Map Gaps

Thomas W. Tucker

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Gap-finding: finding infinitely many surfaces that are gaps

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“orientably regular”?: but then that is different from “oriented regular” or “regular oriented”

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Alternative is to begin with regular as transitive on flags and then A is generated by three involution etc.

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The proof involves some serious group theory from the 1950s related to almost Sylow-cyclic groups (all Sylow p-subgroups are cyclic, except for $p = 2$ have cyclic subgroup of index at most two).

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There may be other patterns here, which is why I've given them in factored form.

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The computer list of chiral gaps for $g - 1 = p$ are completely understood:

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Fact: There is a regular map for every genus g: take sphere, put vertices at north and south poles and join by $2n$ edges.

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Moreover, one can modify the Belolipetsky-Jones map for $p \equiv 1 \pmod 6$ to show if $g \equiv 2 \pmod 6$ and every prime $p \equiv 5 \pmod 6$ in the prime power factorization of $g - 1$ has even exponent, then there is a simple reg map for g.

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But there are no simple (reflexible) regular maps for $g - 1 = p \equiv 1 \pmod 6$ by CST.

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In effect, the finite field examples are the only ones. And they are all chiral except for K_6 in projective plane and K_3 and K_4 in the sphere (or Petrie duals). In particular, the only ones for non-orientable surfaces are K_3, K_4, K_6.

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This morning:

Theorem

If the graph G underlying a regular (reflexible) map contains K_5, then $G = K_6$.

Figure

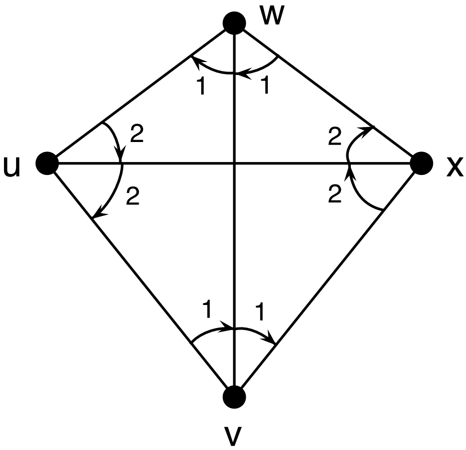

Figure: A picture is worth a thousand words

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Širàň, Tucker, Watkins (2001): turns out S_n does the trick for each type (nice because of S_n all autos are inner).

Gaps for edge-transitive maps

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$2ex$ (reflection instead of half turn): $2, 5, 8, 11, 13, 31$

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Conder claims for type 3, no gaps even for non-degenerate.

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Proof Construct families for different congruence classes.

The gap-filling families

$g = 4k - 1$:
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That leaves $g \equiv 2, 8, 14 \pmod{18}$. Have another more complicated family for $g = 9k - 7$.
For all other $g - 1 \equiv 1 \pmod 6$, there is a variation of the Belolipetsky-Jones maps, which requires even exponent of all $p \equiv 5 \pmod 6$ in prime power factorization of $g - 1$.

Big picture: groups acting on surfaces

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For example, whether an orientable surface S has a simple regular map is equivalent to finding a group $A = \langle x, y : (xy)^2 = 1, \cdots >$ with $Core(x) = \{1\}$ that acts on S preserving orientation.

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Now we want to fix the group A and ask how it might act on different surfaces.

Surface symmetry in 3-space

Here is an easy, but apparently new, theorem concerning rotational symmetry of surfaces in 3-space. It arose because of a question Bojan Mohar asked about DeWitt Godfrey's sculpture "The group of genus two", which was the brainchild of Tomo Pisanski.

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(Tucker 2009, unpublished) (S^4: Seeing Surface Symmetry in Space) The surface of genus g can be immersed in 3-space with n-fold rotational symmetry if and only if $g \equiv 1 \pmod{n}$ or $g = qn - r$ with $0 \leq r < n-1$ and $q \geq r$.

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Example: for $n = 8$, we get $g = 0, 1(2-6)7-9(10-13)14-17(18-20)21-25(26-27)28-33(34)35...$

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Then thicken into surface with genus

$$E - r = [(q - r - 1)n + (r - 1)n] - r = qn - r$$

which has axis for n-fold rotation.

Necessity

Everything about groups acting on surfaces comes down to the Riemann-Hurwitz equation for A acting preserving orientation on the surface of genus g:

$$2 - 2g = n(2 - 2h - 2k(1 - 1/n) = n(2 - 2h - 2k) + 2k$$

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Then

$$g - 1 = n(h - 1 + k) - k \text{ so } g = qn - r \text{ where } q = gh + (k - 1)), r = k - 1$$

Clearly $q \geq r$. If $r \geq n$ it is easy to subtract multiple of n from r (notice here r could be -1.

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Which g are gaps for spatial actions of D_n, S_4, A_5?

Kulkarni's Theorem

Given a group A, if $p^f|A|$ and the largest cyclic subgroup of a Sylow p-group is p^e, let $p^d = p^{f-e}$, the p-deficiency of A. Let P be the product of the p deficiencies. Call A type II if in the Sylow 2-subgroup, the elements of order at most 2^{d-1} form an index two subgroup (yes *form*, not generate). Call A type I otherwise. Then

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Corollary

A acts on almost all surfaces preserving orientation if and only if A is almost Sylow-cyclic and does not contain $Z_2 \times Z_4$.

Proof of Kulkarni's Theorem

Proof If there are b branch points, then

$$2g-2 = |A|(2g'-2) - |A|b + \Sigma|A|/r \equiv 0 \pmod{P}$$

since for branch point of order r, we have P divides $|A|/r$ and of course P divides $|A|$.

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If $|A|$ is odd, just divide through by 2 (mod P). If $|A|$ even, more complicated. Depends on extra relation $\Pi[a_i, b_i]\Pi c_j = 1$ in A, where c_i correspond to branch points, which restricts slightly the nature of the branch points.

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Jones (Glasgow, 1994) has studies the full genus spectrum for Z_p, D_p, Z_{2p}. He also has spectra for free actions, and for actions with guotient of genus h (there are Kulkarni-types theorems for these spectra as well).
Open question: What about orientation-reserving actios or non-orientable surfaces