REGULAR MAPS
WITH NILPOTENT AUTOMORPHISM GROUPS

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(includes joint work with S. F. Du, A. Malnič & R. Nedela, and others)

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Maps

Map

- cellular decomposition of a closed surface into vertices, edges, and faces

Equivalently,

\[ \text{map} = \text{connected graph 2-cell embedded in a surface} \]

Oriented map

- map on an orientable surface with chosen orientation

Map automorphism

- incidence-preserving self-homeomorphism of the underlying surface
- orientation-preserving, if the map is oriented
Regular maps

Flags of a map $M$
  - mutually incident (vertex, edge, face) triples of $M$

By connectivity of the surface, for any two flags $f_1, f_2$ of a map $M$ there exists at most one map automorphism $s$ such that $f_1 \mapsto f_2$

$\implies$

- $|\text{Aut}(M)| \leq \#\text{flags} = 4\#\text{edges}$
Regular maps

Flags of a map $\mathcal{M}$

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$\implies$

- $|\text{Aut}(\mathcal{M})| \leq \#\text{flags} = 4\#\text{edges}$

Definition

A map $\mathcal{M}$ is called regular if

$$|\text{Aut}(\mathcal{M})| = \#\text{flags} = 4\#\text{edges}.$$
Orientably regular maps

If $\mathcal{M}$ is orientable, then

$$|\text{Aut}^+(\mathcal{M})| \leq \frac{1}{2}(\# \text{flags}) = 2\# \text{edges}$$
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**Definition**

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An orientably regular map that is not regular is chiral.
Algebraic regular maps

Every regular map $\mathcal{M}$ can be represented as a quadruple $(G; \rho, \lambda, \tau)$ where

- $G = \langle \rho, \lambda, \tau \rangle$ is a finite 2-generated group with $\rho^2 = \lambda^2 = \tau^2 = 1$ and $\lambda\tau = \tau\lambda$. 

Given an algebraic map $\mathcal{M} = (G; \rho, \lambda, \tau)$, one can reconstruct the topological map as follows:
- vertices . . . orbits of $\rho\tau$
- edges . . . orbits of $\tau\lambda$
- faces . . . orbits of $\rho\lambda$, all acting on the left
- incidence . . . non-empty intersection
- automorphisms = right translations $\sigma_g: g \mapsto xg$, $g \in G$. 

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- incidence  ... non-empty intersection

automorphisms  = right translations $\sigma_g : g \mapsto xg$, $g \in G$. 

Algebraic orientably regular maps

Every **orientably regular** map $\mathcal{M}$ can be represented as a triple $(G; r, l)$ where

- $G = \langle r, l \rangle$ is a finite 2-generated group with $l^2 = 1$

Given an algebraic map $\mathcal{M} = (G; r, l)$, one can reconstruct the **topological map** as follows:

- **vertices** ... orbits of $r$
- **edges** ... orbits of $l$
- **faces** ... orbits of $rl$, all acting on the **left**
- **incidence** ... non-empty intersection

**Automorphisms** = **right** translations $\sigma_g : g \mapsto xg, \quad g \in G$. 
Regular maps with given automorphism group

**Problem.** Given a finite group $G = \langle \rho, \lambda, \tau \rangle$ with $\rho^2 = \lambda^2 = \tau^2 = 1$ and $\lambda \tau = \tau \lambda$, classify all regular maps $M$ with $\text{Aut}(M) \cong G$.

Isomorphism classes of regular maps $M$ with $\text{Aut}(M) \cong G$ correspond to the orbits of $\text{Aut}(G)$ on the generating triples $(\rho, \lambda, \tau)$ of $G$. 

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**Problem$^+$.** Given a group $G = \langle r, l \rangle$ with $l^2 = 1$, classify all orientably regular maps $\mathcal{M}$ with $\text{Aut}^+(\mathcal{M}) \cong G$.

Again, isomorphism classes of orientably regular maps $\mathcal{M}$ with $\text{Aut}(\mathcal{M}) \cong G$ correspond to the orbits of $\text{Aut}^+(G)$ on the generating pairs $(r, l)$ of $G$. 
Regular maps with given automorphism group

- (Malle, Saxl & Weigel, 1994)
  Every non-abelian finite simple group can be generated by two elements of which one is an involution.
  (Situation regarding three involutions two of which commute is more complicated, but known.)
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Classification results have been obtained for certain infinite classes of finite simple or almost simple groups:
- $PSL(2, q)$ and $PGL(2, q)$ (Sah, 1969)
- Suzuki groups (Jones & Silver, 1993)
Regular maps with given automorphism group

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- Classification results have been obtained for certain infinite classes of finite simple or almost simple groups:
  - $\text{PSL}(2, q)$ and $\text{PGL}(2, q)$ (Sah, 1969)
  - Suzuki groups (Jones & Silver, 1993)

- Little is known about regular maps arising from solvable groups.
Nilpotent regular maps
Nilpotent regular maps: nonorientable surfaces

Theorem

Let $\mathcal{M}$ be a regular map on a nonorientable surface such that $\text{Aut}(\mathcal{M})$ is nilpotent. Then $\mathcal{M}$ is a regular embedding of the bouquet $\tilde{\mathcal{B}}_{2^n}$ of $2^n$ loops in the projective plane, or its dual, and $\text{Aut}(\mathcal{M}) \cong \mathbb{D}_{2n+1}$.

Let $\text{Aut}(\mathcal{M}) = G = \langle \rho, \lambda, \tau \rangle$ be nilpotent. $G$ must be a 2-group. By induction on $n$, every nonorientable regular map with $2^n$ edges is either $\tilde{\mathcal{B}}_{2^n}$ or $\tilde{\mathcal{B}}^*_{2^n}$ [Wilson, 1985].

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Let $M$ be a regular map on a nonorientable surface such that $\text{Aut}(M)$ is nilpotent. Then $M$ is a regular embedding of the bouquet $\tilde{B}_{2^n}$ of $2^n$ loops in the projective plane, or its dual, and $\text{Aut}(M) \cong D_{2^{n+1}}$.

We can restrict to orientably regular maps!
FROM NOW ON:

- regular map means orientably regular map

- \( \text{Aut}(\mathcal{M}) \) means \( \text{Aut}^+ (\mathcal{M}) \)
Theorem

Let $\mathcal{M}$ be a regular map whose automorphism group is an abelian group of order $n$. Then either $\mathcal{M} \cong S_n$, or $n = 2m$ and $\mathcal{M} \cong B_m$, or $\mathcal{M} \cong D(m, 1)$. The respective groups are $\mathbb{Z}_n$, and for $n = 2m$, $\mathbb{Z}_{2m}$ and $\mathbb{Z}_m \times \mathbb{Z}_2$. 

$S_n$

$R = (xyz...)$

$B_m, n = 2m$

$R = (xyz...x^{-1}y^{-1}z^{-1}...)$

$D(m, 1), n = 2m$

$R = (xyz...)(x^{-1}y^{-1}z^{-1}...)$
An \textit{n-dipole map} is a regular embedding of the graph $D_n$ having two vertices $u$ and $v$ joined by $n$ parallel edges.
Dipole maps

An \( n \)-dipole map is a regular embedding of the graph \( D_n \) having two vertices \( u \) and \( v \) joined by \( n \) parallel edges.

**Theorem**

Every regular embedding of \( D_n \) arises from the metacyclic group 
\[ G(n, e) = \langle x, y; x^n = y^2 = 1, yxy = x^e \rangle \text{ as the algebraic map} \]
\[ D(n, e) = (G(n, e); x, y). \]
Furthermore, \( D(n, e) \cong D(n, f) \iff e \equiv f \pmod{n}. \)
An \textit{n-dipole map} is a regular embedding of the graph $D_n$ having two vertices $u$ and $v$ joined by $n$ parallel edges.

\textbf{Theorem}

Every regular embedding of $D_n$ arises from the metacyclic group $G(n, e) = \langle x, y; x^n = y^2 = 1, yxy = x^e \rangle$ as the algebraic map $D(n, e) = (G(n, e); x, y)$.

Furthermore, $D(n, e) \cong D(n, f) \iff e \equiv f \pmod{n}$.

\textbf{Combinatorial description:}

- The cyclic order of edges at $v$ is the $e$-th power of the order at $u$.
- It follows from the relations that $e^2 \equiv 1 \pmod{n}$.  

Every regular map with nilpotent automorphism group can be uniquely decomposed into a direct product of two regular maps, a regular map whose automorphism group is a 2-group and a star \( S_m \) of odd valency.
Nilpotent regular maps: Decomposition Theorem

Theorem (Malnič, Nedela & S.)

Every regular map with nilpotent automorphism group can be uniquely decomposed into a direct product of two regular maps, a regular map whose automorphism group is a 2-group and a star $S_m$ of odd valency.

Definition. Let $\mathcal{M}_1 = (G_1; r_1, l_1)$ and $\mathcal{M}_1 = (G_2; r_2, l_2)$ be regular maps. Then $\mathcal{M}_1 \times \mathcal{M}_2 = (G; r, l)$ where $r = (r_1, r_2)$ and $l = (l_1, l_2)$ and $G = \langle r, l \rangle \leq G_1 \times G_2$. 
Nilpotent regular maps: Decomposition Theorem

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Every regular map with nilpotent automorphism group can be uniquely decomposed into a **direct product of two regular maps**, a regular map whose automorphism group is a 2-group and a star $S_m$ of odd valency.

**Definition.** Let $\mathcal{M}_1 = (G_1; r_1, l_1)$ and $\mathcal{M}_2 = (G_2; r_2, l_2)$ be regular maps. Then $\mathcal{M}_1 \times \mathcal{M}_2 = (G; r, l)$ where $r = (r_1, r_2)$ and $l = (l_1, l_2)$ and $G = \langle r, l \rangle \leq G_1 \times G_2$.

If $r_1$ and $r_2$ have coprime orders and the maps are not both bipartite, then $G = G_1 \times G_2$ and the underlying graph of $\mathcal{M}_1 \times \mathcal{M}_2$ coincides with the direct (categorial) product of the underlying graphs of $\mathcal{M}_1$ and $\mathcal{M}_2$. 
Consequences of decomposition

Corollaries

Let $\mathcal{M}$ be a regular map with $\text{Aut}(\mathcal{M})$ nilpotent. The following hold:

- Both $\#$ of vertices and $\#$ of faces are powers of 2.
- Vertex-valency and face-size are both even; if $\mathcal{M}$ is simple, both are powers of 2.
- $\mathcal{M}$ is simple only when $\text{Aut}(\mathcal{M})$ is a 2-group.
- If $\text{Aut}(\mathcal{M})$ is non-abelian, then $\mathcal{M}$ is bipartite;
- Apart from two families of dipole maps and their duals, both vertex-valency and face-size are multiples of 4.
Nilpotent regular maps of class 2: the groups

By Decomposition Theorem, it is sufficient to classify maps arising from 2-groups.

Theorem A (Malniˇ c, Nedela & S.)

Let \( G = \langle x, y \rangle \) be a 2-group of class 2, where \( |x| = 2^n \), \( |y| = 2^m \) and \( n \geq 2 \).

Then \( G \) is one of the following two groups:

\[
G_1(n) = \langle x, y; x^{2^n} = y^2 = 1, [x, y] = x^{2^n-1} \rangle
\]

\[
G_2(n) = \langle x, y, z; x^{2^n} = y^2 = z^2 = [z, x] = [z, y] = 1, z = [x, y] \rangle
\]

Moreover, \( G_2(n)/\langle z^x x^{2^n-1} \rangle \cong G_1(n) \).
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Moreover,

$G_2(n)/\langle zx^{2^{n-1}} \rangle \cong G_1(n)$. 
Nilpotent regular maps of class 2: the maps

**Theorem B (Malnič, Nedela & Š.)**

*Every regular map with automorphism group a 2-group of class 2 is isomorphic to*

\[ M_1(n) = (G_1(n); x, y) \] for some \( n \geq 2 \), or to

\[ M_2(n) = (G_2(n); x, y) \] for some \( n \geq 1 \).
Nilpotent regular maps of class 2: the maps

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\[ M_2(n) = (G_2(n); x, y) \text{ for some } n \geq 1. \]

\[ \begin{align*}
M_2(n + 1) \rightarrow M_2(n) \rightarrow M_2(n - 1) \\
\downarrow \quad \downarrow \quad \downarrow \\
M_1(n + 1) \cong D(2^{n+1}, 2^n + 1) \quad M_1(n) \cong D(2^n, 2^{n-1} + 1) \quad M_1(n - 1) \cong D(2^{n-1}, 2^{n-2} + 1) \\
\downarrow \quad \downarrow \quad \downarrow \\
D(2^n, 1) \rightarrow D(2^{n-1}, 1) \rightarrow D(2^{n-2}, 1)
\end{align*} \]
Nilpotent regular maps of class 2: description of maps

- $\mathcal{M}_1(n) \cong D(2^n, 2^{n-1} + 1)$. For $n \geq 3$ it is self-dual of type $\{2^n, 2^n\}$ and genus $2^{n-1} - 1$. For $n = 2$ it is the spherical map of type $\{2, 4\}$.
Nilpotent regular maps of class 2: description of maps

- $\mathcal{M}_1(n) \cong \mathcal{D}(2^n, 2^{n-1} + 1)$. For $n \geq 3$ it is self-dual of type $\{2^n, 2^n\}$ and genus $2^{n-1} - 1$. For $n = 2$ it is the spherical map of type $\{2, 4\}$.

- $\mathcal{M}_2(n)$ is a regular embedding of a 4-cycle with multiplicity $2^{n-1}$. For $n \geq 2$ it is self-dual of type $\{2^n, 2^n\}$ and genus $2^n - 3$. For $n = 1$ it has type $\{4, 2\}$ and is dual to $\mathcal{M}_1(2)$. 

Nilpotent regular maps of class 2: description of maps

- $M_1(n) \cong D(2^n, 2^{n-1} + 1)$. For $n \geq 3$ it is self-dual of type $\{2^n, 2^n\}$ and genus $2^{n-1} - 1$. For $n = 2$ it is the spherical map of type $\{2, 4\}$.

- $M_2(n)$ is a regular embedding of a 4-cycle with multiplicity $2^{n-1}$. For $n \geq 2$ it is self-dual of type $\{2^n, 2^n\}$ and genus $2^n - 3$. For $n = 1$ it has type $\{4, 2\}$ and is dual to $M_1(2)$.

- Since the maps are uniquely determined by the groups, they admit all orientation-preserving “external” symmetries: they are invariant under all Wilson’s operations $H_j$, $j$ odd. In particular, they are all reflexible and the exponent group is all of $\mathbb{Z}_{2^n}^*$. That is, they are kaleidoscopic.
Nilpotent regular maps of maximal class: the groups

Every 2-group of order $2^{n+1}$ and nilpotency class $n$ is one of the following [Taussky, 1937]:

(i) dihedral group
\[ \mathbb{D}_{2^n} = \langle a, b; x^{2^n} = y^2 = 1, y^{-1}xy = x^{-1} \rangle, \]

(ii) quasi-dihedral group
\[ Q\mathbb{D}_{2^n} = \langle x, y; x^{2^n} = y^2 = 1, y^{-1}xy = x^{2n-1-1} \rangle, \]

(iii) generalised quaternion group
\[ GQ_{2^n} = \langle x, y; x^{2^n} = 1, y^2 = x^{2n-1}, y^{-1}xy = x^{-1} \rangle. \]
Nilpotent regular maps of maximal class: the maps

**Theorem (Hu, Wang)**

Let $\mathcal{M}$ be a regular map whose automorphism group is a $2$-group of order $2^{n+1}$ and nilpotency class $n$. Then $\mathcal{M}$ is one of the following:

(i) the spherical dipole $\mathcal{D}(2^n, -1)$ with $\text{Aut}(\mathcal{M}) \cong \mathbb{D}_{2^n}$ or its dual,

(ii) the dipole $\mathcal{D}(2^n, 2^{n-1} - 1)$ of genus $2^{n-2}$ with $\text{Aut}(\mathcal{M}) \cong Q\mathbb{D}_{2^n}$ or its dual.

(iii) There are no regular maps whose automorphism group is the generalised quaternion group.
Nilpotent regular maps of class 3

Theorem (Ban, Du, Liu, Nedela & S.)

Let $G = \langle x, y \rangle$ be a 2-group of class 3, where $|x| = 2^n$, $|y| = 2$ and $n \geq 2$. Then $G$ is one of seven infinite classes of groups

$$H_1(n) = \langle x, y; x^{2^n} = y^2 = 1, [x, y] = z, [z, x] = [z, y] = w, [w, x] = [w, y] = 1 \rangle,$$

$$H_7(n) = \langle x, y; x^{2^n-1} = t, y^2 = t^2 = 1, [x, y] = z, [z, x] = w, [z, y] = t, [w, x] = [w, y] = [t, x] = [t, y] = 1 \rangle,$$

and two additional groups

$$H_8(2) = \langle x, y; x^4 = wt, y^2 = t^2 = 1, [x, y] = z, [z, x] = w, [z, y] = t, [w, x] = [w, y] = [t, x] = [t, y] = 1 \rangle$$

$$H_9(2) = \langle x, y; x^4 = y^2 = 1, [x, y] = z, [z, x] = 1, [z, y] = t, [t, x] = [t, y] = 1 \rangle.$$ 

Each group $H_i(n)$ gives rise to two exactly non-isomorphic maps, $M_{i,n} = (H_i(n); x, y)$ and the dual $M_{i,n}^* = (H_i(n); xy, y)$. 
Theorem (Du, Nedela & S.)

Let $\mathcal{M} = (G; x, y)$ be a regular map where $G$ is nilpotent of class $c$. Then $\mathcal{M}$ has at most $2^{2^{c-1}}$ vertices.

Corollary

For each nilpotency class $c \geq 1$ there exist only finitely many simple nilpotent regular maps of class $c$. 

Theorem (Du, Nedela & S.)

Let $M = (G; x, y)$ be a regular map where $G$ is nilpotent of class $c$. Then $M$ has at most $2^{2^{c-1}}$ vertices.

Corollary

For each nilpotency class $c \geq 1$ there exist only finitely many simple nilpotent regular maps of class $c$.

Proof.

- By Decomposition Theorem, we may assume that $G$ is a 2-group.
- It is sufficient to show that $|G : \langle x \rangle| \leq 2^{2^{c-1}}$, since $|G : \langle x \rangle| = \#\text{vertices}$.
- Induction on $c$ along the lower central series; involves a lot of commutator calculations.
Reduction to simple nilpotent maps

Theorem

Let \( \mathcal{M} = (G; x, y) \) be a regular map of valency \( d \) and multiplicity \( m \) with underlying graph \( K^{(m)} \). Set \( A = \langle x^{d/m} \rangle \) and \( B = \langle x^{d/m}, y \rangle \). Then:

- \( A \trianglelefteq G \) and \( \mathcal{M}' = (G/A; xA, yA) \) is a regular embedding of \( K \).
- \( \mathcal{M}'' = (B; x^{d/m}, y) \) is a dipole map isomorphic to \( D(m, e) \) for some \( e^2 \equiv 1 \pmod{m} \).
Reduction to simple nilpotent maps

**Theorem**

Let $\mathcal{M} = (G; x, y)$ be a regular map of valency $d$ and multiplicity $m$ with underlying graph $K^{(m)}$. Set $A = \langle x^{d/m} \rangle$ and $B = \langle x^{d/m}, y \rangle$. Then:

- $A \subseteq G$ and $\mathcal{M}' = (G/A; xA, yA)$ is a regular embedding of $K$.
- $\mathcal{M}'' = (B; x^{d/m}, y)$ is a dipole map isomorphic to $\mathcal{D}(m, e)$ for some $e^2 \equiv 1 \pmod{m}$.

If $\mathcal{M}$ is nilpotent, then both $\mathcal{M}'$ and $\mathcal{M}''$ are nilpotent.

In that case, since $\mathcal{M}' = (G/A; xA, yA)$ has a simple underlying graph, $G/A$ is a 2-group.

$\mathcal{M}''$ is a nilpotent dipole – and these can be easily characterised.
Theorem (Malnič, Nedela & S.)

Let \( m = 2^s t \) where \( t \geq 1 \) is odd and \( s \geq 0 \).

- If \( s \leq 1 \), then \( D(m, 1) \) is the only nilpotent regular embedding of \( D_m \).
- For \( s = 2 \) there are two regular embeddings of \( D_m \):
  \( D(m, 1) \) and \( D(m, 1 + m/2) \).
- For \( s \geq 3 \) there are four nilpotent regular embeddings of \( D_m \):
  \( D(m, 1) \), \( D(m, 1 + m/2) \), \( D(m, e) \), and \( D(m, e + m/2) \), where \( e \) is the unique solution of the system \( e \equiv -1 \pmod{2^s} \), \( e \equiv 1 \pmod{t} \).
Reduction to simple nilpotent maps: nilpotent dipoles

Theorem (Malnič, Nedela & S.)

Let \( m = 2^s t \) where \( t \geq 1 \) is odd and \( s \geq 0 \).

- If \( s \leq 1 \), then \( D(m, 1) \) is the only nilpotent regular embedding of \( D_m \).
- For \( s = 2 \) there are two regular embeddings of \( D_m \): \( D(m, 1) \) and \( D(m, 1 + m/2) \).
- For \( s \geq 3 \) there are four nilpotent regular embeddings of \( D_m \): \( D(m, 1) \), \( D(m, 1 + m/2) \), \( D(m, e) \), and \( D(m, e + m/2) \), where \( e \) is the unique solution of the system \( e \equiv -1 \pmod{2^s} \), \( e \equiv 1 \pmod{t} \).

Problem. How can a general nilpotent regular map \( \mathcal{M} \) arise from the corresponding simple map \( \mathcal{M}' \) and the dipole map \( \mathcal{M}'' \)?
THE END

THANK YOU!