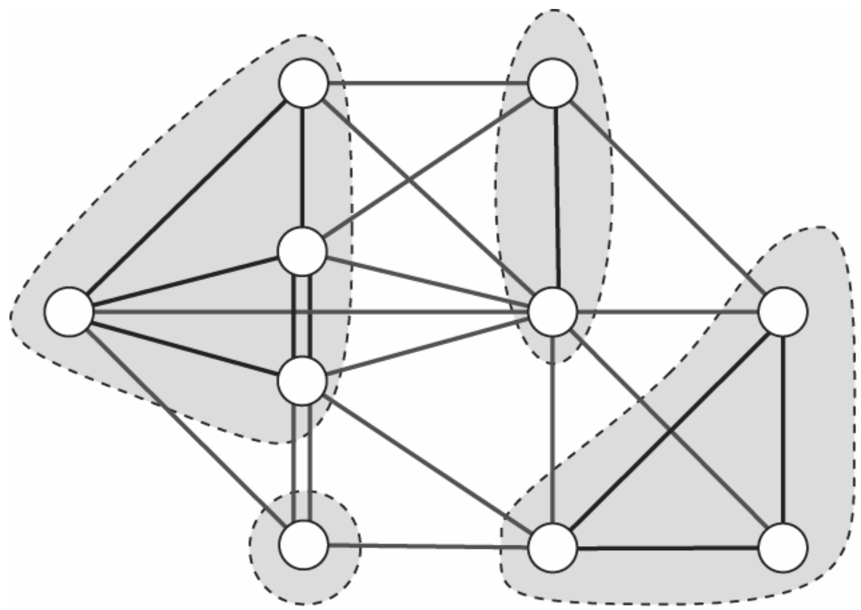


Tutte-Nash-Williams' tree-packing theorem

- Theorem (Tutte 1961, Nash-Williams 1961). An undirected graph G contains k edge-disjoint spanning trees $\Leftrightarrow$

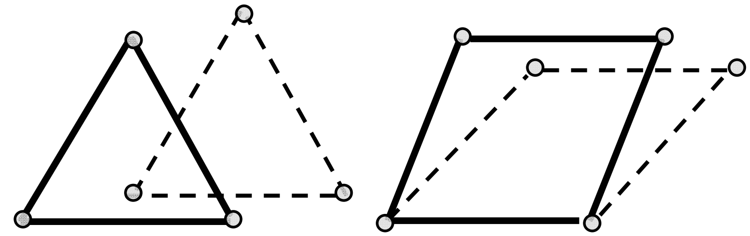
$$\delta_G(\mathcal{P}) \geq k|\mathcal{P}| - k$$

for any partition $\mathcal{P}$ of V into non-empty subsets.



Connection to Rigidity Theory

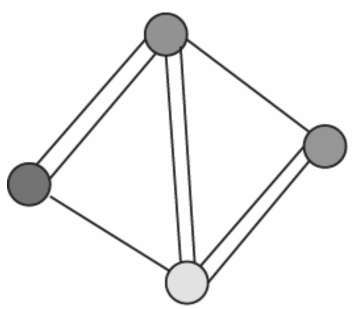
- bar-joint framework (G, p);
  - $G = (V, E)$: a graph
  - $p: \ V \to \mathbb{R}^d$ (joint-configuration)

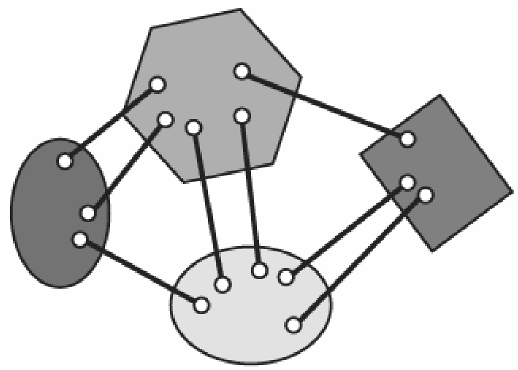


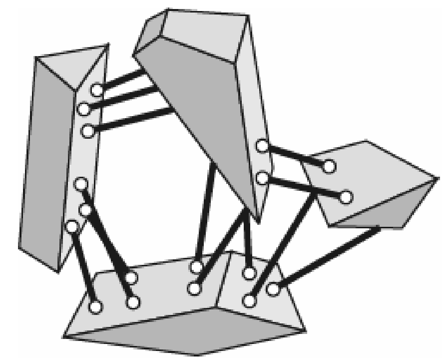
- Theorem. For a graph $G = (V, E)$, the followings are equivalent:
  - (G, p) is minimally rigid for a generic $p: V \to \mathbb{R}^2$
  - $|E| = 2|V| - 3$, and $\forall X \subseteq E$ with $|X| \geq 2$, $|E(X)| \leq 2|X| - 3$ (Laman)
  - Duplicating any edge in G, the resulting graph can be partitioned into 2 edge-disjoint spanning trees. (Recski)
  - G can be decomposed into three trees T_1, T_2, T_3 s.t. each vertex is spanned by exactly two of them and any non-trivial subtrees of T_i supports the same vertex subset (Crapo)

Connection to Rigidity Theory

- body-bar framework (G, b)
  - $G = (V, E)$: a graph
  - $b: e \in E \mapsto$ a line-segment in $\mathbb{R}^d$ (bar-configuration)



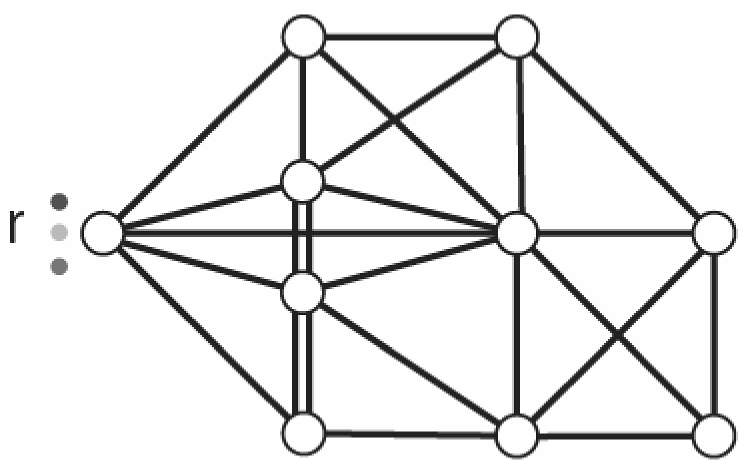


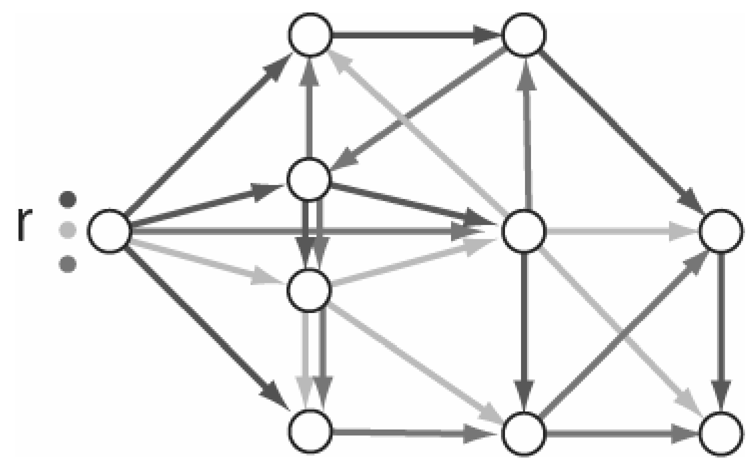


- Tay's theorem (Tay 1984). A d-dimensional body-bar framework (G, b) is rigid for a "generic" bar-configuration b iff G contains $\binom{d+1}{2}$ edge-disjoint spanning trees.
  - A short proof based on tree-decomposition theorem (Whiteley88)
  - A good decomposition theorem implies a good rigidity theorem ??

Rooted-edge-connected Orientability

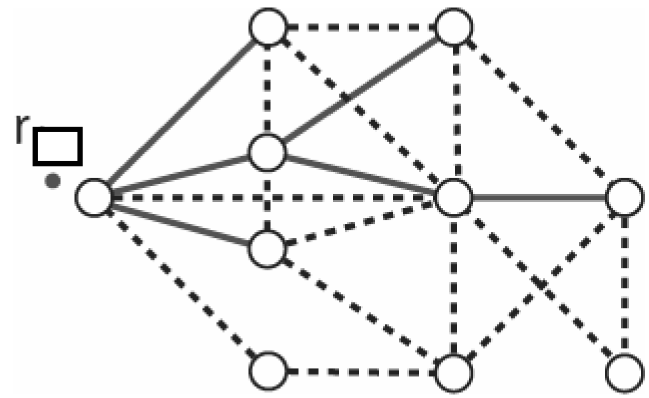
- G contains k edge-disjoint spanning trees
- $\Leftrightarrow$ G is k-rooted-edge-connected orientable
  - G admits an orientation so that there are k edge-disjoint paths from a special node r to each vertex $v \in V \setminus \{r\}$.





Rooted-tree Decomposition

- r-rooted tree : a tree $G' = (V', E')$ with $r \in V'$



- Rooted-tree decomposition w.r.t. a multiset $\{r_1, r_2, \ldots, r_t\}$ of vertices
  - decomposition into edge-disjoint subgraphs $T_1, \ldots, T_t$
  - T_i is an r_i-rooted tree