Better rank estimates for the 3D rigidity matroid

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October 11, 2011

A rigidity question posed by Tibor Jordán in BIRS rigidity workshop 08

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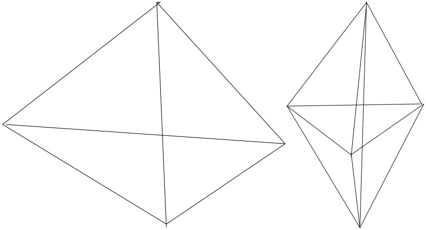

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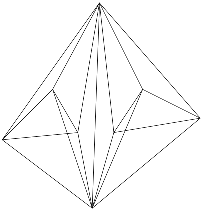

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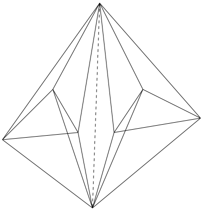

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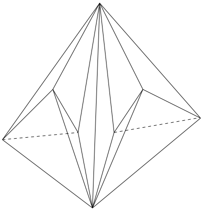

Other names for Maxwell-independent graphs in the literature

- d-sparse, (k,l)-sparse (in this case, $(d,\binom{d+1}{2})$)-sparse) have been used in the literature
- dense and sparse graphs have been used to mean a variety of different things in graph theory;
- we use Maxwell-independence because Maxwell first showed in 1864 that every graph G that is rigid in d dimensions must contain a Maxwell-independent subgraph that has least $d|V| - \binom{d+1}{2}$ edges.

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Our results give affirmative answers to both Question 1 and 2 in $3D$.

Contents

1. Questions and definitions.
2. Answer to Question 1: every maximal Maxwell-independent subgraph of G has size at least rank of the 3D rigidity matroid of G.
3. Conjectures
4. Answer to Question 2: rank inclusion-exclusion formulae giving new bounds on the rank of the 3D rigidity matroid for Maxwell-independent graphs.
5. Similar bounds on the rank of the 3D rigidity matroid for a special class of Maxwell-dependent graphs.
6. Key proofs
7. Open problems

Maxwell-rigidity

A graph $G = (V, E)$ is Maxwell-rigid in d dimensions if there exists a $E' \subseteq E$ such that E' is Maxwell-independent and has size $d|V| - \binom{d+1}{2}$. [1]

[1] Bill Jackson suggests "d-critical", but "critical" is used for many other graph properties.

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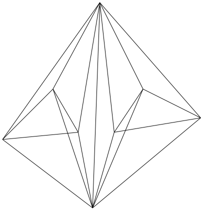

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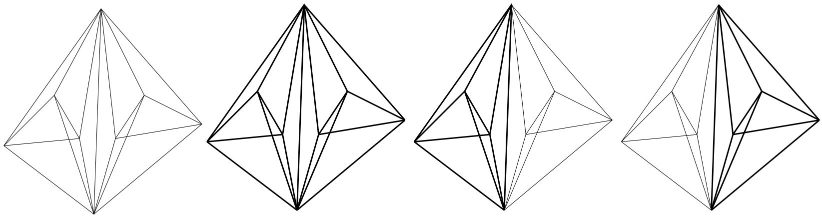
Maxwell-rigid component

Given graph $G = (V, E)$,

- a vertex-maximal, Maxwell-rigid component is a Maxwell-rigid induced subgraph which is not properly contained in any other Maxwell-rigid subgraph of G.
- a proper vertex-maximal, Maxwell-rigid component is a Maxwell-rigid induced subgraph which is not properly contained in any Maxwell-rigid induced proper subgraph of G.



Independence vs Maxwell-independence

Recall: a subgraph of G is independent in d dimensions if its edges give an independent set of rows of a generic d-dimensional rigidity matrix of G.

1. Maxwell's direction: Independence implies Maxwell-independence, true for all dimensions.
2. Laman's direction: Maxwell-independence implies independence, true for 2D.

Laman's theorem ('70)

Theorem (Laman)

In 2D, independence $\Leftrightarrow$ Maxwell-independence.

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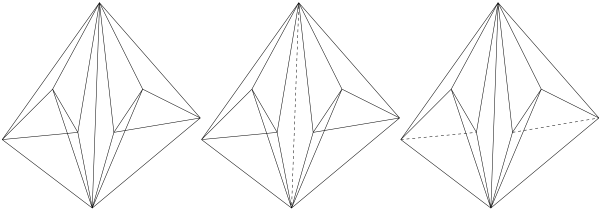
I In 2D, every maximal Maxwell-independent set of G has the same size($= \text{rank}_2(G)$).

II In 2D, a non-edge pair $f = (u, v)$ is implied only if u and v both lie inside a rigid subgraph $(V', E') \subseteq (V, E)$. I.e., every rigidity circuit lies inside a rigid subgraph.

Laman's direction in 3D

In 3D, the two properties in the previous slide are both false.

- In 3D, not all maximal Maxwell-independent sets have the same size.



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Laman's direction in 3D

- In 3D, a non-edge pair f can be implied without lying inside a rigid subgraph, ([Cheng, Sitharam, Streinu, 2009]),i.e., there are rigidity circuits that do not even contain any non-trivial rigid subgraph.

Contents

1. Questions and definitions.
2. **Answer to Question 1: every maximal Maxwell-independent subgraph of G has size at least rank of the 3D rigidity matroid of G.** Our proof is constructive towards an algorithm to find a minimum-sized maximal Maxwell-independent subgraph that contains a maximal independent set of G.
3. Conjectures
4. Answer to Question 2: rank inclusion-exclusion formulae giving new bounds on the rank of the 3D rigidity matroid for Maxwell-independent graphs.
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Main theorem

Theorem (Main)

Let $\mathcal{M}$ be a maximal Maxwell-independent set of a graph $G = (V, E)$ and $\mathcal{I}$ a maximal independent set of G. Then $|\mathcal{M}| \geq |\mathcal{I}|$.

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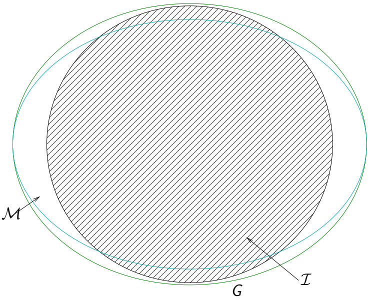
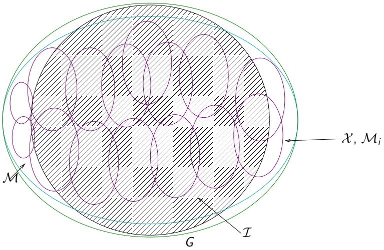


Main theorem and proof idea

Theorem (Main)

Let $\mathcal{M}$ be a maximal Maxwell-independent set of a graph $G = (V, E)$ and $\mathcal{I}$ be a maximal independent set of G. Then $|\mathcal{M}| \geq |\mathcal{I}|$.

- We will decompose $\mathcal{M}$ into a <u>cover</u> $\mathcal{X}$ by vertex-maximal, Maxwell-rigid subgraphs.



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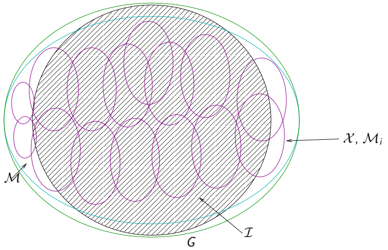

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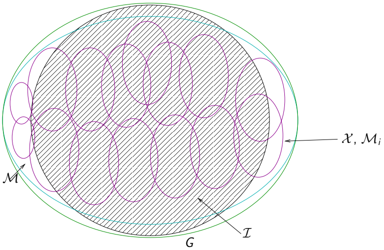
- We show a rank inclusion-exclusion formula on $\mathcal{X}$ that gives an upper bound on $\text{rank}_3(\mathcal{M})$.
- For this, we will construct an independence assignment: a proper subgraph of $\mathcal{M}$ with size matching the formula and containing a maximal independent subgraph of $\mathcal{M}$.



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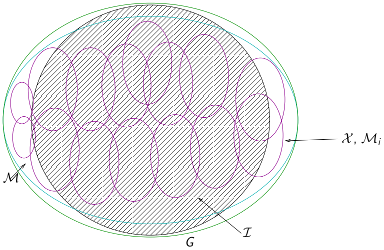
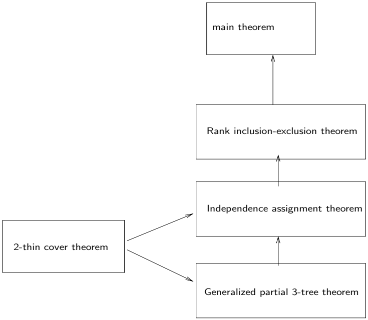
Let $\mathcal{M}$ be a maximal Maxwell-independent set of a graph $G = (V, E)$ and $\mathcal{I}$ be a maximal independent set of G. Then $|\mathcal{M}| \geq |\mathcal{I}|$.

- To construct the independence assignment, we will show that the component graph formed by the cover $\mathcal{X}$ of $\mathcal{M}$ is a generalized partial 3-tree.



Roadmap of our proof for main theorem



Proof of the main theorem

- A <u>cover</u> of a graph $G = (V, E)$ is a collection $\mathcal{X}$ of pairwise incomparable subsets of V, each of size at least two, such that $\cup_{X \in \mathcal{X}} E(X) = E$.

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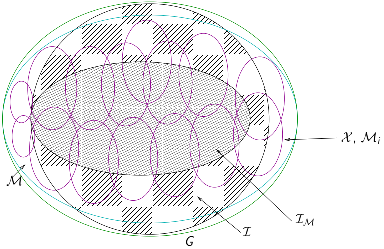
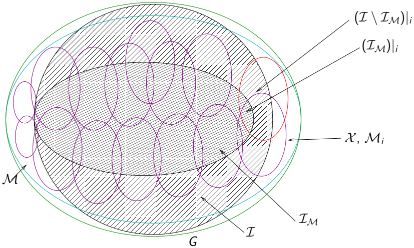
- A cover of a graph $G = (V, E)$ is a collection $\mathcal{X}$ of pairwise incomparable subsets of V, each of size at least two, such that $\cup_{X \in \mathcal{X}} E(X) = E$.
- Let $\mathcal{X} = \{\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n\}$ be any cover by vertex-maximal, Maxwell-rigid components of $\mathcal{M}$. Let $\mathcal{I}_\mathcal{M}$ be a maximal independent set of $\mathcal{M}$ and extend $\mathcal{I}_\mathcal{M}$ to a maximal independent set $\mathcal{I}$ of G.



Proof of the main theorem

- Simple fact: maximality of $\mathcal{M}$ implies that $\mathcal{I} \setminus \mathcal{I}_{\mathcal{M}}$ can be covered by $\mathcal{I} \setminus \mathcal{I}_{\mathcal{M}}$ restricted to each $\mathcal{M}_i$, which we denote $(\mathcal{I} \setminus \mathcal{I}_{\mathcal{M}})|_i$. Hence

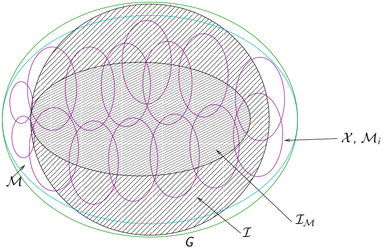
$$|\mathcal{I} \setminus \mathcal{I}_{\mathcal{M}}| \leq \sum_i |(\mathcal{I} \setminus \mathcal{I}_{\mathcal{M}})|_i| \quad (1)$$



Proof of main theorem(cont.)

- Another simple fact: denote by $\mathcal{H}(\mathcal{X})$ pairs of vertices (u, v) such that $\{u, v\} \subseteq (\mathcal{M}_i \cap \mathcal{M}_j)$ for some $1 \leq i < j \leq n$. Denote by n_e the number of components $\mathcal{M}_i$ of $\mathcal{X}$ that share e. Thus

$$|\mathcal{M}| = \sum_i |\mathcal{M}_i| - \sum_{e \in \mathcal{H}(\mathcal{X})} (n_e - 1) \tag{2}$$



Proof of main theorem(cont.)

- Key observation:
  Maxwell-rigidity of $\mathcal{M}_i$ implies that
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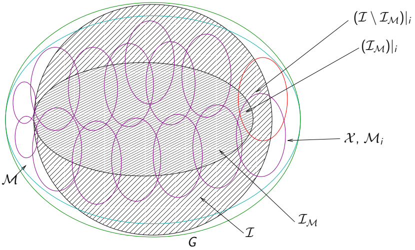

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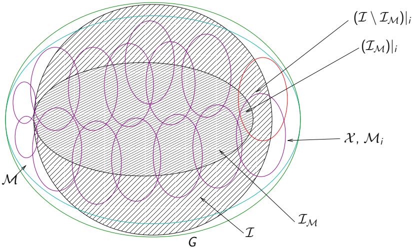

- Applying Equation (2), i.e.,
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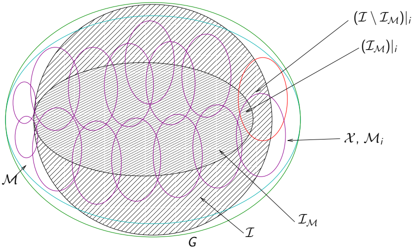

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We prove a rank inclusion-exclusion theorem to lower-bound the first two items of the RHS.

Rank inclusion-exclusion completes proof of main theorem

- We prove a rank inclusion-exclusion theorem to lower-bound the first two items of the RHS of Equation (3) $|\mathcal{M}| \geq \sum_i \mathrm{rank}_3(\mathcal{M}_i) - \sum_{e \in \mathcal{H}(\mathcal{X})}(n_e - 1) + \sum_i |(\mathcal{I} \setminus \mathcal{I}_{\mathcal{M}})|_i|$.

Theorem

Given a Maxwell-independent graph $\mathcal{M}$ and any set $\mathcal{X}$ of vertex-maximal, Maxwell-rigid components $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ that is a cover of $\mathcal{M}$, then the rank inclusion-exclusion of cover $\mathcal{X}$ is at least rank$_3(\mathcal{M})$, i.e,
$\sum_i rank_3(\mathcal{M}_i) - \sum_{e \in \mathcal{H}(\mathcal{X})}(n_e - 1) \geq rank_3(\mathcal{M})$

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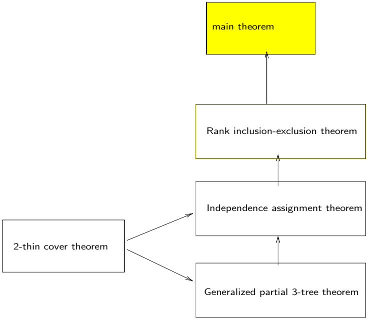
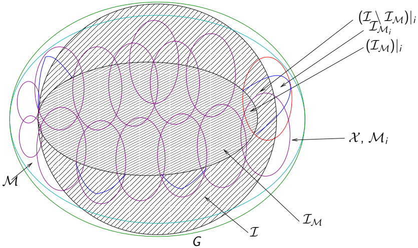
- From Equation (1) $\sum_i |(\mathcal{I} \setminus \mathcal{I}_{\mathcal{M}})|_i| \geq |\mathcal{I} \setminus \mathcal{I}_{\mathcal{M}}|$, the above theorem, and Equation (3) above, we get $|\mathcal{M}| \geq |\mathcal{I}_{\mathcal{M}}| + |\mathcal{I} \setminus \mathcal{I}_{\mathcal{M}}| = |\mathcal{I}|$, <u>thus proving the main theorem</u>.

Roadmap of our proof for main theorem



Rank inclusion-exclusion

- Suppose we can construct an independence assignment $[\mathcal{I}_{\mathcal{M}}, \mathcal{I}_{\mathcal{M}_1}, \ldots \mathcal{I}_{\mathcal{M}_n}]$ where (1) $(\mathcal{I}_{\mathcal{M}})|_i \subseteq \mathcal{I}_{\mathcal{M}_i}$ (2) for each $e \in \mathcal{H}(\mathcal{X})$, $e \in \mathcal{I}_{\mathcal{M}_i}$ for at least $n_e - 1$ of the $\mathcal{M}_i$'s containing e.



- Then we immediately get $\sum_i \text{rank}_3(\mathcal{M}_i) - \sum_{e \in \mathcal{H}(\mathcal{X})}(n_e - 1) \geq \sum_i |(\mathcal{I}_{\mathcal{M}})|_i| \geq |\mathcal{I}_{\mathcal{M}}|$, which is the rank inclusion-exclusion theorem.

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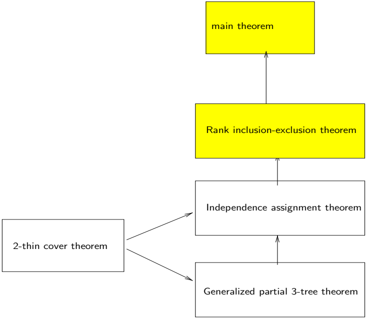

independence assignment

- Constructing an independence assignment is the main technical result of the paper.

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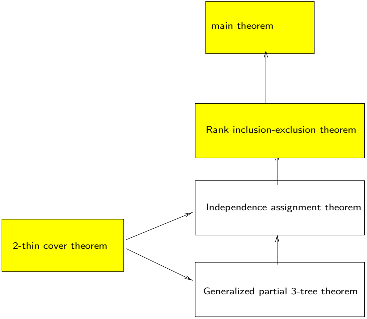
- Constructing an independence assignment is the main technical result of the paper.
- To show the existence of the independence assignment, we show that $\mathcal{X}$ forms a 2-thin cover, i.e., $|\mathcal{M}_i \cap \mathcal{M}_j| \leq 2$ for all $1 \leq i < j \leq n$. We have the following:

Theorem

Let $\mathcal{M}$ and $\mathcal{X}$ be as defined before. Then $\mathcal{X}$ forms a 2-thin cover of $\mathcal{M}$.

Roadmap of our proof for main theorem

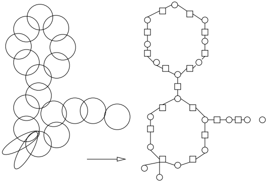


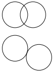
JUST two more definitions before we can get the independence assignment.

Component graph

Definition

Let $G = (V, E)$ be such that its complete set of vertex-maximal, Maxwell-rigid components $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ forms a 2-thin cover. The 2-thin component cover graph, or component graph for short, $\mathscr{C}_G$ of G contains a component node for each component $\mathcal{M}_i$ in $\mathscr{C}_G$ and whenever $\mathcal{M}_i$ and $\mathcal{M}_j$ share an edge in G, their corresponding component nodes in $\mathscr{C}_G$ are connected via an edge node.





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- *Remove all leaves from G to get G_1.*
- *Remove all leaves from G_i to get G_{i+1}.*
- *Continue this process until there is no leaf.*

Generalized partial m-tree

Definition

A graph G is a generalized partial m-tree if the following hold:

- *Call all vertices of G that have degree at most m leaves.*
- *Remove all leaves from G to get G_1.*
- *Remove all leaves from G_i to get G_{i+1}.*
- *Continue this process until there is no leaf.*
- *The remaining kernel graph $\mathscr{K}_G$ is empty.*

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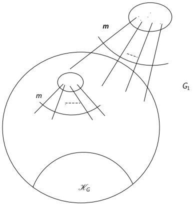
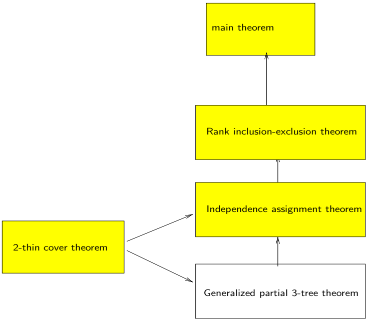


$\mathcal{M}$ has an independence assignment

Theorem

Given Maxwell-independent graph $\mathcal{M}$ and a cover $\mathcal{X}$ by vertex-maximal, Maxwell-rigid components, if the component graph of $\mathcal{M}$ is a generalized partial 9-tree, then $(\mathcal{M}, \mathcal{X})$ has an independence assignment.

Roadmap of our proof for main theorem



Component graph is generalized partial 3-tree

Note that a generalized partial 3-tree is also a generalized partial 9-tree.

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If $(V, \mathcal{M})$ is a Maxwell-independent graph, then its component graph is a generalized partial 3-tree.

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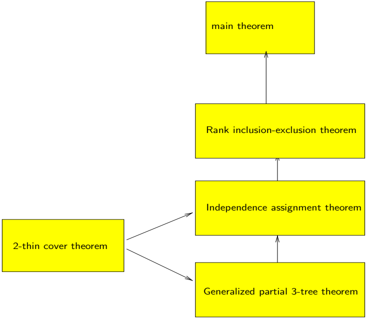

Contents

1. Questions and definitions.
2. Answer to Question 1: every maximal Maxwell-independent subgraph of G has size at least rank of the 3D rigidity matroid of G.
3. **Conjectures**
4. Answer to Question 2: rank inclusion-exclusion formulae giving new bounds on the rank of the 3D rigidity matroid for Maxwell-independent graphs.
5. Similar bounds on the rank of the 3D rigidity matroid for a special class of Maxwell-dependent graphs.
6. Key proofs
7. Open problems

Conjecture: average degree in higher dimension

- Actually we show a stronger theorem

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If $(V, \mathcal{M})$ is a Maxwell-independent graph, then the average degree of any subgraph of its component graph is < 4.

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If $(V, \mathcal{M})$ is a Maxwell-independent graph, then the average degree of any subgraph of its component graph is < 4.

- We can extend 2-thin cover to $(d-1)$-thin cover in d dimensions and define component graph accordingly.

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Conjecture

For a Maxwell-independent graph in d dimensions, the average degree of the component nodes of any subgraph of the component graph (induced by a subset of component nodes) is strictly smaller than $d+1$.

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- We have just shown the bound in 3D. In 2D, this bound is tight.
- In 3D, however, we do not know of an example where all nodes have degree ≥ 3. In fact, we do not even know of an example with average degree ≥ 3.

Conjecture: higher dimensions

- Another natural conjecture is the following:

Conjecture

For any dimension d, the size of any maximal Maxwell-independent set gives an upper bound on the rank of the rigidity matroid of a graph G.

- Bill Jackson extended our result to $d \leq 5$. He used a proof by contradiction, but the main technical theorem (generalized partial 3-tree) is the same. Our proof is constructive towards an algorithm to find a minimum-sized maximal Maxwell-independent subgraph that contains a maximal independent set of G.

Contents

Rank Inclusion-Exclusion of Maxwell-independent graphs

Theorem (rank inclusion-exclusion)

Given a Maxwell-independent graph $\mathcal{M}$ and any set $\mathcal{X}$ of vertex-maximal, Maxwell-rigid components $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ that is a cover of $\mathcal{M}$, then the rank inclusion-exclusion of cover $\mathcal{X}$ is at least $rank_3(\mathcal{M})$, i.e,
$\sum_i rank_3(\mathcal{M}_i) - \sum_{(u,v)\in\mathcal{H}(\mathcal{X})}(n_{(u,v)} - 1) \geq rank_3(\mathcal{M})$

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- Note: $(u, v) \in \mathcal{H}(\mathcal{X})$ turns out to always be an edge (we show 2-thin cover is "strong").

Better bounds for Maxwell-independent graphs

We can get better bounds for Maxwell-independent graphs.

Theorem

Given a Maxwell-independent graph $(V, \mathcal{M})$ *and any set of* *proper* *vertex-maximal, Maxwell-rigid components* $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ *that form a cover* $\mathcal{X}$*, then the rank inclusion-exclusion of* $\mathcal{X}$*, i.e.,* $\sum_i rank_3(\mathcal{M}_i) - \sum_{(u,v)\in\mathcal{H}(\mathcal{X})}(n_{(u,v)} - 1)$ *is at least* $rank_3(\mathcal{M})$.

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- This is better: recursive algorithm requires cover by proper vertex-maximal, Maxwell-rigid components.

Contents

Bounds for non-Maxwell-independent graphs

Definition

Given graph G and a cover $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ of G. The full rank inclusion-exclusion of cover $\mathcal{X}$ in 3D is defined as $\sum_{i}^{m}(3|V_i| - 6) - \sum_{(u,v)\in\mathcal{H}(\mathcal{X})}(n_{(u,v)} - 1)$*, where V_i is the vertex set of X_i and $\mathcal{H}(\mathcal{X})$ and $n_{(u,v)}$ are as defined before.*

Theorem

Given graph $G = (V, E)$, if the complete collection $\mathcal{X}$ of vertex-maximal, Maxwell-rigid components forms a 2-thin cover, then the full rank inclusion-exclusion of the cover $\mathcal{X}$ is an upper bound on $rank_3(G)$, i.e.,

$$\sum_{i}^{m}(3|V_i| - 6) - \sum_{(u,v)\in\mathcal{H}(\mathcal{X})}(n_{(u,v)} - 1) \geq rank_3(G).$$

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- In fact, in 2006 they showed that the minimum taken over all independent 2-thin covers is an upper bound on the rank. They conjectured that their bound is tight when restricted to non-rigid graphs and covers of size at least 2.
- They also defined an iterated, or recursive, version of independent covers and showed that the minimum of the full rank inclusion-exclusion taken over all iterated 2-thin covers is an upper bound on rank.

More remarks

- We show that rank inclusion-exclusion over a specific, non-independent cover gives a rank upper bound for Maxwell-independent graphs.

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- As pointed out, our bound is more useful in an algorithmic sense.

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Given Maxwell-independent graph $\mathcal{M}$ and a cover $\mathcal{X}$ by vertex-maximal, Maxwell-rigid components, if the component graph of $\mathcal{M}$ is a generalized partial 9-tree, then $(\mathcal{M}, \mathcal{X})$ has an independence assignment.

Proof.

- If $\mathcal{M}$ itself is Maxwell-rigid, done.

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  - Add $\mathcal{M}_{k+1}$ to form $\mathscr{C}_{\mathcal{M}}^{k+1}$. We can keep the edges of $\mathcal{I}^k$ in $\mathcal{I}^{k+1}$.

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  - Take $\mathcal{I}_i^{k+1} := \mathcal{I}_i^k$ for $1 \le i \le k$. Then we find a maximal independent set $\mathcal{I}_{k+1}^{k+1}$ within $\mathcal{M}_{k+1}$ that contains all its shared edges $\mathcal{S}$. If $\mathcal{I}^{k+1}$ is not indep, we remove edges from $\mathcal{I}_{k+1}^{k+1} - \mathcal{S}$ until it is indep.

$\square$

Component graph is generalized partial 3-tree

Note that a generalized partial 3-tree is also a generalized partial 9-tree.

Theorem

If $(V, \mathcal{M})$ is a Maxwell-independent graph, then any subgraph of its component graph is a generalized partial 3-tree.

Proof

Without loss of generality, we deal only with the complete component graph $\mathscr{C}_{\mathcal{M}}$.

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If $(V, \mathcal{M})$ is a Maxwell-independent graph, then any subgraph of its component graph is a generalized partial 3-tree.

Proof

Without loss of generality, we deal only with the complete component graph $\mathscr{C}_{\mathcal{M}}$.
Suppose the component graph of $\mathcal{M}$ is not a generalized partial 3-tree, then in the kernel component graph $\mathscr{K}_{\mathcal{M}}$, each component node has degree more than 3. Let K denote $\mathscr{K}_{\mathcal{M}}$'s corresponding subgraph in $\mathcal{M}$.

proof cont.

- Denote by $\mathcal{X} = \{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ the set of vertex-maximal, Maxwell-rigid components of K.
- V_i: vertices of M_i that are shared by other component(s);

Proof of generalized partial 3-tree

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- n_e and n_v: the number of components $\mathcal{M}_i$ of K that share e and v respectively.
- $C_v \subseteq \{1, \ldots, n\}$: the set of indices of components that are incident at v.
- w_v^j: the number of shared edges incident at v in component $\mathcal{M}_j$
- s_v: the number of shared edges that are incident at v.

proof cont.

In order to determine the Maxwell-independence of K, we need to calculate $3|V_K| - |E_K|$, which is

$$\sum_i 6 - 3\sum_{v\in V_s} n_v + \sum_{e\in E_s} n_e + 3|V_s| - |E_s| = \sum_i (6 - 3|V_i| + |E_i|) + 3|V_s| - |E_s|$$

Since $\mathcal{M}$ is Maxwell-independent, we know K is also Maxwell-independent, thus

$$6n - 6 \geq 3\sum_i |V_i| - \sum_i |E_i| - 3|V_s| + |E_s| \tag{4}$$

proof cont.

The collection of all n_v components of K meeting at v forms a subgraph C. $3|V_C| - |E_C|$ can be computed as follows:

- there are n_v components, which contributes $6n_v$;
- v is shared by n_v components, and the contribution is $-(3n_v - 3)$;
- each shared edge in a component $\mathcal{M}_j$ contributes 1 to the count, and in total the shared edges contribute $(\sum_{j \in C_v} w_v^j) - s_v$
- for each shared edge $e = (u, v)$, vertex u contributes $-3[(\sum_{j \in C_v} w_v^j) - s_v]$

Thus $3|V_C| - |E_C|$ is equal to:

$$3n_v - 2[(\sum_{j \in C_v} w_v^j) - s_v] + 3$$

proof cont.

Since C is Maxwell-independent, we know:

$$\begin{aligned} 3n_v - 2[(\sum_{j \in C_v} w_v^j) - s_v] + 3 &\geq 6 \\ 3n_v - 2[(\sum_{j \in C_v} w_v^j) - s_v] &\geq 3 \end{aligned}$$

Summing over all shared vertices in V_s, we have:

$$3 \sum_{v \in V_s} n_v - 2 \sum_{v \in V_s} [(\sum_{j \in C_v} w_v^j) - s_v] \geq 3|V_s|$$

Since $\sum_{v \in V_s} n_v = \sum_i |V_i|$, $\sum_{v \in V_s} (\sum_{j \in C_v} w_v^j) = 2 \sum_i |E_i|$ and $\sum_{v \in V_s} s_v = 2|E_s|$, we know

$$3 \sum_i |V_i| - 4 \sum_i |E_i| - 3|V_s| + 4|E_s| \geq 0$$

proof cont.

Plugging into (4), we have:

$$\begin{aligned} 6n-6 &\geq 3\sum_i |V_i| - \sum_i |E_i| - 3|V_s| + |E_s| \\ &\geq 3\sum_i |V_i| - 4\sum_i |E_i| - 3|V_s| + 4|E_s| \\ &\quad +3(\sum_i |E_i| - |E_s|) \\ &\geq 3(\sum_i |E_i| - |E_s|) \end{aligned}$$

Since $|E_s| \leq \frac{1}{2}\sum_i |E_i|$, we have:

$$6n-6 \geq \frac{3}{2}\sum_i |E_i|$$

proof cont.

Since $\mathcal{M}$ is not a generalized partial 3-tree, we know in K, each component has degree at least 4, i.e. $|E_i| \geq 4$ for $1 \leq i \leq n$. Hence we have:

$$6n - 6 \geq \frac{3}{2} \sum_i 4 = 6n.$$

Contradiction. □

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Stronger notions than Maxwell-independence

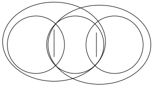
A natural open problem is to improve the bound in main theorem directly by considering other notions of independence that are stronger than Maxwell-independence.

Maximum Maxwell-independent sets

Note that Maxwell-rigid graph requires the *maximum* Maxwell-independent set to be of size $\geq 3|V| - 6$. Although the maximum Maxwell-independent set is trivially as big as the rank (and is not directly relevant to finding good bounds on rank), covers by vertex-maximal, Maxwell-rigid components have played a role in some of the Theorems that give useful bounds on rank. Recall that Hendrickson [Hendrickson, 92] gives an algorithm to test Maxwell-independence by finding a maximal Maxwell-independent set that is automatically maximum in 2D. While an extension of Hendrickson [Hendrickson, 92] to 3D given in [Lomonosov, 04] finds a maximal Maxwell-independent set, it is not guaranteed to be maximum. Thus another question of interest is whether maximum Maxwell-independent sets can be characterized in some natural way.

Guarantee 2-thin

Note that the complete collection of (proper) vertex-maximal, Maxwell-rigid components is far from being a 2-thin cover. For example, in the following figure we have 3 K_5's and the neighboring K_5's share an edge with each other. There are two vertex-maximal, Maxwell-rigid components, each of which consists of 2 K_5's with a shared edge.



Using strong Maxwell-rigidity to guarantee 2-thin

Definition

A graph $G = (V, E)$ is strong Maxwell-rigid if for all maximal Maxwell-independent edge sets $E' \subseteq E$, we have $|E'| \geq 3|V'| - 6$.

Conjecture

Given graph G, any cover $\mathcal{X}$ by a collection of vertex-maximal, strong Maxwell-rigid components is a 2-thin cover.

Thank you!

Report of Banff International Research Station workshop on rigidity and application, 2008

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