Characterizing minimal generic rigid graphs in the *d*-dimensional space

András Recski

Budapest University of Technology and Economics

Toronto, 2011

Characterizing minimal generic rigid graphs in the *d*-dimensional space

András Recski

Budapest University of Technology and Economics

Toronto, 2011

Combinatorial conditions for the rigidity of tensegrity frameworks

La Vacquerie-et-Saint-Martin-de-Castries, Languedoc-Roussillon, 2007

András Recski

Budapest University of Technology and Economics

When is a matrix singular?

Let A be an $n \times n$ matrix.

We can find det A, that is, the signed sum of $n!$ products, in $O(n^3)$ steps.

When is a matrix singular?

Let A be an $n \times n$ matrix.

det A can be determined effectively if the entries are from a field where division can be performed quickly.

When is a matrix singular?

Let A be an $n \times n$ matrix.

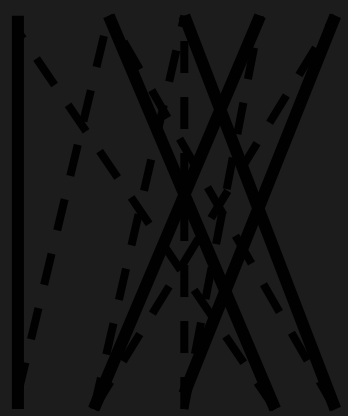
det A can be determined effectively if the entries are from a field where division can be performed quickly.
But what if they are from a field in general, or from a commutative ring ?

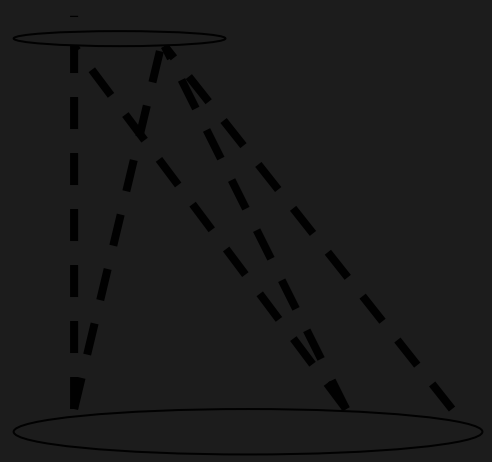
A classical case

D. Kőnig, 1915

If the nonzero entries are distinct variables (or real numbers, algebraically independent over the field of the rationals) then we can describe the zero-nonzero pattern of the matrix by a bipartite graph and check whether the graph has a perfect matching.

$$A_1 = \begin{pmatrix} x_1 & x_2 & 0 & 0 & 0 \\ 0 & 0 & x_3 & x_4 & x_5 \\ 0 & 0 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & 0 & 0 & 0 \\ 0 & x_{11} & x_{12} & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} x_1 & x_2 & 0 & 0 & 0 \\ 0 & 0 & x_3 & x_4 & x_5 \\ 0 & 0 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & 0 & 0 & 0 \\ 0 & x_{11} & 0 & 0 & 0 \end{pmatrix}$$





If the nonzero entries are different variables (or real numbers, algebraically independent over the field of the rationals)

then rank equals term rank.

Systems of Distinct Representatives and Linear Algebra*

Jack Edmonds

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(November 16, 1966)

Some purposes of this paper are: (1) To take seriously the term, "term rank." (2) To make an issue of not "rearranging rows and columns" by not "arranging" them in the first place. (3) To promote the numerical use of Cramer's rule. (4) To illustrate that the relevance of "number of steps" to "amount of work" depends on the amount of work in a step. (5) To call attention to the computational aspect of SDR's, an aspect where the subject differs from being an instance of familiar linear algebra. (6) To describe an SDR instance of a theory on extremal combinatorics that uses linear algebra in very different ways than does totally unimodular theory. (The preceding paper, Optimum Branchings, describes another instance of that theory.)

Key Words: Algorithms, combinatorics, indeterminates, linear algebra, matroids, systems of distinct representatives, term rank.

1. Introduction

'he well-known concept of term rank [5, 6],[1] is wn here to be a special case of linear-algebra rank. s observation is used to provide a simple linear-ebra proof of the well-known SDR theorem. Except familiar linear algebra, the paper is self-contained. ncidentally to SDR's, an algorithm is presented for nputing the determinant or the rank of any matrix r any integral domain. It is a variation of Gaussian ., linear) elimination which has certain advantages. s observed to be an interestingly bad algorithm for nputing term rank.

'he final part of the paper discusses some simple troidal aspects of SDR's.

However, here the word "transversal" will be used differently.)

3. Matrices of Zeros and Ones

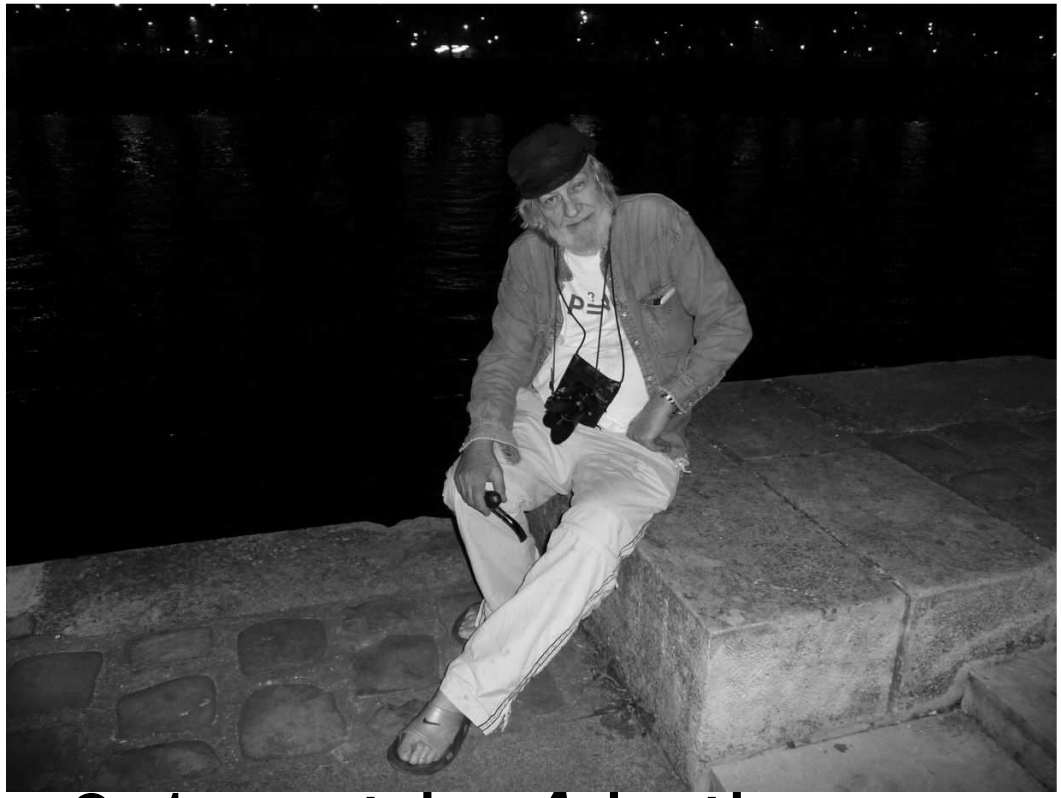
The subject of SDR's is frequently treated in the context of matrices of 0's and 1's. The *incidence matrix* of the family Q of subsets of E is the matrix $A=[a_{ij}]$, $i\epsilon E$, $j\epsilon Q$, such that $a_{ij}=1$ if $i\epsilon j$, and $a_{ij}=0$ otherwise.

A *matching* in a matrix is a subset of its positions (i, j) such that first indices (rows) of members are all different and second indices (columns) of members are all different. A *transversal* (*column transversal*) of a matrix is a matching in the matrix which has a member

Edmonds, 1967

The term rank of a 0,1 -matrix A is the same as the linear algebra rank of the matrix obtained by replacing the 1's in A by distinct indeterminates over any integral domain.

Edmonds, 2009



Edmonds, 1967

The term rank of a 0,1 matrix A is the same as the linear algebra rank of the matrix obtained by replacing the 1's in A by distinct indeterminates over any integral domain.

If the nonzero entries are **not** independent then deciding singularity may be hard.

Such algebraic dependencies often occur in engineering applications.

If the nonzero entries are **not** independent then deciding singularity may be hard.

Such algebraic dependencies often occur in engineering applications.

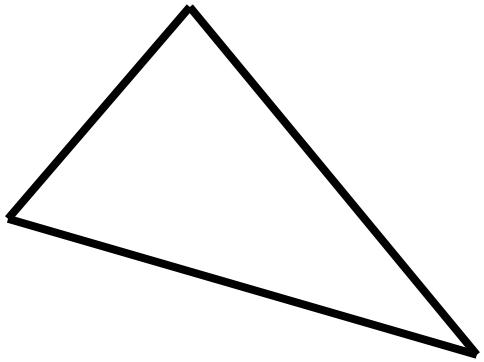
Examples in the analysis of **electric networks** were collected in a survey paper to appear as Chapter 11 of
R. Mahjoub (ed.),
Progress in Combinatorial Optimization,
Wiley, 2012.

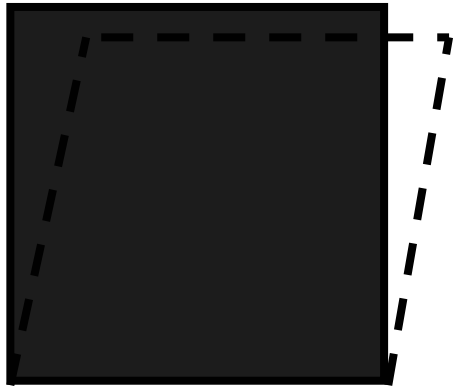
If the nonzero entries are **not** independent then deciding singularity may be hard.

Such algebraic dependencies often occur in engineering applications.

In this talk examples in the analysis of **bar-and-joint frameworks** are discussed.

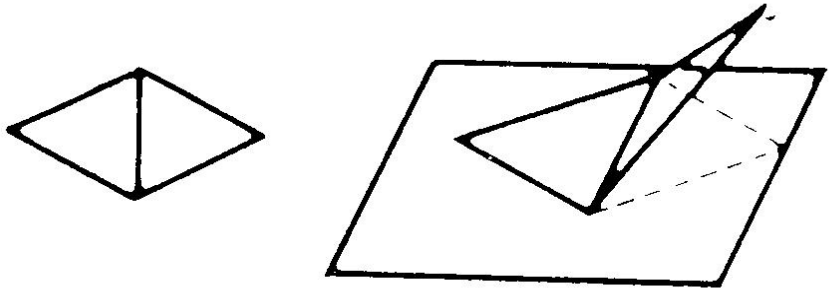
Bar and joint frameworks





Rigid

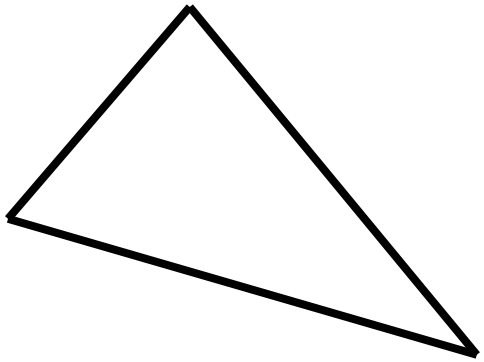
Non-rigid (mechanism)

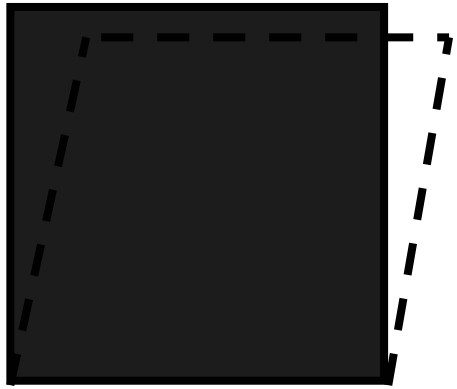


Rigid in the plane

Non-rigid in the space

Bar and joint frameworks





Rigid

Non-rigid (mechanism)

How can we describe the difference?

What is the effect of a rod?

i ——————————— j

$$\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} = \text{constant}$$

$$(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2 + (z_i(t) - z_j(t))^2 = c_{ij}$$

What is the effect of a rod?

i ——————————— j

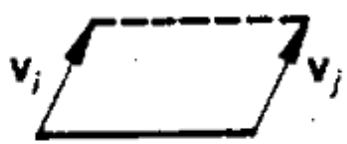
$$\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} = \text{constant}$$

$$(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2 + (z_i(t) - z_j(t))^2 = c_{ij}$$

$$(x_i(t) - x_j(t))(\dot{x}_i(t) - \dot{x}_j(t)) + (y_i(t) - y_j(t))(\dot{y}_i(t) - \dot{y}_j(t)) +$$
$$+ (z_i(t) - z_j(t))(\dot{z}_i(t) - \dot{z}_j(t)) = 0,$$

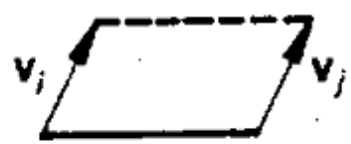
$$(x_i(t) - x_j(t))(\dot{x}_i(t) - \dot{x}_j(t)) + (y_i(t) - y_j(t))(\dot{y}_i(t) - \dot{y}_j(t)) +$$
$$+ (z_i(t) - z_j(t))(\dot{z}_i(t) - \dot{z}_j(t)) = 0,$$

$$(\mathbf{h}_i - \mathbf{h}_j)(\mathbf{v}_i - \mathbf{v}_j) = 0 \text{ for every rod } R_{ij}$$



$$(x_i(t) - x_j(t))(\dot{x}_i(t) - \dot{x}_j(t)) + (y_i(t) - y_j(t))(\dot{y}_i(t) - \dot{y}_j(t)) +$$
$$+(z_i(t) - z_j(t))(\dot{z}_i(t) - \dot{z}_j(t)) = 0,$$

$$(\mathbf{h}_i - \mathbf{h}_j)(\mathbf{v}_i - \mathbf{v}_j) = 0 \text{ for every rod } R_{ij}$$



$$(x_i(t) - x_j(t))\dot{x}_i(t) + (x_j(t) - x_i(t))\dot{x}_j(t) + \ldots$$
$$\ldots + (z_i(t) - z_j(t))\dot{z}_i(t) + (z_j(t) - z_i(t))\dot{z}_j(t) = 0.$$

$$(x_i(t) - x_j(t))\dot{x}_i(t) + (x_j(t) - x_i(t))\dot{x}_j(t) + \dots$$
$$\dots + (z_i(t) - z_j(t))\dot{z}_i(t) + (z_j(t) - z_i(t))\dot{z}_j(t) = 0.$$

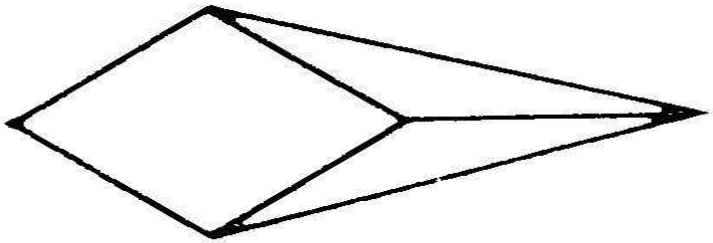
Au=0

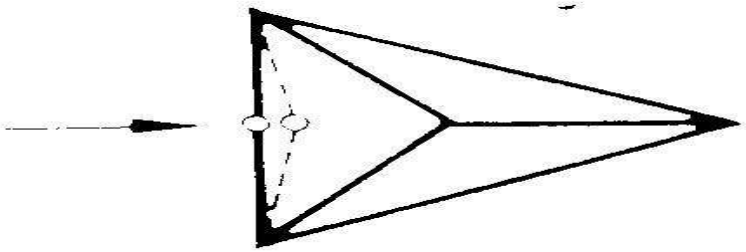
$$\begin{bmatrix} x_1 - x_2 & x_2 - x_1 & 0 & 0 & y_1 - y_2 & y_2 - y_1 & 0 & 0 \\ x_1 - x_3 & 0 & x_3 - x_1 & 0 & y_1 - y_3 & 0 & y_3 - y_1 & 0 \\ x_1 - x_4 & 0 & 0 & x_4 - x_1 & y_1 - y_4 & 0 & 0 & y_4 - y_1 \\ 0 & x_2 - x_3 & x_3 - x_2 & 0 & 0 & y_2 - y_3 & y_3 - y_2 & 0 \\ 0 & x_2 - x_4 & 0 & x_4 - x_2 & 0 & y_2 - y_4 & 0 & y_4 - y_2 \\ 0 & 0 & x_3 - x_4 & x_4 - x_3 & 0 & 0 & y_3 - y_4 & y_4 - y_3 \end{bmatrix}$$

A framework with n joints in the d-dimensional space is defined to be (infinitesimally) rigid if

$$\mathrm{r}(A) = nd - d(d+1)/2$$

In particular: $\mathrm{r}(A) = n - 1$ if $d = 1$, $\mathrm{r}(A) = 2n - 3$ for the plane and $\mathrm{r}(A) = 3n - 6$ for the 3-space.





Rigid

Non-rigid (has an *infinitesimal* motion)

(although the graphs of the two frameworks are isomorphic)

- For certain graphs (like C_4) every realization leads to nonrigid frameworks.
- For others, some of their realizations lead to rigid frameworks.

These latter type of graphs are called *generic rigid.*

- Deciding the rigidity of a framework (that is, of an actual realization of a graph) is a problem in linear algebra.

- Deciding whether a graph is generic rigid is a combinatorial problem.

- Deciding the rigidity of a framework (that is, of an actual realization of a graph) is determining r*(A)* over the field of the reals.

- Deciding whether a graph is generic rigid is determining r*(A)* over a commutative ring.

The matrix A in case of K_4 in the 2-dimensional space

$$\begin{bmatrix} x_1 - x_2 & x_2 - x_1 & 0 & 0 & y_1 - y_2 & y_2 - y_1 & 0 & 0 \\ x_1 - x_3 & 0 & x_3 - x_1 & 0 & y_1 - y_3 & 0 & y_3 - y_1 & 0 \\ x_1 - x_4 & 0 & 0 & x_4 - x_1 & y_1 - y_4 & 0 & 0 & y_4 - y_1 \\ 0 & x_2 - x_3 & x_3 - x_2 & 0 & 0 & y_2 - y_3 & y_3 - y_2 & 0 \\ 0 & x_2 - x_4 & 0 & x_4 - x_2 & 0 & y_2 - y_4 & 0 & y_4 - y_2 \\ 0 & 0 & x_3 - x_4 & x_4 - x_3 & 0 & 0 & y_3 - y_4 & y_4 - y_3 \end{bmatrix}$$

- Special case: minimal generic rigid graphs (when the deletion of any edge destroys rigidity).

- In this case the number of rods must be $r(A) = nd - d(d+1)/2$

- Special case: minimal generic rigid graphs (when the deletion of any edge destroys rigidity).

- In this case the number of rods must be $r(A) = nd - d(d+1)/2$

- Why minimal?

A famous minimally rigid structure:

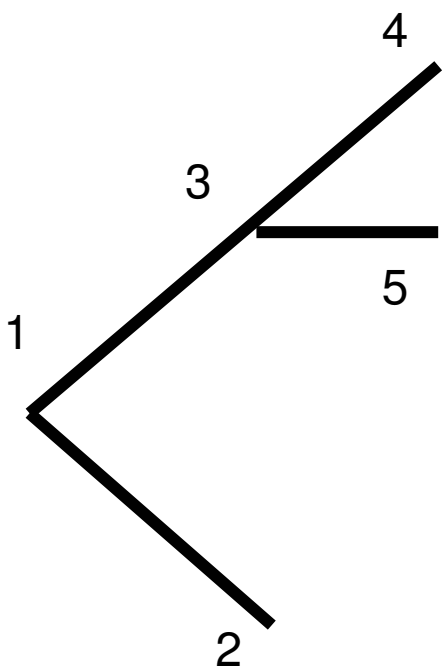
Szabadság Bridge, Budapest

In the 1-dimensional case a framework is rigid if and only if its graph is connected, hence minimal (generic) rigid graphs are the trees.

In the 2-dimensional case minimal generic rigid graphs are well characterized (see later).

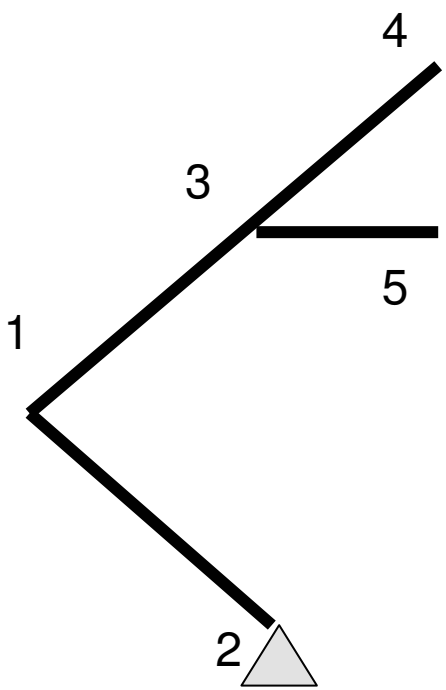
In higher dimensions the problem is open.

A 1-dimensional example



x_1-x_2	x_2-x_1	0	0	0
x_1-x_3	0	x_3-x_1	0	0
0	0	x_3-x_4	x_4-x_3	0
0	0	x_3-x_5	0	x_5-x_3

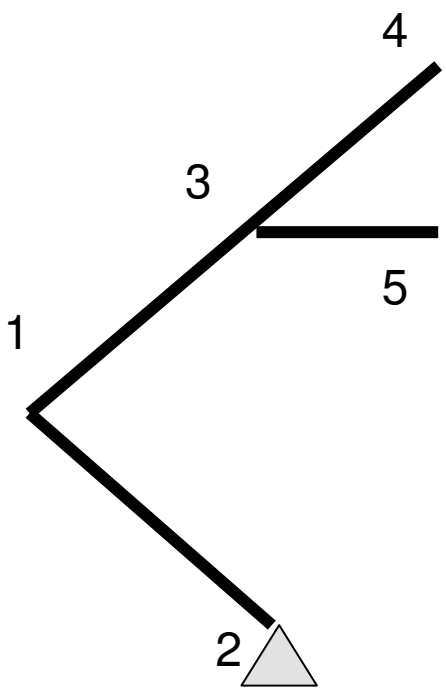
A 1-dimensional example



$$
\begin{bmatrix}
x_1-x_2 & x_2-x_1 & 0 & 0 & 0 \\
x_1-x_3 & 0 & x_3-x_1 & 0 & 0 \\
0 & 0 & x_3-x_4 & x_4-x_3 & 0 \\
0 & 0 & x_3-x_5 & 0 & x_5-x_3
\end{bmatrix}
$$

$$
\begin{bmatrix}
x_1-x_2 & 0 & 0 & 0 \\
x_1-x_3 & x_3-x_1 & 0 & 0 \\
0 & x_3-x_4 & x_4-x_3 & 0 \\
0 & x_3-x_5 & 0 & x_5-x_3
\end{bmatrix}
$$

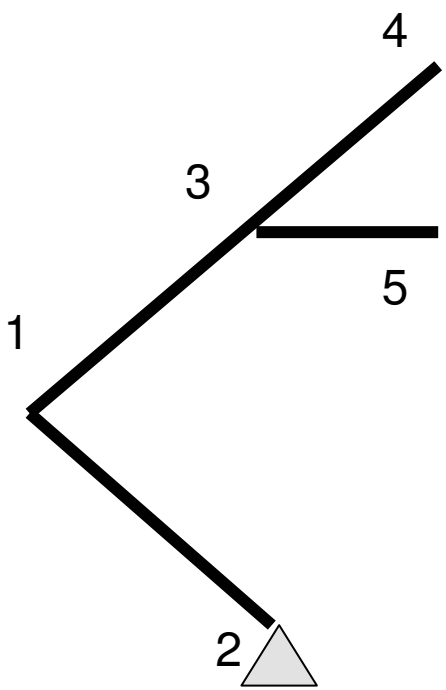
A 1-dimensional example



$$
\begin{bmatrix}
x_1-x_2 & x_2-x_1 & 0 & 0 & 0 \\
x_1-x_3 & 0 & x_3-x_1 & 0 & 0 \\
0 & 0 & x_3-x_4 & x_4-x_3 & 0 \\
0 & 0 & x_3-x_5 & 0 & x_5-x_3
\end{bmatrix}
$$

$$
\begin{bmatrix}
x_1-x_2 & 0 & 0 & 0 \\
x_1-x_3 & x_3-x_1 & 0 & 0 \\
0 & x_3-x_4 & x_4-x_3 & 0 \\
0 & x_3-x_5 & 0 & x_5-x_3
\end{bmatrix}
$$

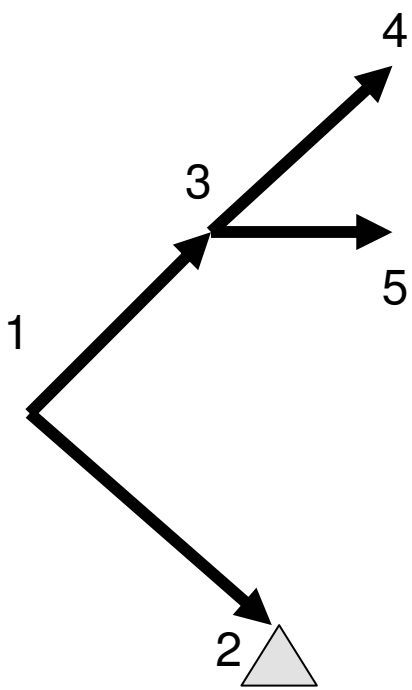
A 1-dimensional example



x_1-x_2	0	0	0
x_1-x_3	x_3-x_1	0	0
0	x_3-x_4	x_4-x_3	0
0	x_3-x_5	0	x_5-x_3

One single „product" – expanded to $2^4 = 16$ products. They correspond to the 16 possible orientations of the tree.

A 1-dimensional example



x_1-x_2	0	0	0
x_1-x_3	x_3-x_1	0	0
0	x_3-x_4	x_4-x_3	0
0	x_3-x_5	0	x_5-x_3

For example, this orientation corresponds to $x_1^2 x_3^2$

The sequence [2,0,2,0,0] of the exponents will be called the profile of the product.

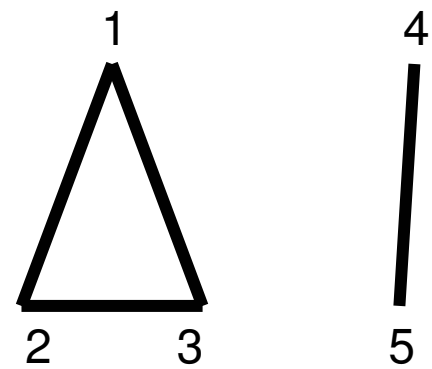
The n-tuple $[\alpha_1, \alpha_2, \ldots \alpha_n]$ with $\Sigma\alpha_i = n-1$ arises as the profile of an expansion member if and only if the tree has an orientation satisfying $d_{out}(v_i) = \alpha_i$ for every vertex v_i

This can be checked in polynomial time (using a matroid intersection algorithm)

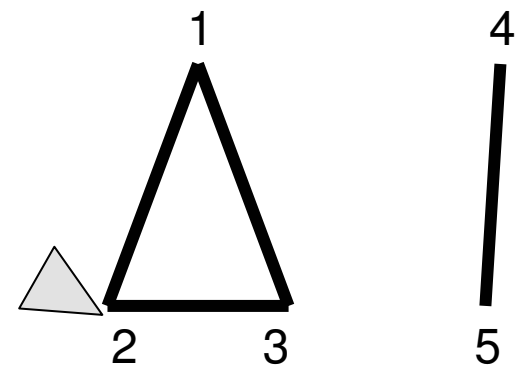
If a graph with *n* vertices and *n-1* edges is not a tree then it is not generic rigid (in the 1-dimensional case).

The determinant may or may not have nonzero expansion members, depending on the selection of the vertex to be fixed.

$$
\begin{matrix}
x_1-x_2 & x_2-x_1 & 0 & 0 & 0 \\
x_1-x_3 & 0 & x_3-x_1 & 0 & 0 \\
0 & x_2-x_3 & x_3-x_2 & 0 & 0 \\
0 & 0 & 0 & x_4-x_5 & x_5-x_4
\end{matrix}
$$

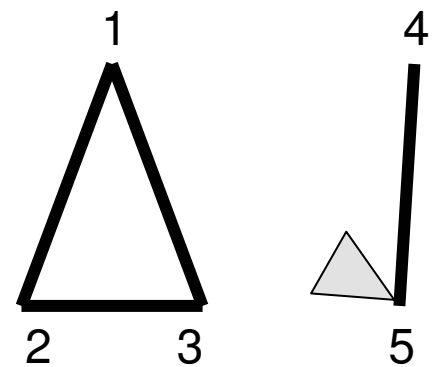


$$
\begin{matrix}
x_1-x_2 & x_2-x_1 & 0 & 0 & 0 \\
x_1-x_3 & 0 & x_3-x_1 & 0 & 0 \\
0 & x_2-x_3 & x_3-x_2 & 0 & 0 \\
0 & 0 & 0 & x_4-x_5 & x_5-x_4
\end{matrix}
$$



If we delete one of the first three columns, the remaining 4X4 matrix will have no nonzero expansion member

$$
\begin{matrix}
x_1-x_2 & x_2-x_1 & 0 & 0 & 0 \\
x_1-x_3 & 0 & x_3-x_1 & 0 & 0 \\
0 & x_2-x_3 & x_3-x_2 & 0 & 0 \\
0 & 0 & 0 & x_4-x_5 & x_5-x_4
\end{matrix}
$$



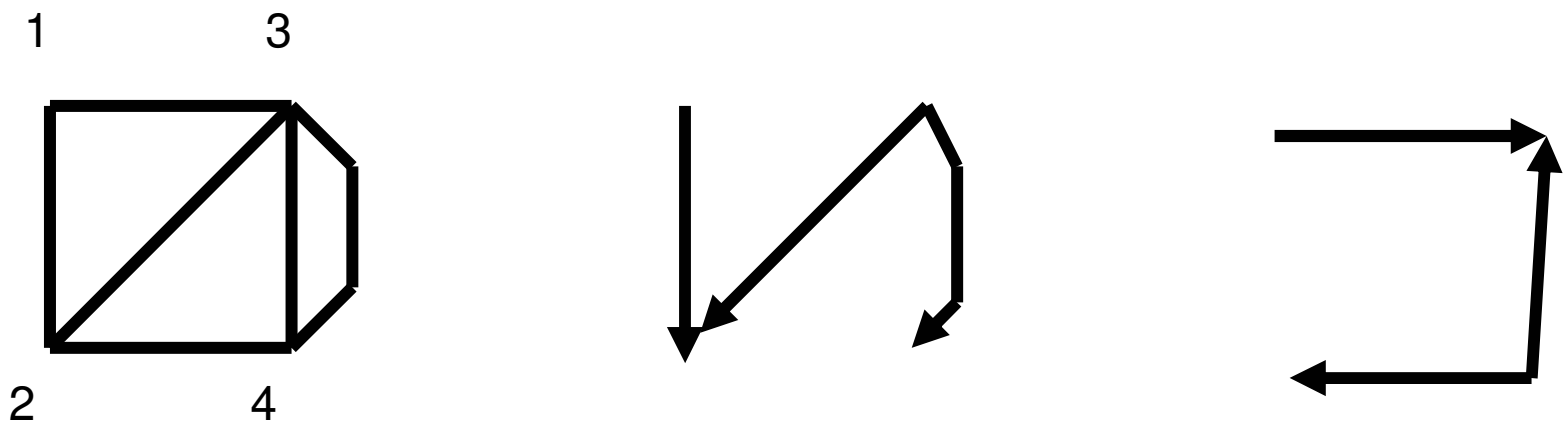
If we delete one of the last two columns, the remaining 4X4 matrix will have 32 nonzero expansion members (which will pairwise cancel out each other).

The reason of the different behaviour:

The profile of a nonzero expansion member uniquely determines the orientation of the graph
if and only if
the graph has no circuits.

Lovász and Yemini (1982)

In the 2-dimensional case a graph G with n vertices and $2n$-3 edges is minimal generic rigid if and only if for any edge e of G the edge set $G+e$ is the union of two trees.



The profile of this pair of trees is
[1,0,2,0; 1,0,0,2]
and this corresponds to the
expansion member
$x_1 x_3^2 y_1 y_4^2$

The corresponding 2-dimensional problem:

Given a graph G with n vertices and $2n - 2$ edges, and two n-tuples $[\alpha_1, \alpha_2, \ldots \alpha_n]$ with $\Sigma\alpha_i = n - 1$ and $[\beta_1, \beta_2, \ldots \beta_n]$ with $\Sigma\beta_i = n - 1$.

Can we decompose G into two trees so that the first one has an orientation satisfying $d_{out}(v_i)=\alpha_i$ for every vertex v_i and the second has an orientation satisfying $d_{out}(v_i)=\beta_i$ for every vertex v_i ?

Results:

- Either decomposability alone or orientability alone can be formulated as the intersection problem of two matroids and hence can be checked in polynomial time (these are well known since the late 1960's).
- The whole problem can be formulated as the intersection problem of three matroids (but this latter problem is NP-hard).

Conjecture

The whole problem is solvable in polynomial time and probably so is its d-dimensional generalization for every d.

Thank you for your attention

recski@cs.bme.hu