Rigidity, tensegrity and applications

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Flavors of rigidity

- (Local) rigidity
- Infinitesimal rigidity = static rigidity
- Global rigidity
- Universal rigidity
- Generic (local, infinitesimal) rigidity
- Generic global rigidity
- Generic universal rigidity

Rigid Objects

- Bar-and-joint frameworks
- Tensegrity frameworks (with struts and cables)
- Bar-and-body-frameworks
- Body-and-hinge frameworks
- Panel-and-hinge frameworks

Applications

- Structural mechanics-Why things don’t fall down.
- Mechanical structures-Why linkages move.
- Stability of structures-How wiggly is my structure.
- Stability of packings-Why granular materials jam.
- Robot arm manipulation.
- Point location-Am I where I think I am?

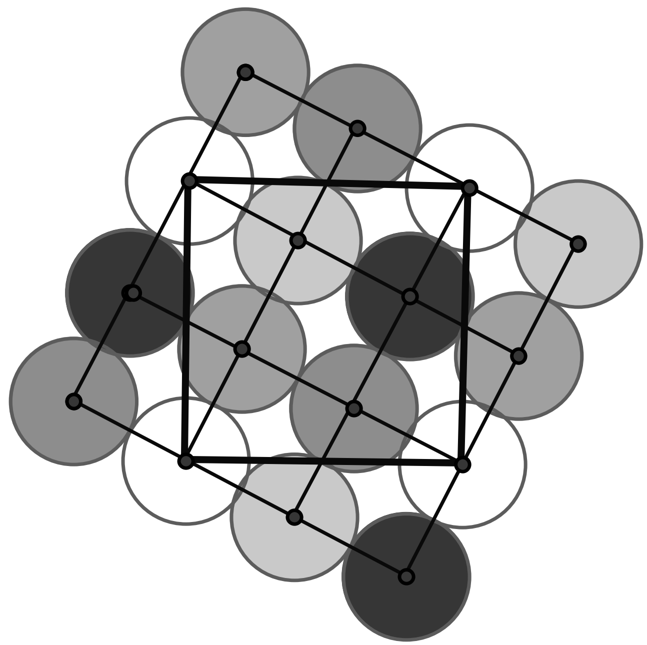
An application: How can you find the most dense disk packings in E^d ?

Questions to ask first

- What is the container?
- Can some sort of finiteness be assumed?
- Can locally most dense packings be used to simplify the analysis?
- Is there a finite algorithm to find a most dense packing? (Perhaps with an infeasible running time.)

What is the container?

One Answer: The flat torus = R^d/Λ, where Λ is a lattice in R^d.



But which lattice/torus?

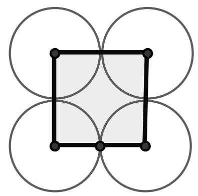
Take your favorite. If you think that the best packing is achieved with a particular lattice/torus, and it works for all covers, then that is the best packing overall.

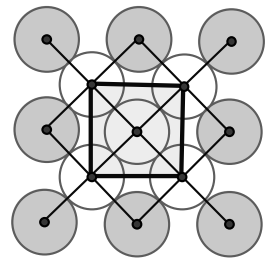
This leads to a subquestion: For a fixed shape of a torus, and an integer n, what is the most dense packing of n (congruent) disks in that torus?

For example

For a square torus. (Work of Will Dickinson)

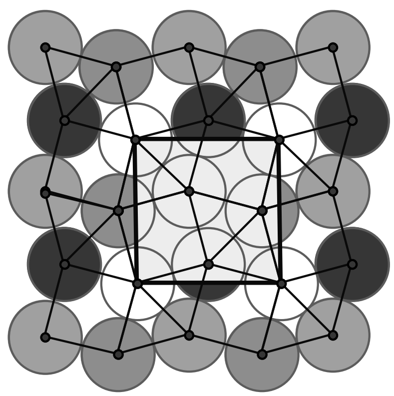
One disk

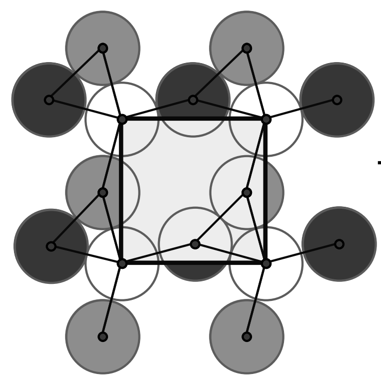




Two disks

Four disks





Three disks

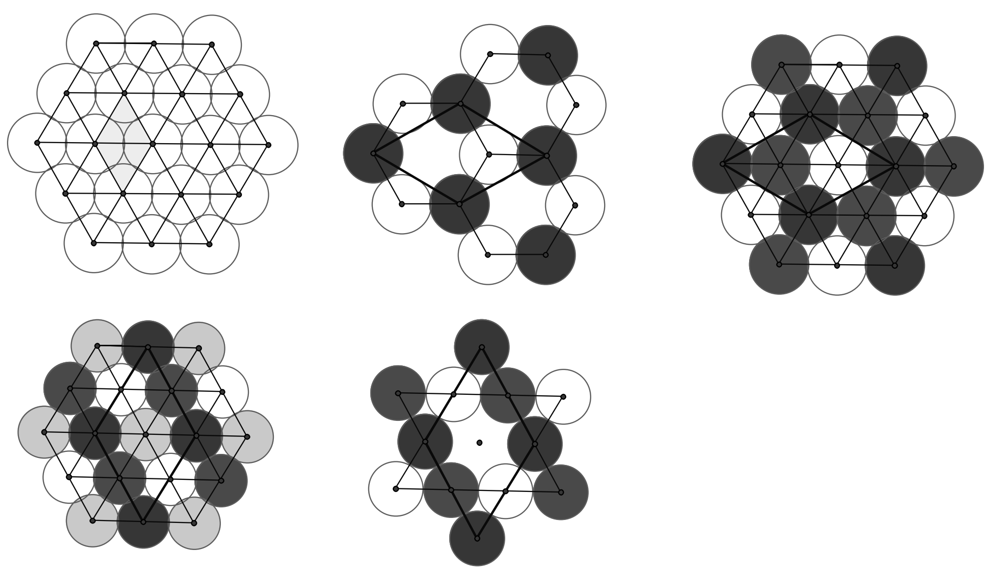
A better example:
The triangular torus

The best packings of n disks in a triangular torus:

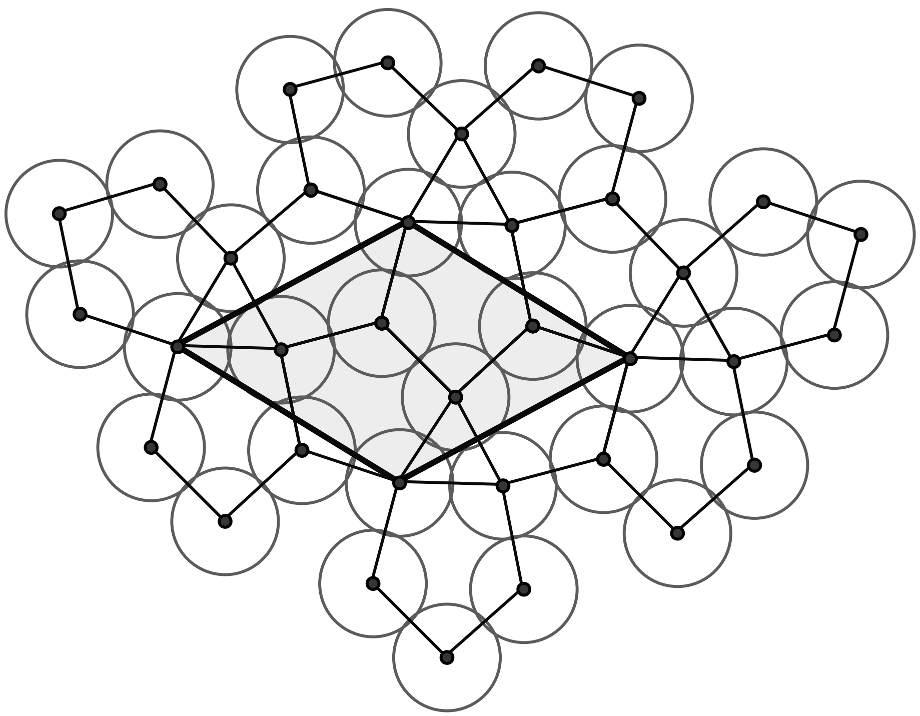
For $n = a^2 + ab + b^2$, a, b integers: A similar sublattice.

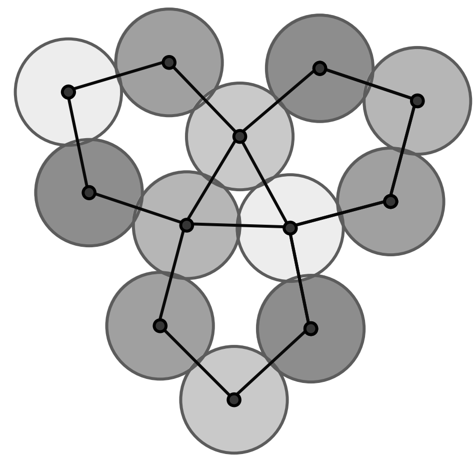
n = 1, 3, 4, 7, 9, 12, 13, 16, 19, …= *lattice triangle numbers*

For $n = a^2 + ab + b^2 - 1$, n not a lattice triangle number: A similar sublattice minus one disk, n = 2, 6, 8, 11, 15, 18, . . .

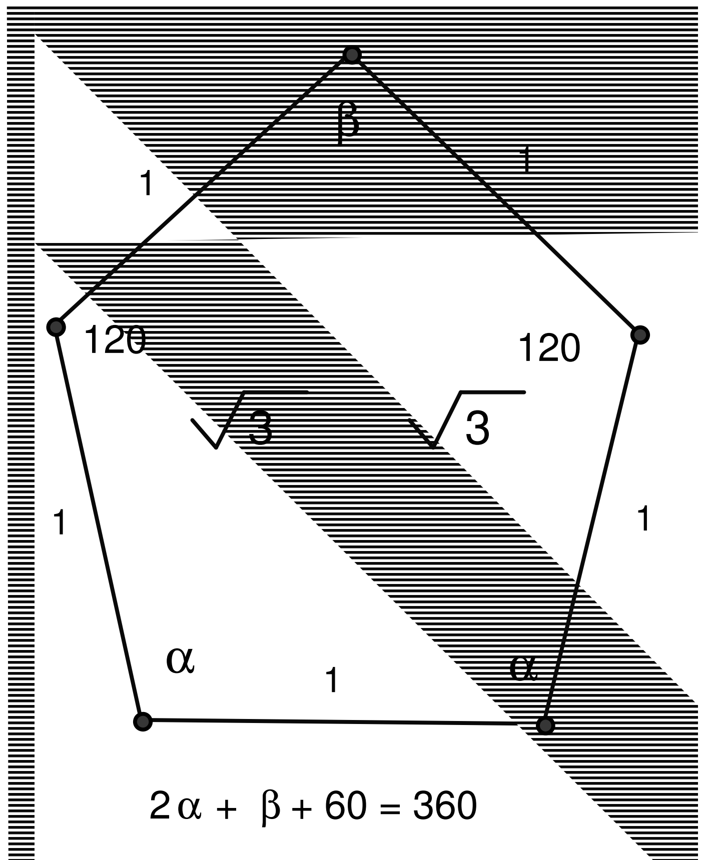


The best packing of 5 disks in a triangular lattice





The dimensions of the pentagon



Conjecture

In the plane, when n + 1 is a lattice triangle number, but n is not a lattice triangle number, then the most dense packing of n congruent disks in a triangular torus is the lattice packing minus one packing disk.

Remark: This conjecture implies L. Fejes Toth's conjecture that the only finite rearrangement of the best equal circle packing in the plane, with one disk removed, is the best packing with one disk removed.

Remark

The area of the pentagon above is $(\sqrt{11} + 2\sqrt{3})r^2$, where r is the radius of the disks. The area of the triangle is $\sqrt{3}r^2$, so if there is more than 1 pentagon, then $13.56\ldots = 2(\sqrt{11} + 2\sqrt{3}) > 6\sqrt{3} = 10.39\ldots$ implies that when n+1 is a triangle lattice number, but n is not a triangle lattice number, a counterexample to the conjecture has at most 1 pentagon as above.

How to find best packings?

1. Find all graphs that are candidates for being graphs of a jammed packing.
2. Find a fixed distance realizations of the graphs in part 1.
3. Choose the packing graph with the largest edge length.

(A word from our sponsor)

Toroidal Bar Frameworks

A toroidal bar framework has an *infinitesimal flex* if there are vectors p_i' associated to each vertex p_i such that

$$(p_i - p_j)\,(p_i' - p_j') = 0$$

A toroidal bar framework is *infinitesimally rigid* if it has only the constant infinitesimal flexes. The term $(p_i - p_j)$ depends on the (homotopy class of) path(s) (lifts) from p_i to p_j in the torus.

Toroidal Strut Frameworks

A toroidal strut framework has an *infinitesimal flex* if there are vectors p_i' associated to each vertex p_i such that

$$(p_i - p_j)\,(p_i' - p_j') \geq 0$$

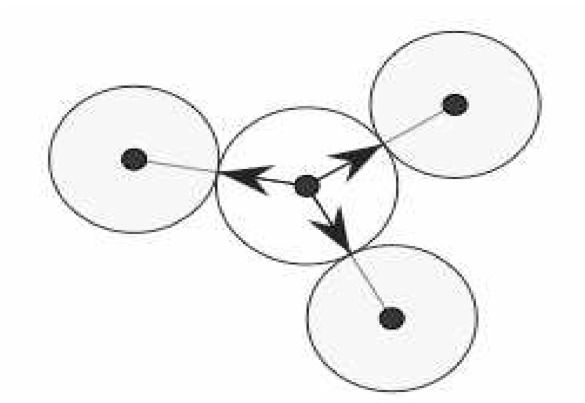
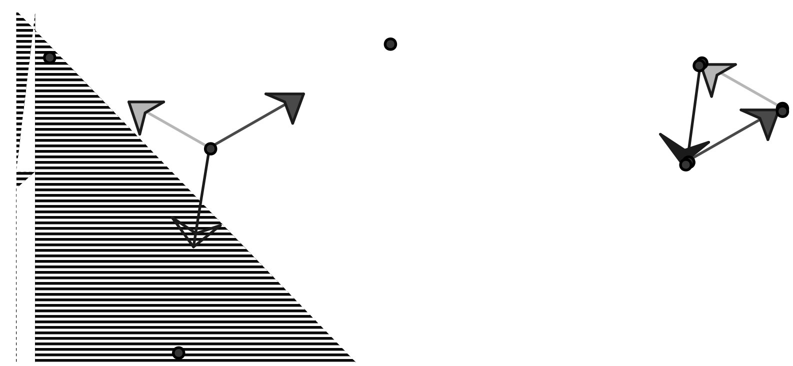
A toroidal strut framework is *infinitesimally rigid* if it has only the constant infinitesimal flexes. The term $(p_i - p_j)$ depends on the (homotopy class of) path(s) (lifts) from p_i to p_j in the torus.

Self-stresses

A toroidal framework has a *self-stress* if there are scalars ω_{ij} associated to each edge $\{i,j\}$ such that at each vertex the weighted vector sum

$$\sum_i \omega_{ij}(p_i - p_j) = 0.$$

A self-stress

This is a graphical interpretation of the equilibrium condition for a self-stress. The picture on the right is a verification that the weighted sum of the stress vectors is zero.

Infinitesimally rigid strut graphs

Think of a packing graph as a *strut tensegrity*. That means that each edge of the graph is allowed to increase in length, but not decrease.

Theorem (Roth-Whiteley): A strut tensegrity on a torus is infinitesimally rigid if and only if it is infinitesimally rigid as a bar framework (i. e. with fixed lengths of edges), and it has a stress with all stresses non-zero of the same sign.

The canonical push

Theorem: A toroidal strut tensegrity is (locally) rigid if and only if it is infinitesimally rigid.

The "if" direction is standard algebraic topology/geometry. The "only if" direction is achieved by linearly extending the infinitesimal direction of the motion.

Proof: If $\mathbf{p'}=(p_1', p_2' \ldots p_n')$ is an infinitesimal flex of the strut tensegrity, then for each $\{i, j\}$ a strut,

$p_i(t) = p_i + tp_i'$, for $t \geq 0$, is a finite flex of the tensegrity (i.e. disk packing) because

$$|p_i(t) - p_j(t)|^2 = ((p_i - p_j) + t(p_i' - p_j'))^2$$

$$=(p_i - p_j)^2 + 2t(p_i - p_j)(p_i' - p_j') + t^2(p_i' - p_j')^2 \geq 0.$$

Counting

Corollary: Every locally maximally dense torus packing has a stressed infinitesimally rigid positively stressed subpacking, the *spine*, and the spine must have at least

$2n-2+1 = 2n-1$ edges in its graph, for n disks.

Method for 1. Finding the graphs

a. The vertex degree is bounded by the local kissing number.
b. The configuration for the graph realization has a self-stress with stresses all the same sign by local rigidity and the local maximal density.
c. The homotopy type of the graph is restricted by the length of short paths in the graph.
d. The self-stressed realizations are globally rigid in a given homotopy class.

Collectively Jammed Packings

A packing is *collectively jammed* if it is rigid with a **fixed** lattice defining the torus.

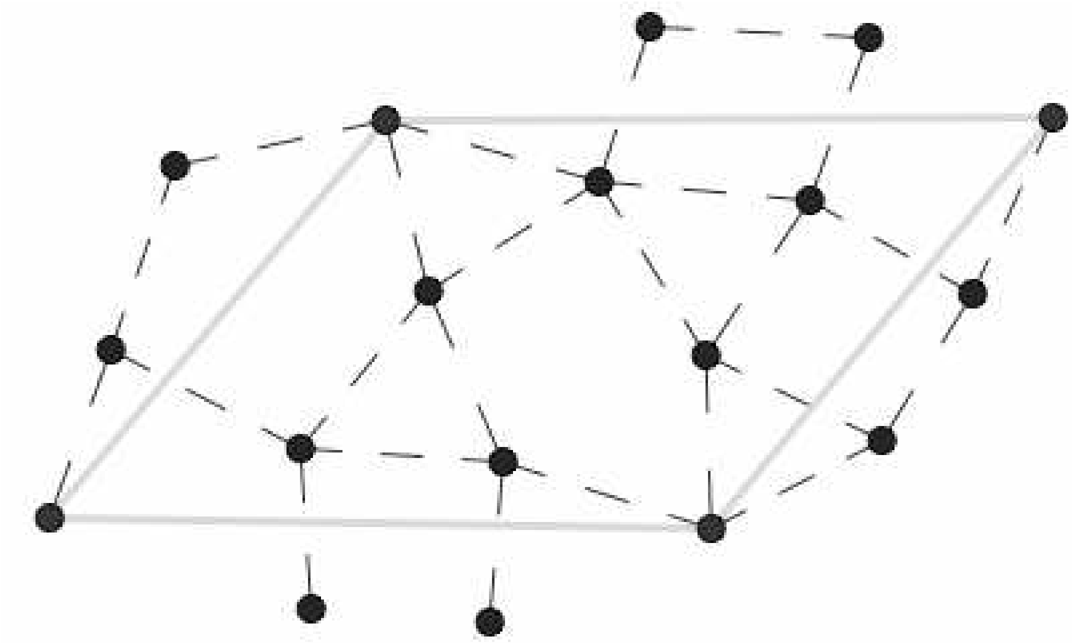
(So by a canonical push its strut graph is infinitesimally rigid and $e \geq 2n-1$.)

Example: The 5 disk packing above is collectively jammed. Its graph has $n=5$ nodes, but only $e=9=2n-1$ edges, the minimum needed to be collectively jammed. A k-fold cover, for $k > 1$, will have kn nodes and $ke=2kn-k < 2kn-1$ edges, which is less than needed to be collectively jammed.

Spider webs-global rigidity

A periodic positively stressed tensegrity of all cables has a globally rigid property.

Theorem: If a toroidal packing graph has a positive equilibrium stress, then any other realization in the same free homotopy class must have at least one cable that is longer.

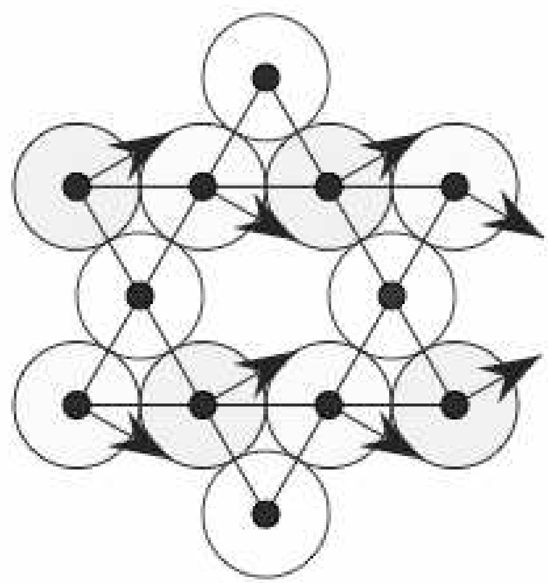


So …

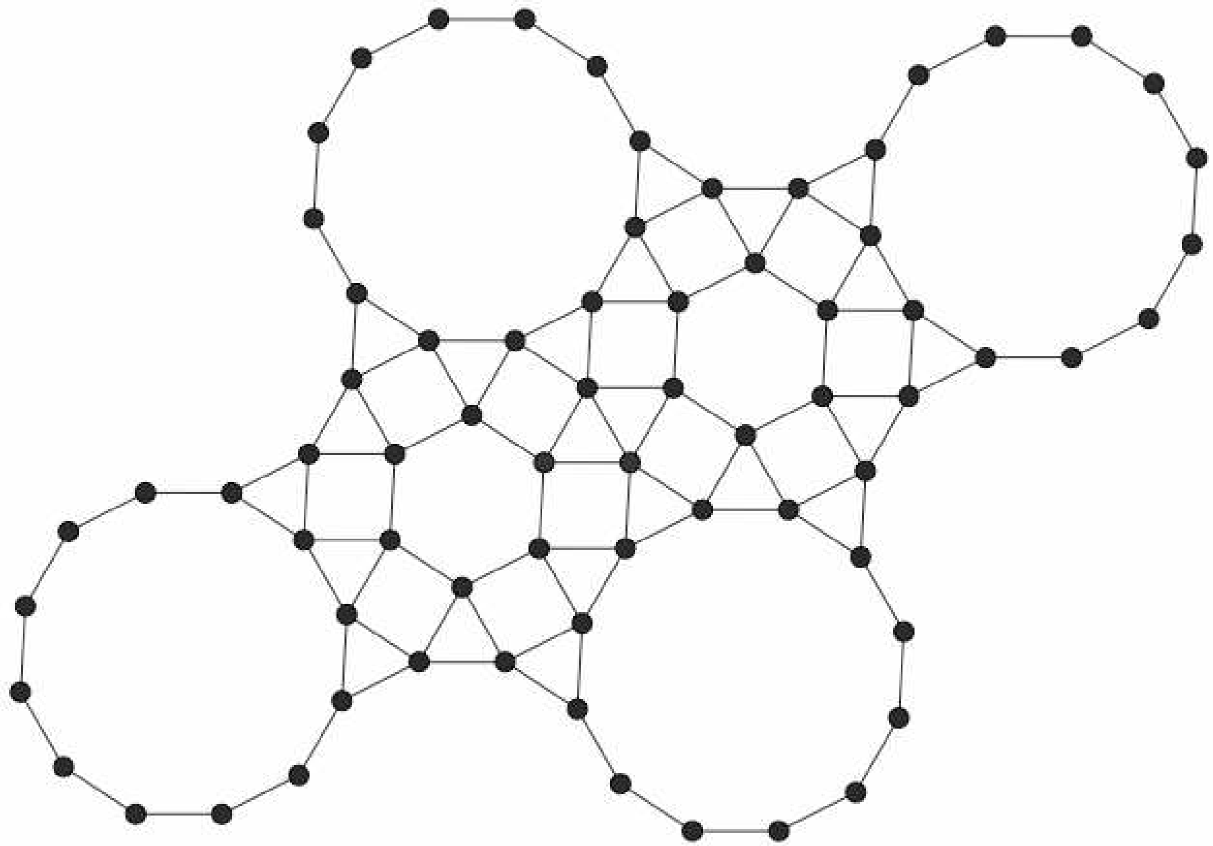
Any locally maximally dense stressed packing of equal disks with that packing graph, in the same homotopy class, must have a disk radius less than the maximum of the edge lengths of any realization of G in that homotopy class.

An example of a flexible packing

The following is the Kagome lattice that has a motion in the fixed triangular lattice. So this packing of n=3 disks in triangular torus is not even locally best, even though the graph has a positive self-stress and the bar framework is generically infinitesimally rigid.



Is this the graph of a collectively jammed packing?



Evidence that this packing is collectively jammed

The number of nodes n=30; and

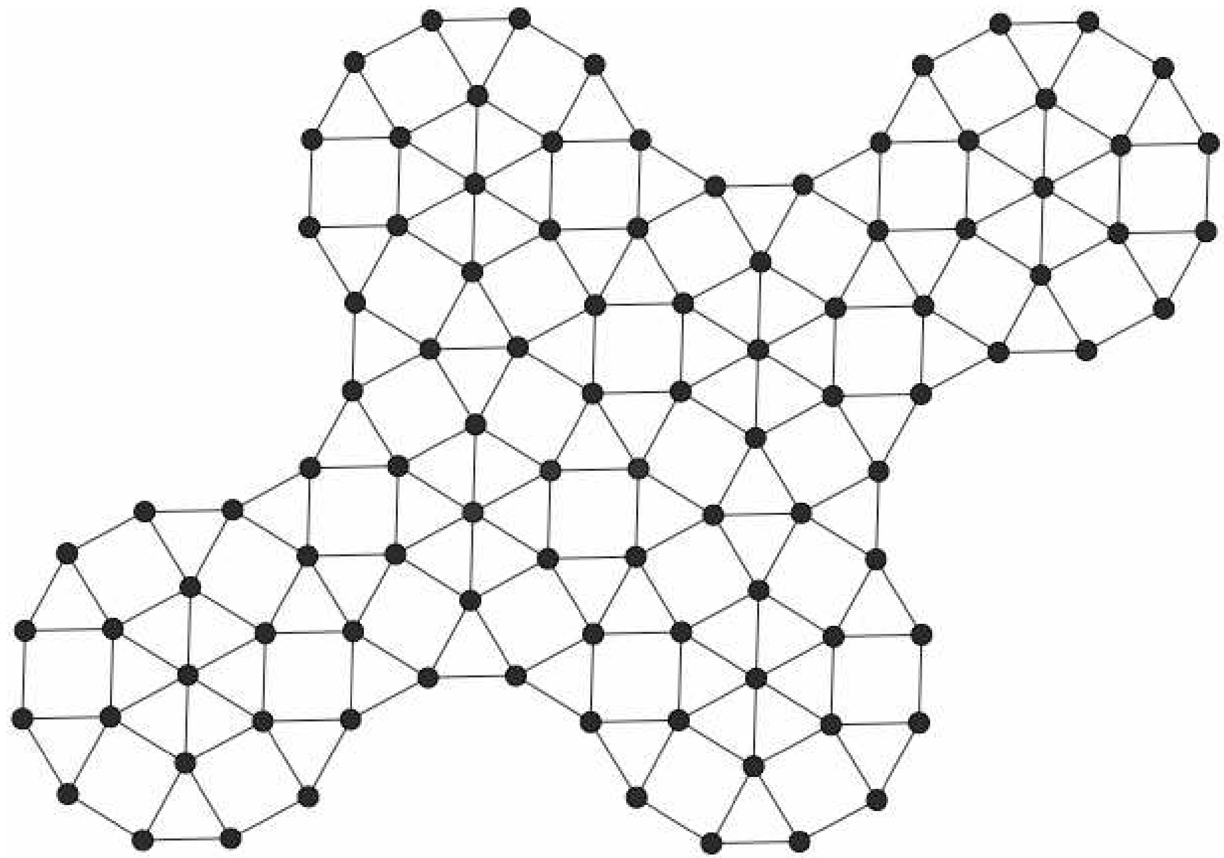
The number of edges (contacts) e=63=2n+3.

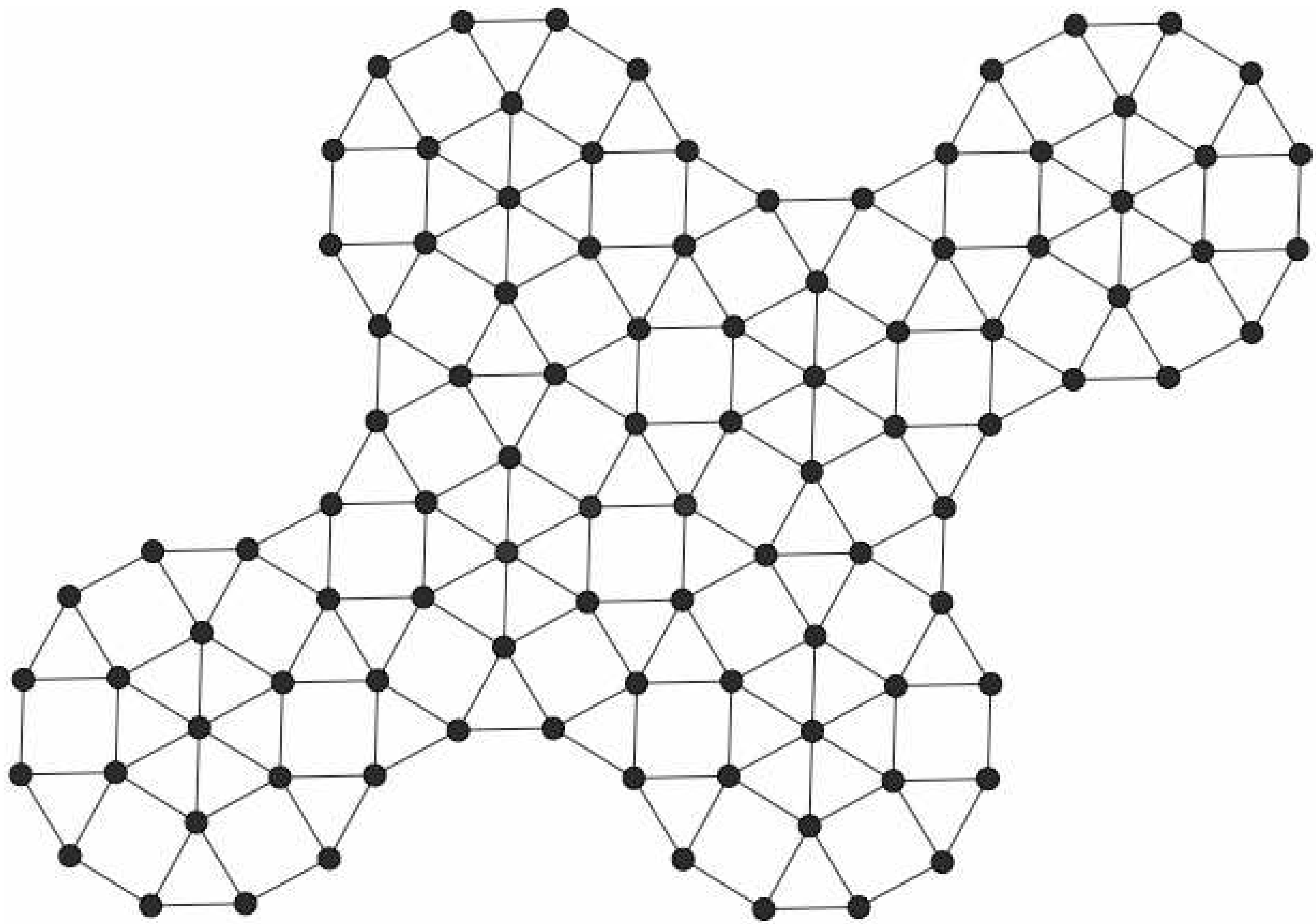
There is a (symmetric) positive self stress, non-zero on each edge.

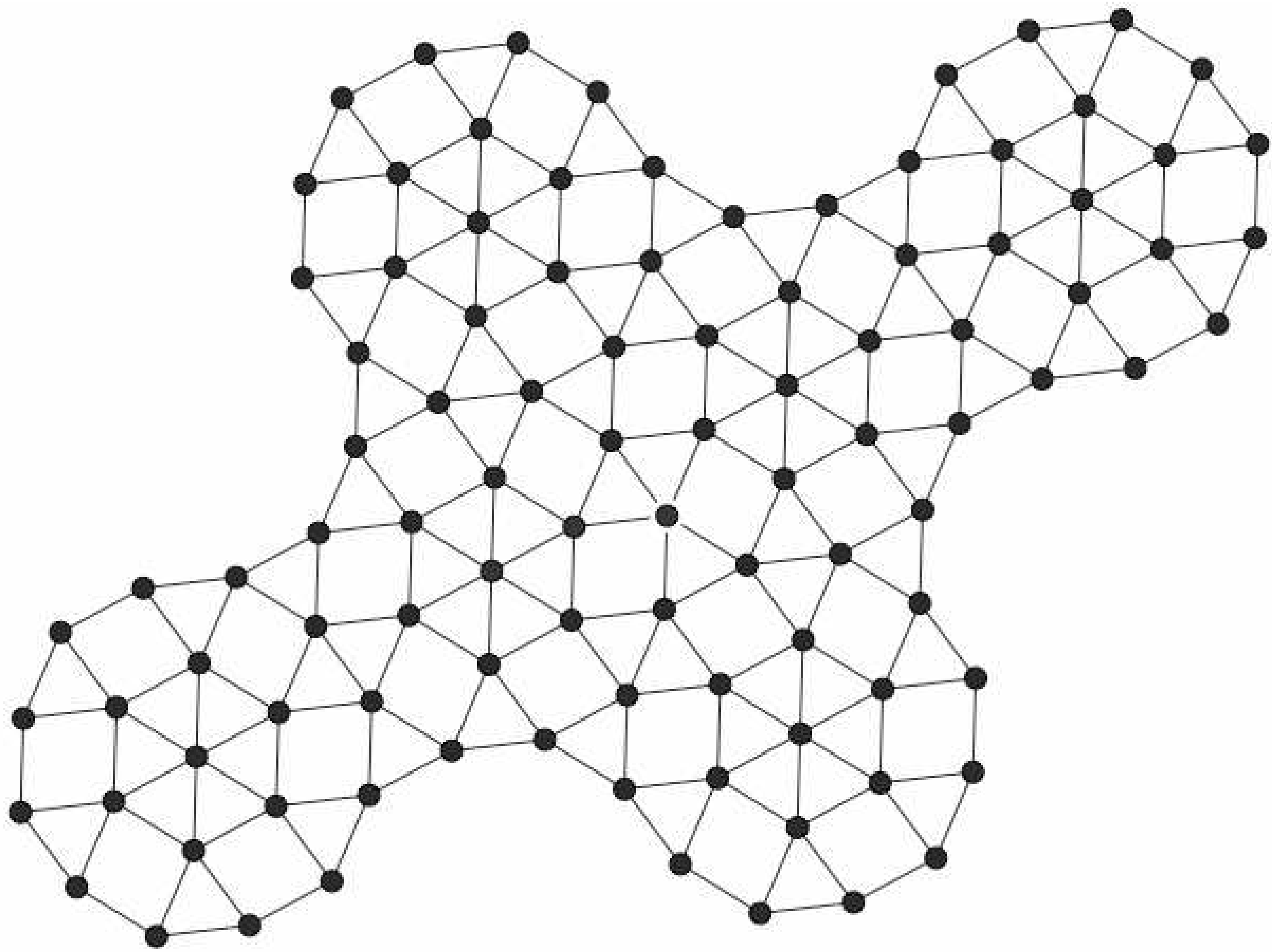
(This packing was suggested by Ruggero Gabbrielli)

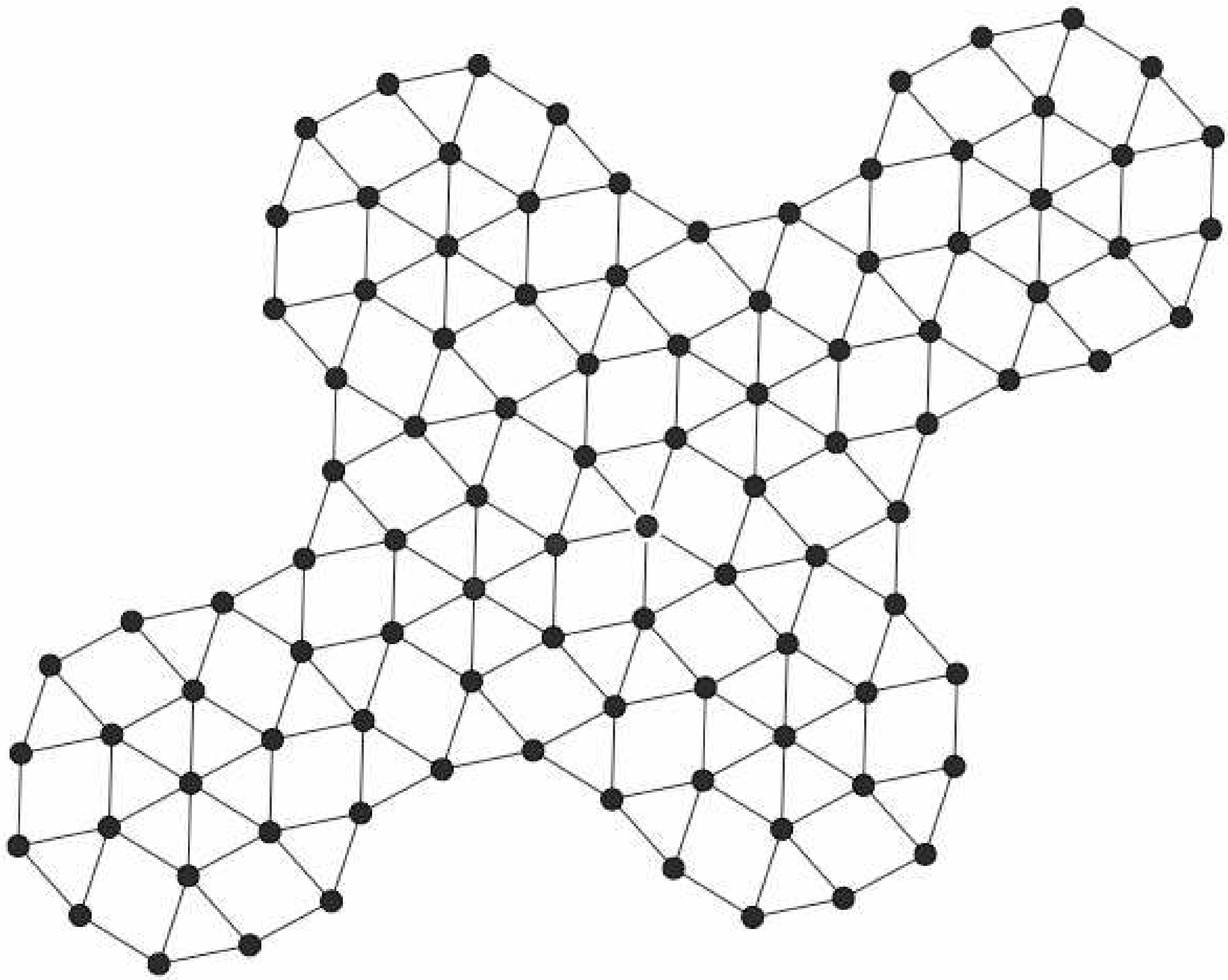
But . . .

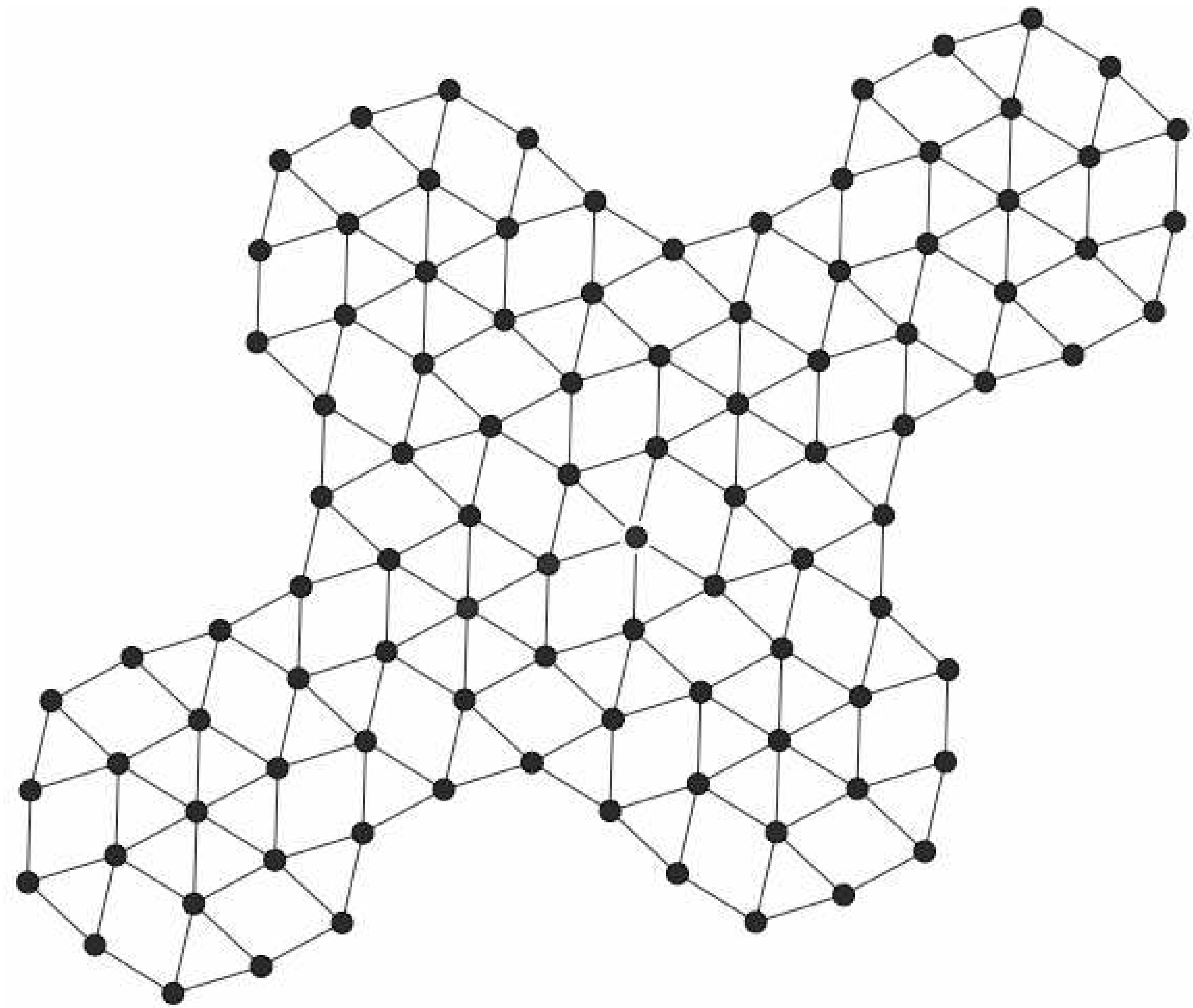
Let's make it harder by adding more disks

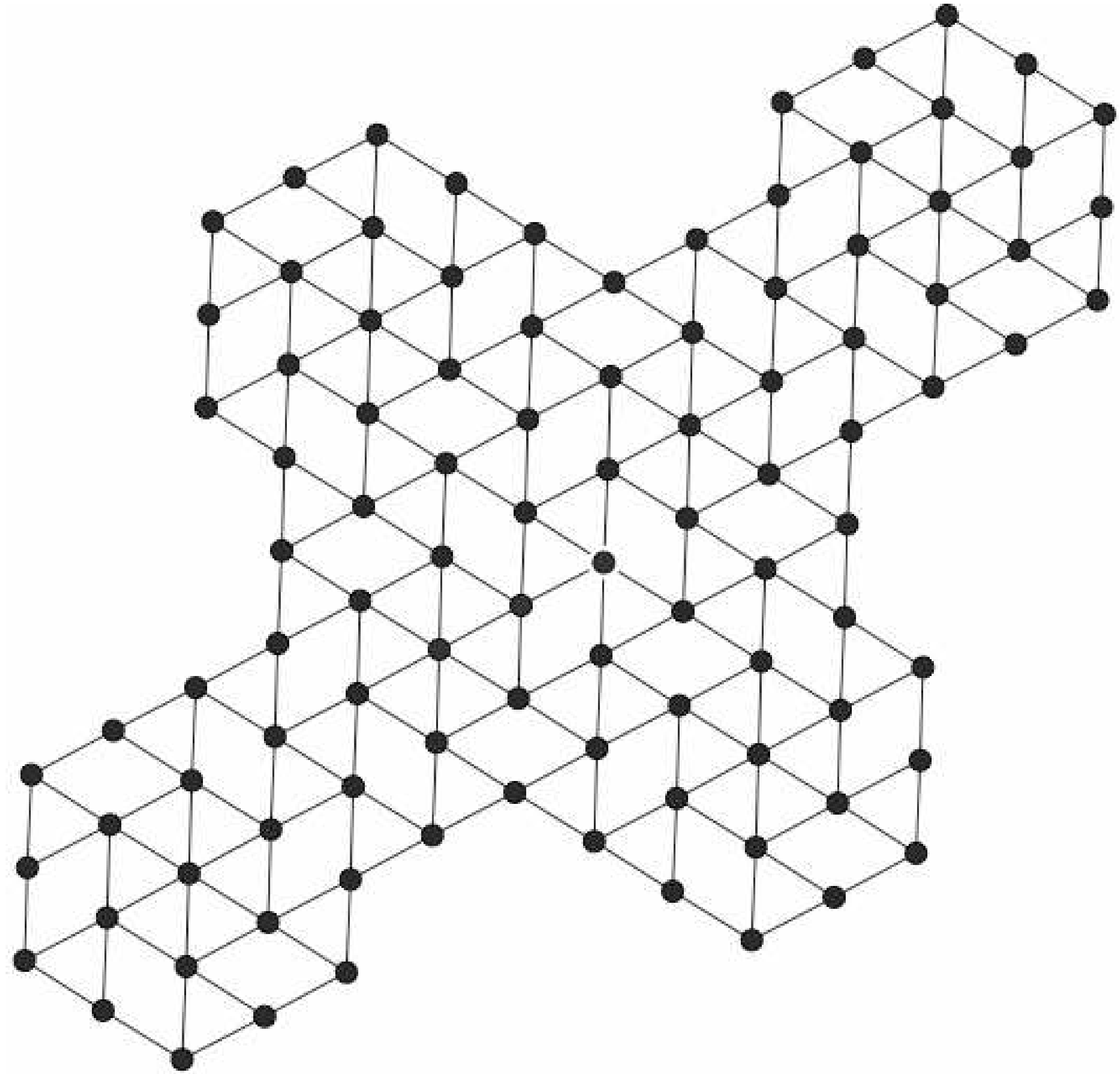












Algorithms

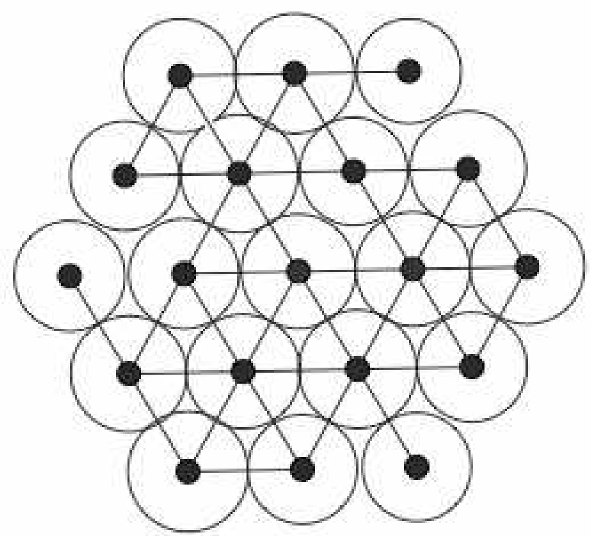
These ideas could help with finding "interesting" packings. Instead of choosing the packing elements and letting them "grow" until they jam, choose the graph of the packing and its homotopy class and "shrink" it until all the edge lengths are equal, adjusting the stresses as you go.

This could help with that plague of packing algorithms . . .

Contact ambiguity

For computer simulations with large numbers of packing elements, how do you know if some pair of packing elements are in contact or if they overlap? This can influence the role of that stresses play in the structural mechanics of the packing.

Is this part of a packing graph?



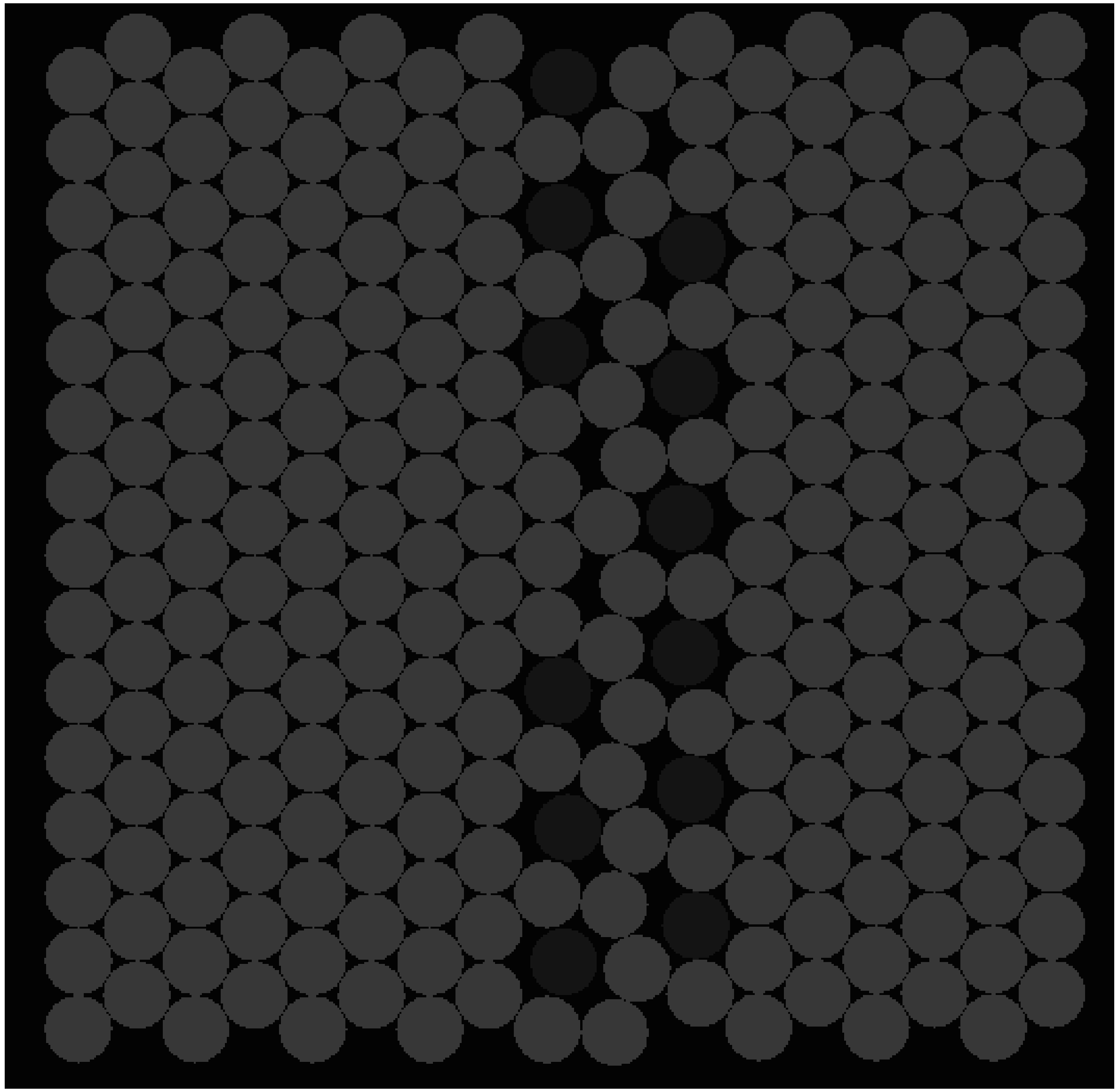
You have to decide where the contact cutoff distance is, but it can be inconsistent with the geometry.

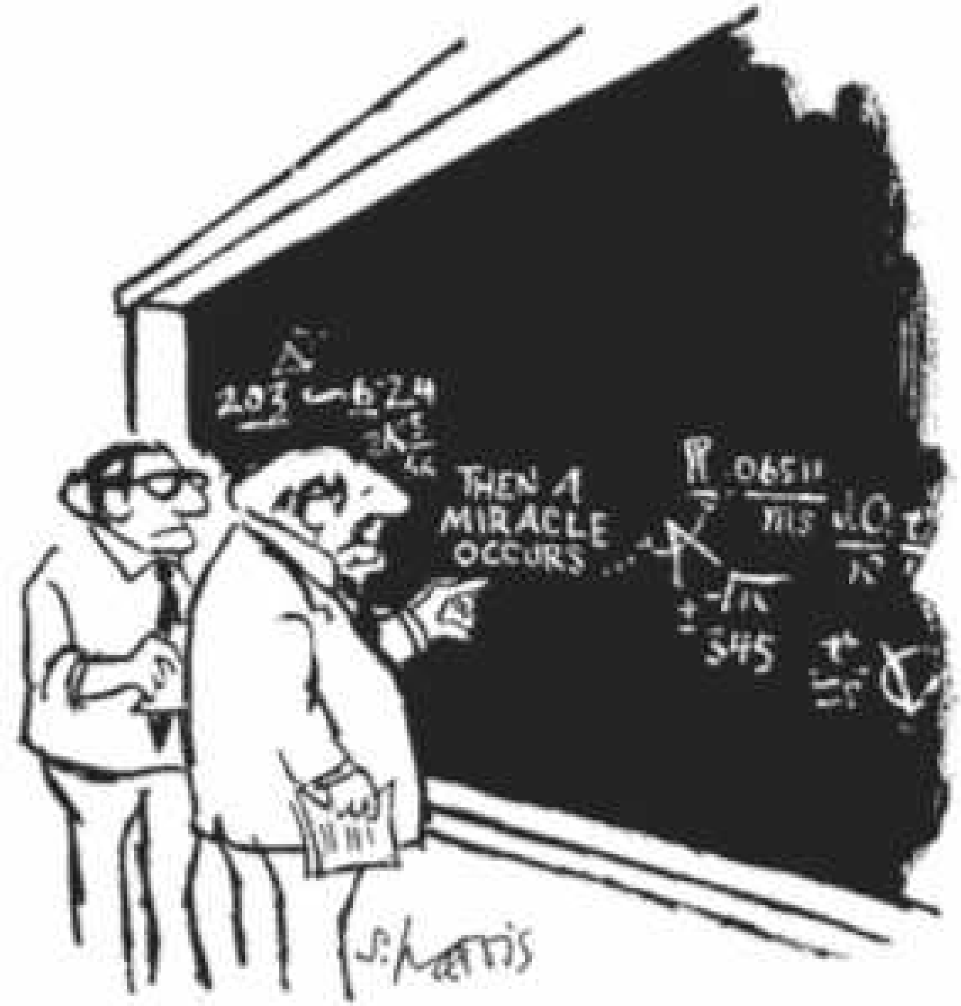
Conjectures and Questions

1. Can there exist a graph of a packing (in the plane) in a triangular torus that is locally rigid, but not saturated? (That is can there be large enough holes to fit another packing disk inside?) Is there a limit to the number of "rattlers in a hole"?
2. For any 2-D torus that has a strictly jammed packing graph, do the "fault lines" necessarily span a non-trivial homology?

Conjectures and Questions

3. How big can the holes be in stressed locally maximally dense packing graph?
4. What is the limit to the maximal internal angle of a polygonal face in a stressed locally maximally dense packing graph? In other words, is there a Boroczky-type example in this context?
5. For each k, what is the smallest number of n equal disks in the plane, such that the graph of the packing is a strictly convex polygon, and k disks are inside the ring of n disks and disjoint from them.





"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."