REGULAR MAPS WITH NILPOTENT AUTOMORPHISM GROUPS

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(includes joint work with S. F. Du, A. Malnič & R. Nedela, and others)

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Maps

Map

- cellular decomposition of a closed surface into vertices, edges, and faces

Equivalently,

map $=$ connected graph 2-cell embedded in a surface

Oriented map

- map on an orientable surface with chosen orientation

Map automorphism

- incidence-preserving self-homeomorphism of the underlying surface
- orientation-preserving, if the map is oriented

Regular maps

Flags of a map $\mathcal{M}$

- mutually incident (vertex,edge,face) triples of $\mathcal{M}$

By connectivity of the surface, for any two flags $\mathbf{f}_1$, $\mathbf{f}_2$ of a map $\mathcal{M}$ there exists at most one map automorphism s. t. $\mathbf{f}_1 \mapsto \mathbf{f}_2$

$\Longrightarrow$

- $|\mathrm{Aut}(\mathcal{M})| \leq \#\text{flags} = 4\#\text{edges}$

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Definition

A map $\mathcal{M}$ is called regular if

$$|\mathrm{Aut}(\mathcal{M})| = \#\text{flags} = 4\#\text{edges}.$$

Orientably regular maps

If $\mathcal{M}$ is orientable, then

$$|\mathrm{Aut}^+(\mathcal{M})| \leq \tfrac{1}{2}(\#\text{flags}) = 2\#\text{edges}$$

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An orientably regular map that is not regular is chiral.

Algebraic regular maps

Every regular map $\mathcal{M}$ can be represented as a quadruple $(G;\, \rho, \lambda, \tau)$ where

- $G = \langle \rho, \lambda, \tau \rangle$ is a finite 2-generated group with $\rho^2 = \lambda^2 = \tau^2 = 1$ and $\lambda\tau = \tau\lambda$.

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Given an algebraic map $\mathcal{M} = (G;\, \rho, \lambda, \tau)$, one can reconstruct the topological map as follows:

- vertices ... orbits of $\rho\tau$
- edges ... orbits of $\tau\lambda$
- faces ... orbits of $\rho\lambda$, all acting on the left
- incidence ... non-empty intersection

automorphisms = right translations $\sigma_g : g \mapsto xg, \quad g \in G$.

Algebraic orientably regular maps

Every orientably regular map $\mathcal{M}$ can be represented as a triple $(G;\, r, l)$ where

- $G = \langle r, l \rangle$ is a finite 2-generated group with $l^2 = 1$

Given an algebraic map $\mathcal{M} = (G;\, r, l)$, one can reconstruct the topological map as follows:

- vertices . . . orbits of r
- edges . . . orbits of l
- faces . . . orbits of rl, all acting on the left
- incidence . . . non-empty intersection

automorphisms = right translations $\sigma_g \colon g \mapsto xg, \quad g \in G.$

Regular maps with given automorphism group

Problem. Given a finite group $G = \langle \rho, \lambda, \tau \rangle$ with $\rho^2 = \lambda^2 = \tau^2 = 1$ and $\lambda\tau = \tau\lambda$, classify all regular maps $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) \cong G$.

Isomorphism classes of regular maps $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) \cong G$ correspond to the orbits of $\mathrm{Aut}(G)$ on the generating triples (ρ, λ, τ) of G.

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Problem$^+$. Given a group $G = \langle r, l \rangle$ with $l^2 = 1$, classify all orientably regular maps $\mathcal{M}$ with $\mathrm{Aut}^+(\mathcal{M}) \cong G$.

Again, isomorphism classes of orientably regular maps $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) \cong G$ correspond to the orbits of $\mathrm{Aut}^+(G)$ on the generating pairs (r, l) of G.

Regular maps with given automorphism group

- (Malle, Saxl & Weigel, 1994)
  Every non-abelian finite simple group can be generated by two elements of which one is an involution.
  (Situation regarding three involutions two of which commute is more complicated, but known.)

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  - $PSL(2, q)$ and $PGL(2, q)$ (Sah, 1969)
  - Suzuki groups (Jones & Silver, 1993)

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  - $PSL(2, q)$ and $PGL(2, q)$ (Sah, 1969)
  - Suzuki groups (Jones & Silver, 1993)

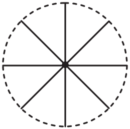
- Little is known about regular maps arising from solvable groups.

Nilpotent regular maps

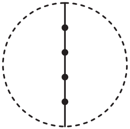
Nilpotent regular maps: nonorientable surfaces

Theorem

Let $\mathcal{M}$ be a regular map on a nonorientable surface such that $\mathrm{Aut}(\mathcal{M})$ is nilpotent. Then $\mathcal{M}$ is a regular embedding of the bouquet $\widetilde{\mathcal{B}}_{2^n}$ of 2^n loops in the projective plane, or its dual, and $\mathrm{Aut}(\mathcal{M}) \cong \mathbb{D}_{2^{n+1}}$.



$\widetilde{\mathcal{B}}_4$



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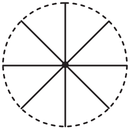

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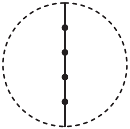

$\widetilde{\mathcal{B}}_4^*$

Let $\mathrm{Aut}(\mathcal{M}) = G = \langle \rho, \lambda, \tau \rangle$ be nilpotent.

- G must be a 2-group.
- By induction on n, every nonorientable regular map with 2^n edges is either $\widetilde{\mathcal{B}}_{2^n}$ or $\widetilde{\mathcal{B}}_{2^n}^*$ [Wilson, 1985].

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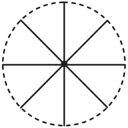

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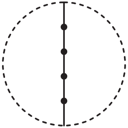


$\widetilde{\mathcal{B}}_4^*$

$\Longrightarrow$ We can restrict to orientably regular maps!

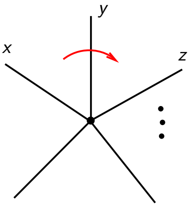
FROM NOW ON:

- regular map means orientably regular map
- $\mathrm{Aut}(\mathcal{M})$ means $\mathrm{Aut}^{+}(\mathcal{M})$

Orientable surfaces: abelian regular maps

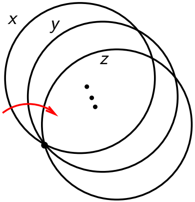
Theorem

Let $\mathcal{M}$ be a regular map whose automorphism group is an abelian group of order n. Then either $\mathcal{M} \cong \mathcal{S}_n$, or $n = 2m$ and $\mathcal{M} \cong \mathcal{B}_m$, or $\mathcal{M} \cong \mathcal{D}(m, 1)$. The respective groups are $\mathbb{Z}_n$, and for $n = 2m$, $\mathbb{Z}_{2m}$ and $\mathbb{Z}_m \times \mathbb{Z}_2$.



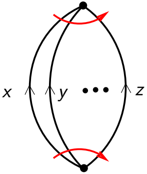
$\mathcal{S}_n$

$R = (xyz...)$



$\mathcal{B}_m, \; n = 2m$

$R = (xyz...x^{-1}y^{-1}z^{-1}...)$



$\mathcal{D}(m, 1), n = 2m$

$R = (xyz...)(x^{-1}y^{-1}z^{-1}...)$

Dipole maps

An n-dipole map is a regular embedding of the graph D_n having two vertices u and v joined by n parallel edges.

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Theorem

Every regular embedding of D_n arises from the metacyclic group $G(n,e) = \langle x, y;\ x^n = y^2 = 1, yxy = x^e \rangle$ as the algebraic map $\mathcal{D}(n,e) = (G(n,e);\ x, y)$.
Furthermore, $\mathcal{D}(n,e) \cong \mathcal{D}(n,f) \iff e \equiv f \pmod{n}$.

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Combinatorial description:

- The cyclic order of edges at v is the e-th power of the order at u.
- It follows from the relations that $e^2 \equiv 1 \pmod{n}$.

Nilpotent regular maps: Decomposition Theorem

Theorem (Malnič, Nedela & S.)

Every regular map with nilpotent automorphism group can be uniquely decomposed into a direct product of two regular maps, a regular map whose automorphism group is a 2-group and a star $\mathcal{S}_m$ of odd valency.

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Definition. Let $\mathcal{M}_1 = (G_1;\ r_1, l_1)$ and $\mathcal{M}_1 = (G_2;\ r_2, l_2)$ be regular maps. Then $\mathcal{M}_1 \times \mathcal{M}_2 = (G;\ r, l)$ where $r = (r_1, r_2)$ and $l = (l_1, l_2)$ and $G = \langle r, l \rangle \leq G_1 \times G_2$.

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If r_1 and r_2 have coprime orders and the maps are not both bipartite, then $G = G_1 \times G_2$ and the underlying graph of $\mathcal{M}_1 \times \mathcal{M}_2$ coincides with the direct (categorial) product of the underlying graphs of $\mathcal{M}_1$ and $\mathcal{M}_2$.

Consequences of decomposition

Corollaries

Let $\mathcal{M}$ be a regular map with $\mathrm{Aut}(\mathcal{M})$ nilpotent. The following hold:

- *Both # of vertices and # of faces are powers of 2.*
- *Vertex-valency and face-size are both even;*
  if $\mathcal{M}$ is simple, both are powers of 2.
- *$\mathcal{M}$ is simple only when $\mathrm{Aut}(\mathcal{M})$ is a 2-group.*
- *If $\mathrm{Aut}(\mathcal{M})$ is non-abelian, then $\mathcal{M}$ is bipartite;*
- *Apart from two families of dipole maps and their duals,*
  both vertex-valency and face-size are multiples of 4.

Nilpotent regular maps of class 2: the groups

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Theorem A (Malnič, Nedela & S.)

Let $G = \langle x, y \rangle$ be a 2-group of class 2, where $|x| = 2^n$, $|y| = 2$ and $n \geq 2$. Then G is one of the following two groups:

$$G_1(n) = \langle x, y;\, x^{2^n} = y^2 = 1, [x, y] = x^{2^{n-1}} \rangle$$

$$G_2(n) = \langle x, y, z;\, x^{2^n} = y^2 = z^2 = [z, x] = [z, y] = 1, z = [x, y] \rangle.$$

Moreover,

$$G_2(n)/\langle zx^{2^{n-1}} \rangle \cong G_1(n).$$

Nilpotent regular maps of class 2: the maps

Theorem B (Malnič, Nedela & S.)

Every regular map with automorphism group a 2-group of class 2 is isomorphic to

$\mathcal{M}_1(n) = (G_1(n); x, y)$ *for some* $n \geq 2$, *or to*

$\mathcal{M}_2(n) = (G_2(n); x, y)$ *for some* $n \geq 1$.

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$$\begin{tikzcd}
\mathcal{M}_2(n+1) \arrow[r] \arrow[d] & \mathcal{M}_2(n) \arrow[r] \arrow[d] \arrow[ddl] & \mathcal{M}_2(n-1) \arrow[d] \arrow[ddl] \\
\begin{array}{c}\mathcal{M}_1(n+1) \cong \\ \mathcal{D}(2^{n+1}, 2^n+1)\end{array} \arrow[d] & \begin{array}{c}\mathcal{M}_1(n) \cong \\ \mathcal{D}(2^n, 2^{n-1}+1)\end{array} \arrow[d] & \begin{array}{c}\mathcal{M}_1(n-1) \cong \\ \mathcal{D}(2^{n-1}, 2^{n-2}+1)\end{array} \arrow[d] \\
\mathcal{D}(2^n, 1) \arrow[r] & \mathcal{D}(2^{n-1}, 1) \arrow[r] & \mathcal{D}(2^{n-2}, 1)
\end{tikzcd}$$

Nilpotent regular maps of class 2: description of maps

- $\mathcal{M}_1(n) \cong \mathcal{D}(2^n, 2^{n-1}+1)$. For $n \geq 3$ it is self-dual of type $\{2^n, 2^n\}$ and genus $2^{n-1}-1$. For $n = 2$ it is the spherical map of type $\{2, 4\}$.

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- $\mathcal{M}_2(n)$ is a regular embedding of a 4-cycle with multiplicity 2^{n-1}. For $n \geq 2$ it is self-dual of type $\{2^n, 2^n\}$ and genus $2^n - 3$. For $n = 1$ it has type $\{4, 2\}$ and is dual to $\mathcal{M}_1(2)$.

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- $\mathcal{M}_2(n)$ is a regular embedding of a 4-cycle with multiplicity 2^{n-1}. For $n \geq 2$ it is self-dual of type $\{2^n, 2^n\}$ and genus $2^n - 3$. For $n = 1$ it has type $\{4, 2\}$ and is dual to $\mathcal{M}_1(2)$.
- Since the maps are uniquely determined by the groups, they admit all orientation-preserving "external" symmetries: they are invariant under all Wilson's operations H_j, j odd. In particular, they are all reflexible and the exponent group is all of $\mathbb{Z}_{2^n}^*$. That is, they are kaleidoscopic.

Nilpotent regular maps of maximal class: the groups

Every 2-group of order 2^{n+1} and nilpotency class n is one of the following [Taussky, 1937]:

(i) dihedral group
$\mathbb{D}_{2^n} = \langle a, b;\, x^{2^n} = y^2 = 1, y^{-1}xy = x^{-1} \rangle$,

(ii) quasi-dihedral group
$Q\mathbb{D}_{2^n} = \langle x, y;\, x^{2^n} = y^2 = 1, y^{-1}xy = x^{2^{n-1}-1} \rangle$,

(iii) generalised quaternion group
$GQ_{2^n} = \langle x, y;\, x^{2^n} = 1, y^2 = x^{2^{n-1}}, y^{-1}xy = x^{-1} \rangle$.

Nilpotent regular maps of maximal class: the maps

Theorem (Hu, Wang)

Let $\mathcal{M}$ be a regular map whose automorphism group is a 2-group of order 2^{n+1} and nilpotency class n. Then $\mathcal{M}$ is one of the following:

(i) *the spherical dipole* $\mathcal{D}(2^n, -1)$ *with* $\mathrm{Aut}(\mathcal{M}) \cong \mathbb{D}_{2^n}$ *or its dual,*

(ii) *the dipole* $\mathcal{D}(2^n, 2^{n-1} - 1)$ *of genus* 2^{n-2} *with* $\mathrm{Aut}(\mathcal{M}) \cong Q\mathbb{D}_{2^n}$ *or its dual.*

(iii) *There are no regular maps whose automorphism group is the generalised quaternion group.*

Nilpotent regular maps of class 3

Theorem (Ban, Du, Liu, Nedela & S.)

Let $G = \langle x, y\rangle$ be a 2-group of class 3, where $|x| = 2^n$, $|y| = 2$ and $n \geq 2$. Then G is one of seven infinite classes of groups

$$H_1(n) = \langle x, y;\ x^{2^n} = y^2 = 1, [x, y] = z, [z, x] = [z, y] = w, \\ [w, x] = [w, y] = 1\rangle, \quad \dots,$$

$$H_7(n) = \langle x, y;\ x^{2^{n-1}} = t, y^2 = t^2 = 1, [x, y] = z, [z, x] = w, \\ [z, y] = t, [w, x] = [w, y] = [t, x] = [t, y] = 1\rangle$$

and two additional groups

$$H_8(2) = \langle x, y;\ x^4 = wt, y^2 = t^2 = 1, [x, y] = z, [z, x] = w, [z, y] = t, \\ [w, x] = [w, y] = [t, x] = [t, y] = 1\rangle$$

$$H_9(2) = \langle x, y;\ x^4 = y^2 = 1, [x, y] = z, [z, x] = 1, [z, y] = t, \\ [t, x] = [t, y] = 1\rangle.$$

Each group $H_i(n)$ gives rise to two exactly non-isomorphic maps,

$$\mathcal{M}_{i,n} = (H_i(n); x, y) \text{ and the dual } \mathcal{M}^*_{i,n} = (H_i(n); xy, y).$$

Nilpotent maps with simple underlying graph

Theorem (Du, Nedela & S.)

Let $\mathcal{M} = (G; x, y)$ *be a regular map where* G *is nilpotent of class* c. *Then* $\mathcal{M}$ *has at most* $2^{2^{c-1}}$ *vertices.*

Corollary

For each nilpotency class $c \geq 1$ *there exist only finitely many simple nilpotent regular maps of class* c.

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Proof.

- By Decomposition Theorem, we may assume that G is a 2-group.
- It is sufficient to show that $|G : \langle x \rangle| \leq 2^{2^{c-1}}$, since $|G : \langle x \rangle| = \#\text{vertices}$.
- Induction on c along the lower central series; involves a lot of commutator calculations.

□

Reduction to simple nilpotent maps

Theorem

Let $\mathcal{M} = (G; x, y)$ be a regular map of valency d and multiplicity m with underlying graph $K^{(m)}$. Set $A = \langle x^{d/m} \rangle$ and $B = \langle x^{d/m}, y \rangle$. Then:

- *$A \trianglelefteq G$ and $\mathcal{M}' = (G/A; xA, yA)$ is a regular embedding of K.*
- *$\mathcal{M}'' = (B; x^{d/m}, y)$ is a dipole map isomorphic to $\mathcal{D}(m, e)$ for some $e^2 \equiv 1 \pmod{m}$.*

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- If $\mathcal{M}$ is nilpotent, then both $\mathcal{M}'$ and $\mathcal{M}''$ are nilpotent.
- In that case, since $\mathcal{M}' = (G/A; xA, yA)$ has a simple underlying graph, G/A is a 2-group.
- $\mathcal{M}''$ is a nilpotent dipole – and these can be easily characterised.

Reduction to simple nilpotent maps: nilpotent dipoles

Theorem (Malnič, Nedela & S.)

Let $m = 2^s t$ where $t \geq 1$ is odd and $s \geq 0$.

- *If $s \leq 1$, then $\mathcal{D}(m,1)$ is the only nilpotent regular embedding of D_m.*
- *For $s = 2$ there are two regular embeddings of D_m: $\mathcal{D}(m,1)$ and $\mathcal{D}(m,1+m/2)$.*
- *For $s \geq 3$ there are four nilpotent regular embeddings of D_m: $\mathcal{D}(m,1)$, $\mathcal{D}(m,1+m/2)$, $\mathcal{D}(m,e)$, and $\mathcal{D}(m,e+m/2)$, where e is the unique solution of the system $e \equiv -1 \pmod{2^s}$, $e \equiv 1 \pmod{t}$.*

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- *For $s = 2$ there are two regular embeddings of D_m: $\mathcal{D}(m, 1)$ and $\mathcal{D}(m, 1 + m/2)$.*
- *For $s \geq 3$ there are four nilpotent regular embeddings of D_m: $\mathcal{D}(m, 1)$, $\mathcal{D}(m, 1 + m/2)$, $\mathcal{D}(m, e)$, and $\mathcal{D}(m, e + m/2)$, where e is the unique solution of the system $e \equiv -1 \pmod{2^s}$, $e \equiv 1 \pmod{t}$.*

Problem. How can a general nilpotent regular map $\mathcal{M}$ arise from the corresponding simple map $\mathcal{M}'$ and the dipole map $\mathcal{M}''$?

THE END

THANK YOU!