External symmetries of regular and orientably regular maps

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OU and STU

Workshop on Symmetry in Graphs, Maps and Polytopes

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Highly symmetric maps

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Platonic
solids:

Highly symmetric maps

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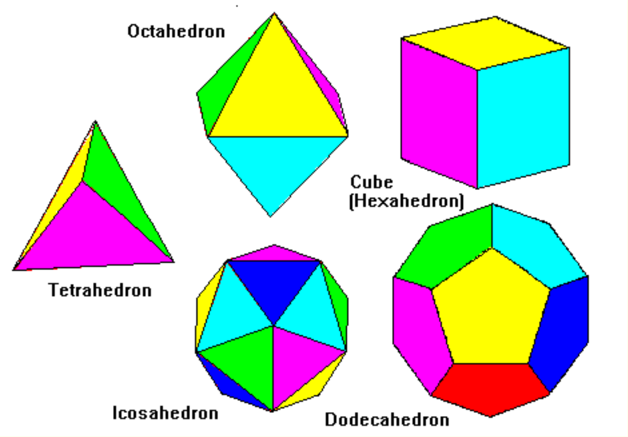

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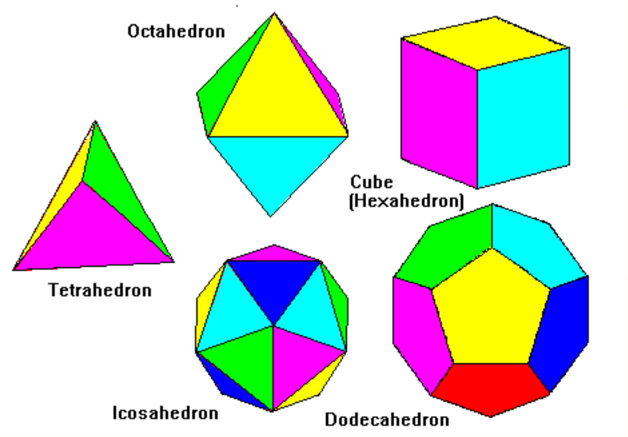

Regular maps are generalizations of Platonic solids to arbitrary surfaces.

Maps and tessellations

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In the remaining cases, the geometry of the underlying plane for $U(d, \ell)$ is Euclidean if $1/d + 1/\ell = 1/2$, and hyperbolic if $1/d + 1/\ell < 1/2$.

Tessellations

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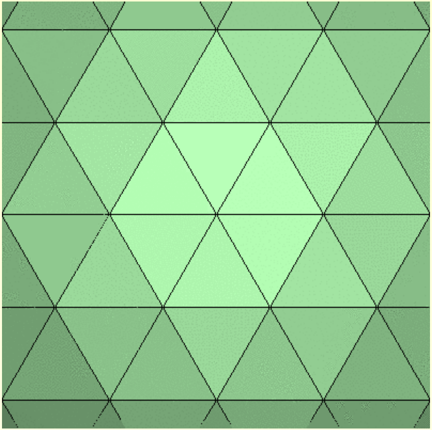
Examples of tessellations $U(d, \ell)$: $U(6,3)$ and $U(5,4)$



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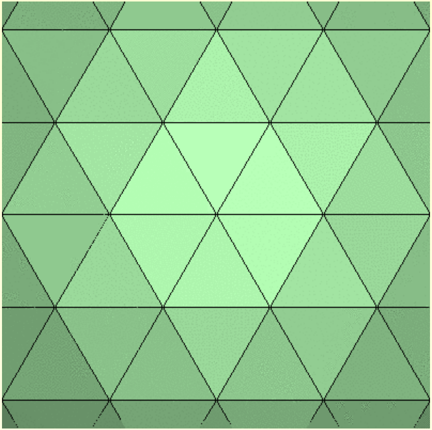

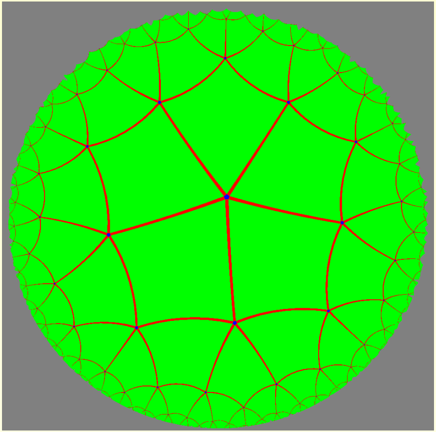

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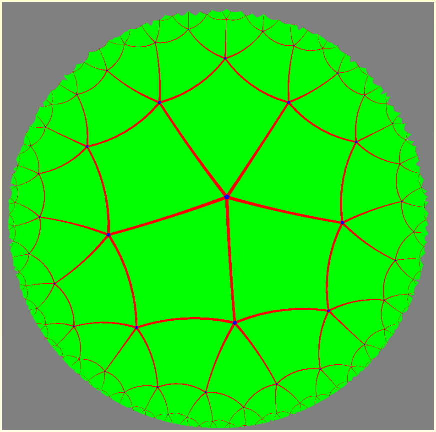

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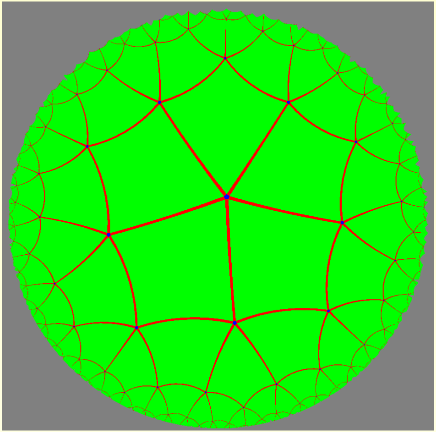

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$$\langle X, Y;\ X^d = Y^2 = (XY)^\ell = 1\rangle$$

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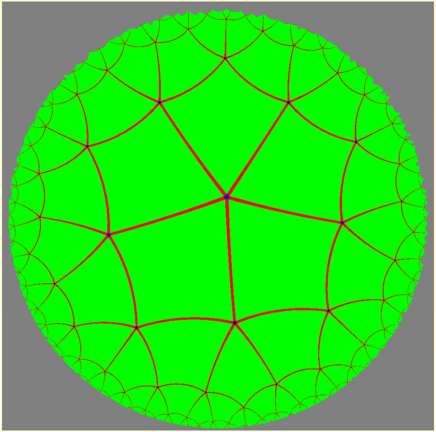

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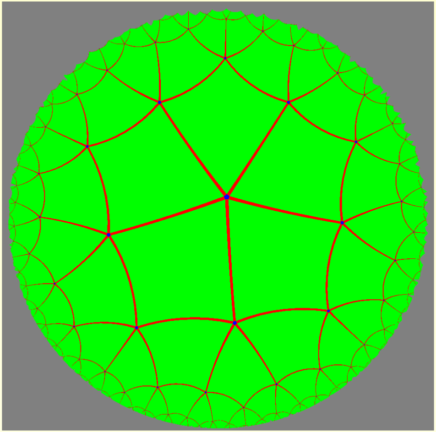

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The group acts regularly on directed edges of $U(d,\ell)$.

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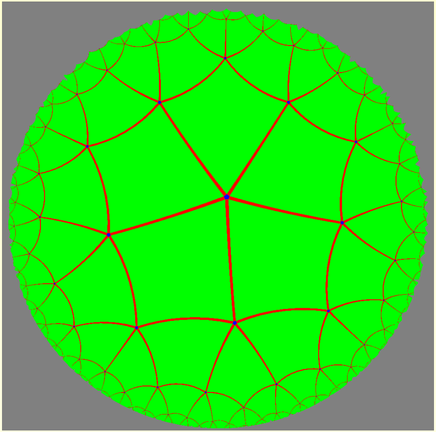

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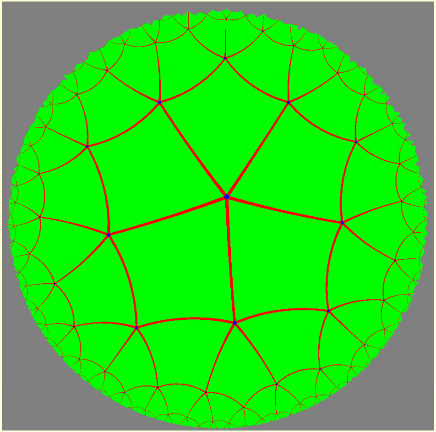

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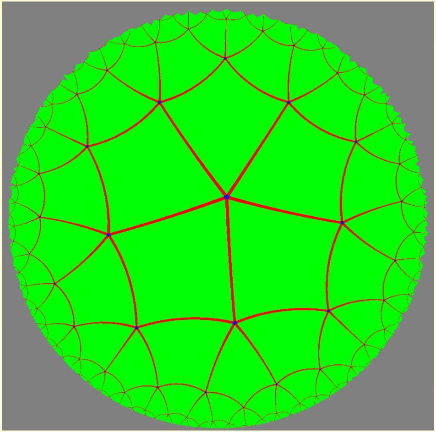

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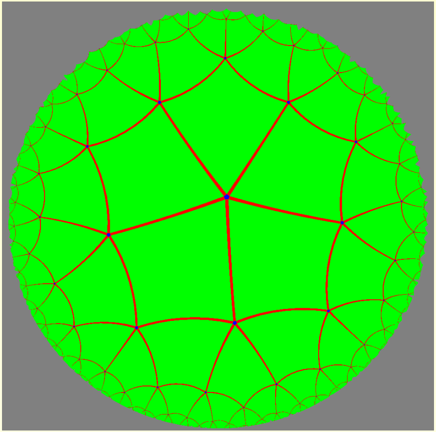

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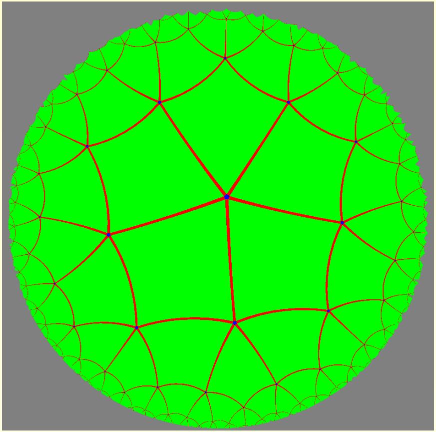

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Note: $T(d,\ell) \triangleleft_2 ET(d,\ell)$.

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For fixed (d, ℓ), all maps of type (d', ℓ') with $d' \mid d$ and $\ell' \mid \ell$ can be identified with *quotient spaces* $U(d, \ell)/H$ of tessellations by subgroups $H < ET(d, \ell)$.

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If $H < ET(d, \ell)$, then the group $Aut(U(d, \ell)) \cong ET(d, \ell)$ projects onto the map $M = U(d, \ell)/H$ if and only if H is normal in $ET(d, \ell)$, and then $Aut(M) \cong ET(d, \ell)/H$.

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and so it can be identified with $Aut(M) := G$ with presentation

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Since this group is residually finite, it contains infinitely many normal, torsion-free subgroups of finite index. Let N be such a subgroup and let

$$K = \bigcap_{\pi\in S_3} \pi(N)$$

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Theorem [Jones, Poulton, '10] *There exist infinitely many finite regular maps admitting the external symmetry PD (triality) but no duality.*

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Orientably regular maps with the 'full' exp group Z_d^* are kaleidoscopic.

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This solves Wilson's conjecture.

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Theorem A and Conder's list give kaleidoscopic regular maps with trinity symmetry for the following degrees and orders of automorphism groups:

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'Unbounded' groups generated by exponents and dualities

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THANK YOU.